MATHEMATICS

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DICTIONARY

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PREFACE

The guiding objective in the preparation of this dictionary has been to make it useful for students, engineers, and others using mathematics in their professions, as well as to make it sufficiently complete to be a valuable reference book for anyone needing to understand some particular mathematical concept or wishing to extend his knowledge of mathematics.

This dictionary is an enlargement and revision of the *Mathematics Dictionary,* (1949), which was an enlargement of the *James Mathematics Dictionary* (1942). The *Mathematics Dictionary* (1949) contained an exhaustive coverage of terms in the range beginning with arithmetic and extending through the calculus. Also included were basic terms in differential geometry, theory of functions of real and complex variables, advanced calculus, differential equations, theory of group and matrices, theory of summability, point-set topology, integral equations, calculus of variations, analytic mechanics, theory of potential, and statistics, as well as many miscellaneous terms of importance in applications and in the structure of sequences of mathematical courses. This book has now been largely revised and many corrections made. Many additional terms have been added in the above fields, as well as miscellaneous terms chosen to increase the value of the dictionary. Additions include the basic terms in the fields of modern algebra, number theory, topology, vector spaces, the theory of games and linear and dynamic programming, numerical analysis, and computing machines.

The appendix contains many useful tables as well as an extensive list of mathematical symbols. Formulas of many kinds appear in the context.

Leading words are printed in bold-face capitals beginning at the left margin, followed by an abbreviation indicating the part (or parts) of speech of the entry word —as determined by its definition and its uses in the following subheadings. Subheadings are printed in bold-face at the beginning of paragraphs and in alphabetical order on the basis of their leading words.

Citations give the leading word in capitals (unless the citation is under the leading word) followed by a dash and then by the subheading, if necessary, as: ANGLE—adjacent angle.

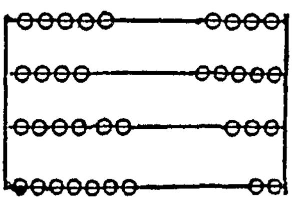
Although this is by no means a mere word dictionary, neither is it an encyclopedia. It is rather a correlated condensation of mathematical concepts, designed for time-saving reference work. Nevertheless the general reader can come to an understanding of concepts in which he has not been schooled by looking up the unfamiliar terms in the definition at hand and following this procedure down to familiar concepts.

Comments on definitions and discussions of any phase of this dictionary are invited.

GLENN JAMES
ROBERT C. JAMES

A

AB'A-CUS, *n.* [*pl.* **abaci**]. A counting frame to aid in arithmetic computation; an instructive plaything for children, used as an aid in teaching place value; a primitive predecessor of the modern computing machine. One form consists of a rectangular frame carrying as many parallel wires as there are digits in the largest number to be dealt with. Each wire contains nine beads free to slide on it. A bead on the lowest wire counts unity, on the next higher wire 10, on the next higher 100, etc. Two beads slid to the right on the lowest wire, three on the next higher, five on the next and four on the next denote 4532.



ABEL. Abel's identity. The identity

$$\sum_{i=1}^{n} a_i u_i \equiv s_1(a_1-a_2)+s_2(a_2-a_3)+\cdots+s_{n-1}(a_{n-1}-a_n)+s_n a_n,$$

where

$$s_n=\sum_{i=1}^{n} u_i.$$

This is easily obtained from the evident identity:

$$\sum_{i=1}^{n} a_i u_i \equiv a_1 s_1 + a_2(s_2-s_1)+\cdots+a_n(s_n-s_{n-1}).$$

Abel's inequality. If $u_n \geq u_{n+1} > 0$ for all positive integers n, then $\left|\sum_{n=1}^{r} a_n u_n\right| \leq Lu_1$, where L is the largest of the quantities: $|a_1|$, $|a_1+a_2|$, $|a_1+a_2+a_3|$, $\cdots$ $|a_1+a_2+\cdots+a_r|$. This inequality can be easily deduced from Abel's identity.

Abel's method of summation. The method of *summation* for which a series $\sum_0^\infty a_n$ is *summable* and has *sum* S if $\lim_{x\to 1-} \sum_0^\infty a_n x^n = S$ exists. A convergent series is summable by this method [see below, Abel's theorem on power series (2)]. Also called *Euler's method of summation.* See SUMMATION—summation of a divergent series.

Abel's problem. Suppose a particle is constrained (without friction) to move along a certain path in a vertical plane under the force of gravity. Abel's problem is to find the path for which the time of descent is a given function $f(x)$ of x, where x is the horizontal axis and the particle starts from rest. This reduces to the problem of finding a solution $s(x)$ of the *Volterra integral equation of the first kind* $f(x)=\int_0^x \frac{s(t)}{\sqrt{2g(x-t)}}\,dt$, where $s(x)$ is the length of the path. If $f'(x)$ is continuous, a solution is $s(x)=\frac{\sqrt{2g}}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{1/2}}\,dt.$

Abel's tests for convergence. (1) If the series Σu_n converges and $\{a_n\}$ is a bounded monotonic sequence, then $\Sigma a_n u_n$ converges. (2) If $\sum_{n=1}^{k} u_n$ is equal to or less than a properly chosen constant for all k and $\{a_n\}$ is a positive, monotonic decreasing sequence which approaches zero as a limit, then $\Sigma a_n u_n$ converges. (3) If a series of complex numbers Σa_n is convergent, and the series $\Sigma(v_n - v_{n+1})$ is absolutely convergent, then $\Sigma a_n v_n$ is convergent. (4) If the series $\Sigma a_n(x)$ is uniformly convergent in an interval (a, b), $v_n(x)$ is positive and monotonic decreasing for any value of x in the interval, and there is a number k such that $v_0(x) < k$ for all x in the interval, then $\Sigma a_n(x)v_n(x)$ is uniformly convergent (this is called Abel's test for uniform convergence).

Abel's theorem on power series. (1) If a power series, $a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$, converges for $x=c$, it converges absolutely for $|x|<|c|$. (2) If a power series converges to $f(x)$ for $|x|<1$ and to s for $x=1$, then $\lim_{x\to 1} f(x)=s$ $(0 \leq x \leq 1)$. The latter theorem is variously designated, most explicitly by "Abel's

Theorem on Continuity up to the Circle of Convergence."

A-BEL'IAN, *adj.* **Abelian group.** See GROUP.

A-BRIDGED', *adj.* **abridged multiplication.** See MULTIPLICATION.

Plücker's abridged notation. A notation used for studying curves. Consists of the use of a single symbol to designate the expression (function) which, equated to zero, has a given curve for its locus; hence reduces the composition of curves to the study of polynomials of the first degree. *E.g.*, if $L_1=0$ denotes $2x+3y-5=0$ and $L_2=0$ denotes $x+y-2=0$, then $k_1L_1+k_2L_2=0$ denotes the family of lines passing through their common point (1, 1). See PENCIL—pencil of lines through a point.

AB-SCIS'SA, *n.* [*pl.* **abscissas or abscissae**]. The horizontal coordinate in a two-dimensional system of rectangular coordinates; usually denoted by x. Also used in a similar sense in systems of oblique coordinates. See CARTESIAN—Cartesian coordinates.

AB'SO-LUTE, *adj.* **absolute constant, continuity, convergence, inequality, maximum (minimum), symmetry.** See CONSTANT; CONTINUOUS; CONVERGENCE; INEQUALITY; MAXIMUM; SYMMETRIC—symmetric function.

absolute number. A number represented by figures such as 2, 3, or $\sqrt{2}$, rather than by letters as in algebra.

absolute property of a surface. Same as INTRINSIC PROPERTY OF A SURFACE.

absolute term in an expression. A term which does not contain a variable. *Syn.* Constant term. In the expression ax^2+bx+c, c is the only absolute term.

absolute value of a complex number. See MODULUS—modulus of a complex number.

absolute value of a real number. Its value without regard for sign; its numerical value. The number 2 is the absolute value of both $+2$ and -2.

absolute value of a vector. See VECTOR—absolute value of a vector.

AB'STRACT, *adj.* **abstract mathematics.** See MATHEMATICS—pure mathematics.

abstract number. Any number as such, simply as a number, without reference to any particular objects whatever except in so far as these objects possess the number property. Used to emphasize the distinction between a number, as such, and concrete numbers. See NUMBER, and CONCRETE.

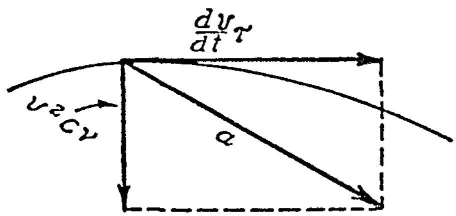
abstract word or symbol. (1) A word or symbol that is not concrete; a word or symbol denoting a concept built up from consideration of many special cases; a word or symbol denoting a property common to many individuals or individual sets, as yellow, hard, two, three, etc. (2) A word or symbol which has no specific reference in the sense that the concept it represents exists quite independently of any specific cases whatever and may or may not have specific reference.

AC-CEL'ER-A'TION, *n.* The time rate of change of velocity. Since velocity is a directed quantity, the acceleration **a** is a vector equal to $\lim_{\Delta t\to 0}\frac{\Delta \mathbf{v}}{\Delta t}=\frac{d\mathbf{v}}{dt}$, where $\Delta\mathbf{v}$ is the increment in the velocity **v** which the moving object acquires in t units of time. Thus, if an airplane moving in a straight line with the speed of 2 miles per minute increases its speed until it is flying at the rate of 5 miles per minute at the end of the next minute, its **average acceleration** during that minute is 3 miles per minute per minute. If the increase in speed during this one minute interval of time is uniform, the average acceleration is equal to the actual acceleration. If the increase in speed in this example is not uniform, the instantaneous acceleration at the time t_1 is determined by evaluating the limit of the quotient $\frac{\Delta\mathbf{v}}{\Delta t}$ as the time interval $\Delta t=t-t_1$ is made to approach zero by making t approach t_1. For a particle moving along a curved path the velocity **v** is directed along the tangent to the path and the acceleration **a** can be shown to be given by the formula

$$\mathbf{a}=\frac{dv}{dt}\boldsymbol{\tau}+v^2c\boldsymbol{\nu},$$

where $\frac{dv}{dt}$ is the derivative of speed v along the path c is the curvature of the path at any point, and $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are vectors of unit

lengths directed along the tangent and normal to the path. The first of these terms, $\frac{dv}{dt}$, is called the tangential component, and the second, v^2c, the normal (or centripetal) component of acceleration. If the path is a straight line, the curvature c is zero, and hence the acceleration vector will be directed along the path of motion. If the path is not rectilinear, the direction of the acceleration vector is determined by its tangential and normal components as shown in the figure.



acceleration of Coriolis. If S' is a reference frame rotating with the angular velocity ω about a fixed point in another reference frame S, the acceleration $\mathbf{a}$ of a particle, as measured by the observer fixed in the reference frame S, is given by the sum of three terms: $\mathbf{a}=\mathbf{a}'+\mathbf{a}_t+\mathbf{a}_c$, where $\mathbf{a}'$ is the acceleration of the particle relative to S', $\mathbf{a}_t$ is the acceleration of the moving space, and $\mathbf{a}_c=2\omega\times\mathbf{v}'$ is the *acceleration of Coriolis.* The symbol $\omega\times\mathbf{v}'$ denotes the vector product of the angular velocity ω, and the velocity $\mathbf{v}'$ relative to S', so that the acceleration of Coriolis is normal to the plane determined by the vectors ω and $\mathbf{v}'$ and has the magnitude $2v'\sin(\omega, \mathbf{v}')$. The acceleration of Coriolis is also called the complementary acceleration.

acceleration of a falling body. The acceleration with which a body falls *in vacuo* at a given point on or near a given point on the earth's surface. This acceleration, frequently denoted by g, varies by less than one percent over the entire surface of the earth. Its "average value" has been defined by the International Commission of Weights and Measures as 9.80665 meters (or 32.174 feet) per second per second. Its value at the poles is 9.8321 and at the equator 9.7799. *Syn.* Acceleration of gravity.

angular acceleration. The time rate of change of angular velocity. If the angular velocity is represented by a vector ω directed along the axis of rotation, then the angular acceleration α, in the symbolism of calculus, is given by $\alpha=\frac{d\omega}{dt}$. See VELOCITY—angular velocity.

centripetal, normal, and tangential components of acceleration. See above, ACCELERATION.

uniform acceleration. Acceleration in which there are equal changes in the velocity in equal intervals of time. *Syn.* Constant acceleration.

AC'CENT, *n.* A mark above and to the right of a quantity (or letter), as in a' or x'; the mark used in denoting that a letter is primed. See PRIME—prime as a symbol.

AC-CU'MU-LAT'ED, *adj.* **accumulated value.** Same as AMOUNT at simple or compound interest. The accumulated value (or amount) of an annuity at a given date is the sum of the compound amounts of the annuity payments to that date.

AC-CU'MU-LA'TION, *adj., n.* Same as ACCUMULATED VALUE.

accumulation of discount on a bond. Writing up the book value of a bond on each dividend date by an amount equal to the interest on the investment (interest on book value at yield rate) minus the dividend. See VALUE—book value.

accumulation factor. The name sometimes given to the binomial $(1+r)$, or $(1+i)$, where r, or i, is the rate of interest. The formula for compound interest is $A=P(1+r)^n$, where A is the amount accumulated at the end of n periods from an original principal P at a rate r. See COMPOUND—compound amount, and TABLE III in the appendix.

accumulation point. An accumulation point of a set of points is a point P such that there is at least one point of the set distinct from P in any neighborhood of the given point; a point which is the limit of a sequence of points of the set (for spaces which satisfy the first axiom of countability). An accumulation point of a sequence is a point P such that there are an infinite number of terms of the sequence in any neighborhood of P; *e.g.*, the sequence

$1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \cdots$ has two accumulation points, the numbers 0 and 1 (also see SEQUENCE—accumulation point of a sequence). *Syn.* Cluster point, limit point. See BOLZANO—Bolzano-Weierstrass theorem, and CONDENSATION—condensation point.

accumulation problem. The determination of the amount when the principal, or principals, interest rate, and time for which each principal is invested are given. See TABLES III and IV in the appendix.

accumulation schedule of bond discount. A table showing the accumulation of bond discounts on successive dates. Interest and book values are usually listed also.

AC-CU′MU-LA′TOR, *n.* In a computing machine, an adder or counter that augments its stored number by each successive number it receives.

AC′CU-RA-CY, *n.* Correctness, usually referring to numerical computations. The **accuracy of a table** may mean either: (1) The number of significant digits appearing in the numbers in the table (*e.g.*, in the mantissas of a logarithm table); (2) the number of correct places in computations made with the table. (This number of places varies with the form of computation, since errors may repeatedly combine so as to become of any size whatever.)

AC′CU-RATE, *adj.* Exact, precise, without error. One speaks of an accurate statement in the sense that it is correct or true and of an accurate computation in the sense that it contains no numerical error. **Accurate to a certain decimal place** means that all digits preceding and including the given place are correct and the next place has been made zero if less than 5 and 10 if greater than 5 (if it is equal to 5, the most usual convention is to call it zero or 10 as is necessary to leave the last digit even). *E.g.*, 1.26 is accurate to two places if obtained from either 1.264 or 1.256 or 1.255. See ROUNDING—rounding off numbers.

AC′NODE, *n.* See POINT—isolated point.

A-COUS′TI-CAL, *adj.* **acoustical property of conics.** See ELLIPSE—focal property of ellipse, HYPERBOLA—focal property of hyperbola, and PARABOLA—focal property of parabola.

A′CRE, *n.* The unit commonly used in the United States in measuring land; contains 43,560 square feet, 4,840 square yards, or 160 square rods.

AC′TION, *n.* A concept in advanced dynamics defined by the line integral $A = \int_{P_1}^{P_2} m\mathbf{v} \cdot d\mathbf{r}$ called the **action integral,** where m is the mass of the particle, $\mathbf{v}$ is its velocity, and $d\mathbf{r}$ is the vector element of the arc of the trajectory joining the points P_1 and P_2. The dot in the integrand denotes the scalar product of the momentum vector $m\mathbf{v}$ and $d\mathbf{r}$. The *action* A plays an important part in the development of dynamics from variational principles. See below, principle of least action.

law of action and reaction. The basic law of mechanics asserting that two particles interact so that the forces exerted by one on another are equal in magnitude, act along the line joining the particles, and are opposite in direction. See NEWTON—Newton's laws of motion.

principle of least action. Of all curves passing through two fixed points in the neighborhood of the natural trajectory, and which are traversed by the particle at a rate such that for each (at every instant of time) the sum of the kinetic and potential energies is a constant, that one for which the action integral has an extremal value is the natural trajectory of the particle. See ACTION.

A-CUTE′, *adj.* **acute angle.** An angle numerically smaller than a right angle; usually refers to positive angles less than a right angle.

acute triangle. See TRIANGLE.

AD′DEND, *n.* One of a set of numbers to be added, as 2 or 3 in the sum $2+3$.

AD′DER, *n.* In a computing machine, any arithmetic component that performs the operation of addition of positive numbers. An arithmetic component that performs the operations of addition and

subtraction is said to be an **algebraic adder.** See ACCUMULATOR, and COUNTER.

AD-DI'TION, *n.* See SUM.

addition of angles, directed line segments, integers, fractions, irrational numbers, mixed numbers, matrices, and vectors. See various headings under SUM.

addition of complex numbers. See COMPLEX—complex numbers.

addition of decimals. The usual procedure for adding decimals is to place digits with like place value under one another, *i.e.*, place decimal points under decimal points, and add as with integers, putting the decimal point of the sum directly below those of the addends. See SUM—sum of real numbers.

addition formulas of trigonometry. See TRIGONOMETRY.

addition of series. See SERIES.

addition of similar terms in algebra. The process of adding the coefficients of terms which are alike as regards their other factors: $2x+3x=5x$, $3x^2y-2x^2y=x^2y$ and $ax+bx=(a+b)x$. See DISSIMILAR TERMS.

addition of tensors. See TENSOR.

algebraic addition. See SUM—algebraic sum, sum of real numbers.

arithmetic addition. See SUM—arithmetic sum of two quantities.

proportion by addition (and addition and subtraction). See PROPORTION.

ADD'I-TIVE, *adj.* **additive function.** A function f which has the property that $f(x+y)$ is defined and equals $f(x)+f(y)$ whenever $f(x)$ and $f(y)$ are defined. A *continuous* additive function is necessarily *homogeneous*. A function f is **subadditive** or **superadditive** according as

$$f(x_1+x_2)\leqq f(x_1)+f(x_2),$$

or

$$f(x_1+x_2)\geqq f(x_1)+f(x_2),$$

for all x_1, x_2, and x_1+x_2 in the domain of definition of f (this domain is usually taken to be an interval of the form $0\leqq x\leqq a$).

additive set function. A function which assigns a number $\phi(X)$ to each set X of a family F of sets is **additive** (or **finitely additive**) if the *union* of any two members of F is a member of F and

$$\phi(X\cup Y)=\phi(X)+\phi(Y)$$

for any disjoint members X and Y of F. The function ϕ is **completely additive** (or **countably additive**) if the union of any finite or countable set of members of F is a member of F and

$$\phi(\cup X_i)=\Sigma\phi(X_i)$$

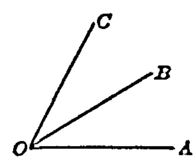
for any finite or countable collection of sets $\{X_i\}$ which are *pairwise disjoint* and belong to F. If $\phi(\cup X_i)\leqq\Sigma\phi(X_i)$, then ϕ is said to be **subadditive** (it is then not necessary to assume the sets are pairwise disjoint). See MEASURE—measure of a set.

AD'I-A-BAT'IC, *adj.* **adiabatic curves.** Curves showing the relation between pressure and volume of substances which are assumed to have adiabatic expansion and contraction.

adiabatic expansion (or contraction). (*Thermodynamics*) A change in volume without loss or gain of heat.

AD IN'FI-NI'TUM. Continuing without end (according to some law); denoted by three dots, as $\cdots$; used, principally, in writing infinite series, infinite sequences, and infinite products.

AD-JA'CENT, *adj.* **adjacent angles.** Two angles having a common side and common vertex and lying on opposite sides of their common side. In the figure, AOB and BOC are adjacent angles.



AD-JOINED', *adj.* **adjoined number.** See FIELD—number field.

AD'JOINT, *adj., n.* **adjoint of a differential equation.** For a homogeneous differential equation

$$L(y)\equiv p_0\frac{d^ny}{dx^n}+p_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+p_{n-1}\frac{dy}{dx}+p_ny=0,$$

the adjoint is the differential equation

$$\bar{L}(y)\equiv(-1)^n\frac{d^n(p_0y)}{dx^n}+(-1)^{n-1}\frac{d^{n-1}(p_1y)}{dx^{n-1}}+\cdots-\frac{d(p_{n-1}y)}{dx}+p_ny=0.$$

This relation is symmetric, $L=0$ being the adjoint of $\bar{L}=0$. A function is a solution of one of these equations if and only if it is an *integrating factor* of the other. There is an expression $P(u, v)$ for which

$$vL(u)-u\bar{L}(v)\equiv\frac{dP(u, v)}{dx}.$$

$P(u, v)$ is linear and homogeneous in $u, u', \cdots, u^{(n-1)}$, and in $v, v', \cdots, v^{(n-1)}$. It is known as the **bilinear concomitant.** An equation is **self-adjoint** if $L(y)\equiv\bar{L}(y)$. *E.g.*, Sturm-Liouville differential equations and Legendre differential equations are self-adjoint.

adjoint of a matrix. The *transpose* of the matrix obtained by replacing each element by its *cofactor*; the matrix obtained by replacing each element a_{rs} (in row r and column s) by the cofactor of the element a_{sr} (in row s and column r). The adjoint is defined only for square matrices. The *Hermitian conjugate* matrix is frequently called the adjoint matrix by writers on quantum mechanics.

adjoint of a transformation. For a bounded linear transformation T which maps a **Hilbert space** H into H (with domain of T equal to H), there is a unique bounded linear transformation T^*, the *adjoint* of T, such that the inner products (Tx, y) and (x, T^*y) are equal for all x and y of H. It follows that $\|T\|=\|T^*\|$. Two linear transformations T_1 and T_2 are said to be *adjoint* if $(T_1x, y)=(x, T_2y)$ for any x in the domain of T_1 and y in the domain of T_2. If T is a linear transformation whose domain is dense in H, there is a unique transformation T^* (called the *adjoint* of T) such that T and T^* are adjoint and, if S is any other transformation adjoint to T, then the domain of S is contained in the domain of T^* and S and T^* coincide on the domain of S. For a **finite dimensional space** and a transformation T which maps vectors $x=(x_1, x_2, \cdots, x_n)$ into $Tx=(y_1, y_2, \cdots, y_n)$ with $y_i=\sum_j a_{ij}x_j$ (for each i), the adjoint of T is the transformation for which $T^*x=(y_1, y_2, \cdots, y_n)$ with $y_i=\sum_j \bar{a}_{ji}x_i$ and the matrices of the coefficients of T and of T^* are *Hermitian conjugates* of each other. If T is a bounded linear transformation which maps a **Banach space** X into a Banach space Y and X^* and Y^* are the *first conjugate spaces* of X and Y, then the adjoint of T is the linear transformation T^* which maps Y^* into X^* and is such that $T^*(g)=f$ (for f and g members of X^* and Y^*, respectively) if f is the continuous linear functional defined by $f(x)=g[T(x)]$. For two bounded linear transformations T_1, and T_2, the adjoints of T_1+T_2 and $T_1\cdot T_2$ are $T_1^*+T_2^*$ and $T_2^*\cdot T_1^*$, respectively. If T has an inverse whose domain is all of H (or Y), then $(T^*)^{-1}=(T^{-1})^*$. For Banach spaces, the adjoint T^{**} of T^* is a mapping of X^{**} into Y^{**} which is a norm-preserving extension of T (T maps a subset of X^{**}, which is isometric with X, into Y^{**}). For Hilbert space, T^{**} is identical with T if T is bounded with domain H; T^{**} is a linear extension of T otherwise. See SELF—self-adjoint transformation.

adjoint space. See CONJUGATE—conjugate space.

AF-FINE′, *adj.* **affine transformation.** (1) A transformation of the form

$$x'=a_1x+b_1y+c_1, \quad y'=a_2x+b_2y+c_2,$$

where $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1b_2-a_2b_1\neq 0.$

(2) A transformation of the form given in (1) except that the determinant of the coefficients may or may not be zero (it is **singular** or **nonsingular** according as this determinant is zero or nonzero). The determinant of the coefficients is denoted by Δ. The following are important special cases of the affine transformation, $\Delta\neq 0$: (a) a **translation** ($x'=x+a$, $y'=y+b$); (b) a **rotation** ($x'=x\cos\theta+y\sin\theta$, $y'=-x\sin\theta+y\cos\theta$); (c) a **stretching and shrinking** ($x'=kx$, $y'=ky$), called *transformations of similitude* or *homothetic transformations*; (d) **reflections** in the x-axis and y-axis, respectively, ($x'=x$, $y'=-y$ or $x'=-x$, $y'=y$); (e) **simple elongations and compressions** ($x'=x$, $y'=ky$ or $x'=kx$, $y'=y$), sometimes called *one-dimensional strains* and *one-dimensional elongations and compressions.* The affine transformation carries parallel lines into parallel lines, finite points into finite points and leaves the line at infinity fixed. An affine transformation can always be factored into the product of transformations

belonging to the above special cases. A **homogeneous affine transformation** is an affine transformation in which the constant terms are zero; an affine transformation which does not contain a translation as a factor. Its form is

$$x'=a_1x+b_1y, \quad y'=a_2x+b_2y,$$
$$\Delta=a_1b_2-a_2b_1\neq 0.$$

An **isogonal affine transformation** is an affine transformation which does not change the size of angles. It has the form

$$x'=a_1x+b_1y+c_1,$$
$$y'=a_2x+b_2y+c_2,$$

where either $a_1=b_2$ and $a_2=-b_1$ or $-a_1=b_2$ and $a_2=b_1$.

AGE, *n.* **age at issue.** (*Life Insurance.*) The age of the insured at his birthday nearest the policy date.

age year. (*Life Insurance.*) A year in the lives of a group of people of a certain age. The age year l_x refers to the year from x to $x+1$, the year during which the group is x years old.

AG'GRE-GA'TION, *n.* **Signs of aggregation:** Parenthesis, (); bracket, []; brace, {}; and vinculum or bar, ——. Each means that the terms enclosed are to be treated as a single term. *E.g.*, $3(2-1+4)$ means 3 times 5, or 15. See various headings under DISTRIBUTIVE.

AGNESI. witch of Agnesi. Same as WITCH.

AHMES (RHYND or RHIND) PAPYRUS. Probably the oldest mathematical book known, written about 1550 B.C.

A'LEPH, *n.* The first letter of the Hebrew alphabet, written ℵ.

aleph-null or aleph-zero. The *cardinal number* of countably infinite sets, written $\aleph_0$. See CARDINAL—cardinal number.

AL'GE-BRA, *n.* (1) A generalization of arithmetic. *E.g.*, the arithmetic facts that $2+2+2=3\times 2$, $4+4+4=3\times 4$, etc., are all special cases of the (general) algebraic statement that $x+x+x=3x$, where x is any number. Letters denoting any number, or any one of a certain set of numbers, such as all real numbers, are related by laws that hold for any numbers in the set; *e.g.*, $x+x=2x$ for all x (all numbers). On the other hand, conditions may be imposed upon a letter, representing any one of a set, so that it can take on but one value, as in the study of equations; *e.g.*, if $2x+1=9$, then x is restricted to 4. Equations are met in arithmetic, although not so named. For instance, in percentage one has to find one of the unknowns in the equation, interest = principal × rate, or $I=p\times r$, when the other two are given. (2) A system of logic expressed in algebraic symbols, or a Boolean algebra (see BOOLEAN). (3) See below, algebra over a field.

algebra over a field. A set R is an algebra over the field F if R is a *ring* and multiplication of elements of R by "scalars" belonging to F is defined and satisfies: $(a+b)x=ax+bx$, $a(x+y)=ax+ay$, $a(bx)=(ab)x$, $1\cdot x=x$, and $(ax)(bx)=(ab)(xy)$, where a and b are any members of F and x and y are any members of R. The algebra is a **commutative algebra,** or an **algebra with unit element,** according as the ring is a commutative ring, or a ring with unit element. The set of real numbers is a commutative algebra with unit element over the field of rational numbers; for any positive integer n, the set of all square matrices of order n with complex numbers (or real numbers) as elements is an algebra (noncommutative) over the field of real numbers.

algebra of subsets. An algebra of subsets of a set X is a class of subsets of X which contains the *complement* of each of its members and the *union* of any two of its members (or the *intersection* of any two of its members). It is called a σ-algebra if it also contains the union of any sequence of its members. An algebra of subsets is a *Boolean algebra* relative to the operations of union and intersection. A *ring* of subsets of a set X is an algebra of subsets of X if and only if it contains X as a member. For any class C of subsets of a set X, the intersection of all algebras (or σ-algebras) which contain C is the smallest algebra (σ-algebra) which contains C and is said to be the algebra (σ-algebra) generated by C. For the real line (or n-dimensional space) examples of σ-algebras are the system of all measurable sets, the system of all Borel

sets, and the system of all sets having the property of Baire. See RING—ring of sets.

Banach algebra. An algebra over the field of real numbers (or complex numbers) which is also a real (or complex) Banach space for which $\|xy\| \leqq \|x\| \cdot \|y\|$ for all x and y. It is called a **real** or a **complex Banach algebra** according as the field is the real or the complex number field. The set of all functions which are continuous on the closed interval $[0, 1]$ is a Banach algebra over the field of real numbers if $\|f\|$ is defined to be the largest value of $f(x)$ for $0 \leqq x \leqq 1$. *Syn.* Normed vector ring.

Boolean algebra. See BOOLEAN.

measure algebra. See MEASURE—measure ring and measure algebra.

fundamental theorem of algebra. See FUNDAMENTAL—fundamental theorem of algebra.

AL'GE-BRAIC, *adj.* **algebraic adder.** See ADDER.

algebraic addition. See SUM—algebraic sum, sum of real numbers.

algebraic curve. See CURVE.

algebraic deviation. See DEVIATION.

algebraic expression, equation, function, operation, etc. An expression, etc., containing or using only algebraic symbols and operations, such as $2x+3$, x^2+2x+4, or $\sqrt{2}-x+y=3$. See FUNCTION—algebraic function.

algebraic multiplication. See MULTIPLICATION.

algebraic number. (1) Any ordinary positive or negative number; any real directed number. (2) Any number which is a root of a polynomial equation with rational coefficients; the degree of the polynomial is said to be the **degree** of the algebraic number α, and the equation is the **minimal equation** of α, if α is not a root of such an equation of lower degree. An **algebraic integer** is an algebraic number which satisfies some *monic* equation

$$x^n + a_1 x^{n-1} + \cdots + a^n = 0,$$

with *integers* as coefficients. The minimal equation of an algebraic integer is also monic. A rational number is an algebraic integer if and only if it is an ordinary integer. The set of all algebraic numbers is an integral domain (see DOMAIN—integral domain).

algebraic operations. Addition, subtraction, multiplication, division, evolution, and involution (extracting roots and raising to a power), no infinite processes being used.

algebraic proofs and solutions. Proofs and solutions which use algebraic symbols and no operations other than those which are algebraic. See above, algebraic operations.

algebraic subtraction. See SUBTRACTION.

algebraic symbols. Letters representing numbers, and the various operational symbols indicating *algebraic operations.* See MATHEMATICAL SYMBOLS in the appendix.

irrational algebraic surface. See IRRATIONAL.

AL'GO-RITHM, *n.* Some special process of solving a certain type of problem.

Euclid's Algorithm. A method of finding the greatest common divisor (G.C.D.) of two numbers—one number is divided by the other, then the second by the remainder, the first remainder by the second remainder, the second by the third, etc. When exact division is finally reached, the last divisor is the greatest common divisor of the given numbers (integers). In algebra the same process can be applied to polynomials. *E.g.*, to find the highest common factor of 12 and 20, we have $20 \div 12$ is 1 with remainder 8; $12 \div 8$ is 1 with remainder 4; and $8 \div 4 = 2$; hence 4 is the G.C.D.

AL'IEN-A'TION, *n.* **coefficient of alienation.** See CORRELATION—normal correlation.

A-LIGN'MENT, *adj.* **alignment chart.** Same as NOMOGRAM.

AL'I-QUOT PART. Any exact divisor of a quantity; any factor of a quantity; used almost entirely when dealing with integers. *E.g.*, 2 and 3 are *aliquot parts* of 6.

AL'MOST, *adj.* **almost everywhere.** See MEASURE—measure zero.

almost periodic. See PERIODIC.

AL'PHA, *n.* The first letter in the Greek alphabet, written α.

AL'PHA-BET, GREEK. See the APPENDIX.

AL-TER'NANT, *n.* A determinant for which there are n functions $f_1, f_2, \cdots, f_n$ (if the determinant is of order n) and n quantities $r_1, r_2, \cdots, r_n$ for which the element in the ith column and jth row is $f_i(r_j)$ for each i and j (this determinant with rows and columns interchanged is also called an alternant). The Vandermonde determinant is an alternant (see DETERMINANT).

AL'TER-NATE, *adj.* **alternate angles.** Angles on opposite sides of a transversal cutting two lines, each having one of the lines for one of its sides. See ANGLE—angles made by a transversal.

alternate exterior angles. *Alternate angles* neither of which lies between the two lines cut by a transversal. See ANGLE—angles made by a transversal.

alternate interior angles. Either of the two pairs of *alternate angles* lying between the two lines cut by a transversal. See ANGLE—angles made by a transversal.

AL-TER-NAT'ING, *adj.* **alternating group.** See GROUP—alternating group.

alternating series. A series whose terms are alternately positive and negative, as

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots+(-1)^{n-1}/n+\cdots,$$

An alternating series converges if each term is numerically equal to or less than the preceding and if the nth term approaches zero as n increases without limit. This is a sufficient but not a necessary set of conditions—the term-by-term sum of any two convergent series converges and, if one series has all negative terms and the other all positive terms, their indicated sum may be a convergent alternating series and not have its terms monotonically decreasing. The series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{9}-\tfrac{1}{8}+\tfrac{1}{27}-\tfrac{1}{16}+\cdots$$

is such a series. See NECESSARY—necessary condition for convergence.

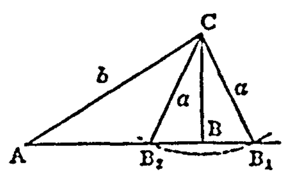
AL'TER-NA'TION, *n.* **proportion by alternation.** See PROPORTION.

AL'TI-TUDE, *n.* See CONE, CYLINDER, PARABOLIC—parabolic segment, PARALLELOGRAM, PARALLELEPIPED, PRISM, PYRAMID, RECTANGLE, SEGMENT—spherical segment, TRAPEZOID, TRIANGLE, ZONE.

altitude of a celestial point. Its angular distance above, or below, the observer's horizon, measured along a great celestial circle (vertical circle) passing through the point, the zenith, and the nadir. The altitude is taken positive when the object is above the horizon and negative when below. See figure under HOUR—hour-angle and hour-circle.

AM-BIG'U-OUS, *adj.* Not uniquely determinable.

ambiguous case in the solution of triangles. For a plane triangle, the case in which two sides and the angle opposite one of them is given. One of the other angles is then found by use of the law of sines; but there are always two angles less than 180° corresponding to any given value of the sine (unless the sine be unity, in which case the angle is 90° and the triangle is a right triangle). When the sine gives two distinct values of the angle, two triangles result if the side opposite is less than the side adjacent to the given angle (assuming the data are not such that there is no triangle possible, a situation that may arise in any case, ambiguous or nonambiguous). In the figure, angle A and sides a and b are given ($a<b$); triangles AB_1C and AB_2C are both solutions. If $a=b\sin A$, the right triangle ABC is the unique solution.



For a spherical triangle, the ambiguous case is the case in which a side and the opposite angle are given (the given parts may then be either two sides and an angle opposite one side, or two angles and the side opposite one angle).

A-MER'I-CAN, *adj.* **American experience table of mortality.** See MORTALITY.

AM'I-CA-BLE, *adj.* **amicable numbers.** Two numbers, each of which is equal to the sum of all the exact divisors of the other

except the number itself. *E.g.*, 220 and 284 are amicable numbers, for 220 has the exact divisors 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110, whose sum is 284; and 284 has the exact divisors 1, 2, 4, 71, and 142, whose sum is 220.

A-MOR′TI-ZA′TION, *n.* **amortization of a debt.** The discharge of the debt, including interest, by periodic payments, usually equal, which continue until the debt is paid without any renewal of the contract. The mathematical principles are the same as those used for annuities.

amortization equation. An equation relating the amount of an obligation to be amortized, the interest rate, and the amount of the periodic payments. See AMORTIZE, and TABLE IV in the appendix.

amortization of a premium on a bond. Writing down (decreasing) the book value of the bond on each dividend date by an amount equal to the difference between the dividend and the interest on the investment (interest on the book value at the yield rate). See VALUE—book value.

amortization schedule. A table giving the annual payment, the amount applied to principal, amount applied to interest, and the balance of principal due. See AMORTIZE.

A-MOUNT′, *n.* **amount of a sum of money at a given date.** The sum of the principal and interest (simple or compound) to the date; designated as *amount at simple interest* or *amount at compound interest* (or *compound amount*), according as interest is simple or compound. In practice the word *amount* without any qualification usually refers to amount at compound interest. See TABLE III in the appendix.

amount of an annuity. See ACCUMULATED—accumulated value of an annuity at a given date.

compound amount. See COMPOUND.

AM′PERE, *n.* A unit of measure of electric current. The legal standard of current since 1950 is the **absolute ampere**, which is the current in each of two long parallel wires which carry equal currents and for which there is a force of $2 \cdot 10^{-7}$ newton per meter acting on each wire. The legal standard of current before 1950 was the **international ampere**, which is the current which when passed through a standard solution of silver nitrate deposits silver at the rate of .001118 gram per sec. One international ampere equals 0.999835 absolute ampere. See COULOMB, and OHM.

AM′PLI-TUDE, *n.* **amplitude of a complex number.** The angle that the vector representing the complex number makes with the positive horizontal axis. *E.g.*, the *amplitude* of $2+2i$ is 45°. See POLAR—polar form of a complex number.

amplitude of a curve. The greatest numerical value of the ordinates of a periodic curve. The *amplitude* of $y=\sin x$ is 1; of $y=2\sin x$ is 2.

amplitude of a point. See POLAR—polar coordinates in the plane.

amplitude of simple harmonic motion. See HARMONIC—simple harmonic motion.

AN′A-LOGUE, *adj.* **analogue computer.** See COMPUTER.

A-NAL′O-GY, *n.* A form of inference sometimes used in mathematics to set up new theorems. It is reasoned that, if two or more things agree in some respects, they will probably agree in others. Exact proofs must, of course, be made to determine the validity of any theorems set up by this method.

A-NAL′Y-SIS, *n.* [*pl.* analyses]. That part of mathematics which uses, for the most part, algebraic and calculus methods—as distinguished from such subjects as synthetic geometry, number theory, and group theory.

analysis of a problem. The exposition of the principles involved; a listing, in mathematical language, of the data given in the statement of the problem, other related data, the end sought, and the steps to be taken.

analysis of variance. See VARIANCE.

analysis situs. The field of mathematics now called *topology*.

diophantine analysis. See DIOPHANTINE.

***n*-way analysis.** (*Statistics.*) A general joint classification of values based on n joint factors gives an n-way analysis.

one-way analysis. (*Statistics.*) One-way analysis is an analysis in which factors

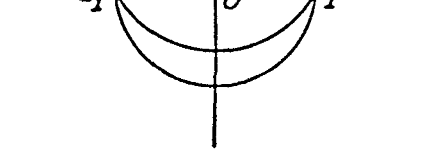
investigated as possible contributors to variances are classified under one general head, *e.g.*, male and female under sex.

proof by analysis. Proceeding from the thing to be proved to some known truth, as opposed to synthesis which proceeds from the true to that which is to be proved. The most common method of *proof by analysis* is, in fact, by *analysis* and *synthesis*, in that the steps in the analysis are required to be reversible.

two-way analysis. (*Statistics.*) *Two-way* analysis is an analysis in which two major factors jointly classify the observed values, *e.g.*, sex (male and female) and marital status (married, single, other).

unitary analysis. A system of analysis that proceeds from a given number of units to a unit, then to the required number of units. Consider the problem of finding the cost of 7 tons of hay if $2\frac{1}{2}$ tons cost \$25.00. Analysis: If $2\frac{1}{2}$ tons cost \$25.00, 1 *ton costs* \$10.00. Hence 7 tons cost \$70.00.

AN-A-LYT'IC, *adj.* **analytic continuation of an analytic function of a complex variable.** If $w=f(z)$ is given to be a single-valued analytic function in a domain D, then possibly there is a function $F(z)$ analytic in a domain of which D is a proper subdomain, and such that $F(z)=f(z)$ in D. If so, the function $F(z)$ is necessarily unique. The process of obtaining $F(z)$ from $f(z)$ is



called analytic continuation. *E.g.*, the function $f(z)$ defined by $f(z)=1+z+z^2+z^3+\cdots$ is thereby defined only for $|z|<1$, the radius of convergence of the series being 1 and the circle of convergence having center at 0. The series represents the function $1/(1-z)$, but if this function is given a new representation, say by

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(\frac{1}{2}i)}{n!}\left(z-\frac{1}{2}i\right)^n,$$

where the coefficients are determined from the original series, the new circle of convergence extends outside the old one (see the figure). The given function $f(z)$, usually given as a power series (but not necessarily so), is called a function-element of $F(z)$. The analytic continuation might well lead to a many-sheeted Riemann surface of definition of $F(z)$. See MONOGENIC—monogenic analytic function.

analytic curve. A curve in n-dimensional Euclidean space which, in the neighborhood of each of its points, admits a representation of the form $x_j=x_j(t)$, $j=1, 2, \cdots n$, where the $x_j(t)$ are real analytic functions of the real variable t. If in addition we have

$$\sum_{j=1}^{n}(x_i')^2\neq 0,$$ the curve is said to be a regular analytic curve and the parameter t is a regular parameter for the curve. For three-dimensional space, an analytic curve is a curve which has a parametric representation $x=x(t)$, $y=y(t)$, $z=z(t)$, for which each of these functions is an analytic function of the real variable t; it is a regular analytic curve if dx/dt, dy/dt and dz/dt do not vanish simultaneously. See POINT—ordinary point of a curve.

analytic function of a complex variable. A single-valued function $w=f(z)$, or a multiple-valued function considered as a single-valued function on its Riemann surface, which has a derivative at each point of its *domain* (a non-null connected open set) of definition D is said to be analytic in D. It can be shown that an analytic function $f(z)$ of a complex variable has continuous derivatives of all orders and can be expanded as a Taylor series in a neighborhood of each point z_0 of D:

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n.$$

A function is sometimes said to be analytic in D if it is continuous in D and has a derivative at all except at most a finite number of points of D. If $f(z)$ has a derivative at all points of D, it is said to be a regular function, or a regular analytic function, or a

holomorphic function, in D. See CAUCHY—Cauchy-Riemann partial differential equations, and MONOGENIC.

analytic function of a real variable. A function, $f(x)$, is analytic for $x=h$ if it can be represented by a Taylor's series in powers of $(x-h)$, which is equal to the function for any x in some neighborhood of h. The function is said to be analytic in the interval (a, b) if the above is true for every h in the interval (a, b). See TAYLOR—Taylor's series.

analytic geometry. See GEOMETRY—analytic geometry.

analytic at a point. A single-valued function $f(z)$ of the complex variable z is said to be analytic at the point z_0 if there is a neighborhood N of z_0 such that $f'(z)$ exists at every point of N. *I.e.*, $f(z)$ is analytic at z_0 if it is analytic in a neighborhood of z_0. *Syn.* Holomorphic, regular, or monogenic at a point. See above, analytic function of a complex variable.

analytic proof or solution. A proof or solution which depends upon that sort of procedure in mathematics called analysis; a proof which consists, essentially, of algebraic (rather than geometric) methods and/or of methods based on limiting processes (such as the methods of differential and integral calculus).

analytic structure for a space. A covering of a *locally Euclidean space* of dimension n by a set of open sets each of which is homeomorphic to an open set in n-dimensional Euclidean space E_n and which are such that whenever any two of these open sets overlap the coordinate transformation (in both directions) is given by analytic functions (*i.e.*, functions which can be expanded in power series in some neighborhood of any point). If neighborhoods U and V overlap and P is in their intersection, then the homeomorphisms of U and V with open sets of n-dimensional Euclidean space define coordinants $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ for P and the functions $x_i = x_i(y_1, \cdots, y_n)$ and $y_i = y_i(x_1, \cdots, x_n)$ are the functions required to be analytic. The analytic structure is **real** or **complex** according as the coordinates of points in E_n are taken as real or complex numbers. See EUCLIDEAN—locally Euclidean space, and MANIFOLD.

a-point of an analytic function. An a-point of the analytic function $f(z)$ of the complex variable z is a *zero point* of the analytic function $f(z)-a$. The order of an a-point is the order of the zero of $f(z)-a$ at the point. See ZERO—zero point of an analytic function of a complex variable.

normal family of analytic functions. See NORMAL.

quasi-analytic function. For a sequence of positive numbers $M_1, M_2, \cdots$ and a closed interval $[a, b]=I$, the *class of quasi-analytic functions* is the set of all functions which possess derivatives of all orders on I and which are such that for each function f there is a constant k such that

$$|f^{(n)}(x)| < k^n M_n \quad \text{for} \quad n \geqq 1 \quad \text{and} \quad x \in I,$$

provided this set of functions has the property that if f is a member of the set and $f^{(n)}(x_0)=0$ for $n \geqq 0$ and $x_0 \in I$, then $f(x) \equiv 0$ on I. If $M_n = n!$, or $M_n = n^n$, then the corresponding class of functions is precisely the class of all analytic functions on I. Every function which possesses derivatives of all orders on I (*e.g.*, e^{-1/x^2} on $[0, 1]$) is the sum of two functions each of which belongs to a quasi-analytic class. Even if the class defined by $M_1, \cdots$ and I is not quasi-analytic, certain subclasses are sometimes said to be quasi-analytic if they do not contain a nonzero function f for which $f^{(n)}(x_0)=0$ for $n \geqq 0$ and $x_0 \in I$. Quasi-analyticity is one of the most important properties of analytic functions, but there exist classes of quasi-analytic functions which contain nonanalytic functions.

singular point of an analytic function. See SINGULAR.

AN'A-LYT'I-CAL-LY, *adj.* Performed by analysis, by analytic methods, as opposed to synthetic methods.

AN'A-LY-TIC'I-TY, *n.* **point of analyticity.** A point at which a function $f(z)$ of the complex variable z is analytic.

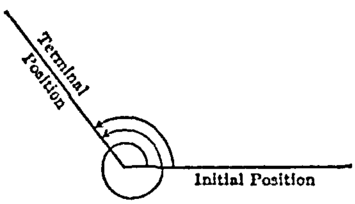
ANCHOR RING or TORUS. A surface the shape of a doughnut with a hole in it; a surface generated by the rotation, in space, of a circle about an axis in its plane but not cutting the circle. If r is the radius of the circle, k the distance from the center to the axis of revolution, in this case the z-axis, and the equation of the generating

circle is $(y-k)^2+z^2=r^2$, then the equation of the anchor ring is

$$(\sqrt{x^2+y^2}-k)^2+z^2=r^2.$$

Its volume is $2\pi^2kr^2$ and the area of its surface is $4\pi^2kr$.

AN'GLE, *n.* In *geometry*, the inclination to each other (the divergence) of two straight lines; the figure formed by two straight lines drawn from the same point (the vertex of the angle). In *trigonometry*, a figure which has been formed by one straight line (called the **terminal line**, or side) having been revolved about a fixed point on a stationary straight line (called the **initial line**, or side). If the motion is counterclockwise, the angle is said to be **positive**; if clockwise, it is said to be **negative**. "Angle" is used for "plane angle."

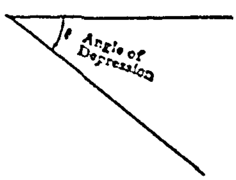


acute angle. See ACUTE.

addition of angles. See SUM—sum of angles.

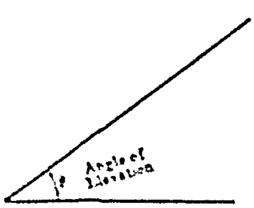
adjacent angles. See ADJACENT—adjacent angles.

angle of depression. The angle between the horizontal plane and the oblique line



joining the observer's eye to some object lower than (beneath) the line of his eye.

angle of elevation. The angle between the horizontal plane and the oblique line

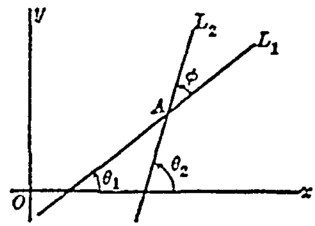


from the observer's eye to a given point above the line of his eye.

angle of friction. See FRICTION.

angle of inclination of a line. The positive angle, less than 180°, measured *from the positive x-axis to the given line.*

angle of intersection. The angle of intersection of two lines in a plane is defined thus: The angle from line L_1, say, to line L_2 is the smallest positive angle through which L_1 can be revolved counterclockwise about the point of intersection of the lines to



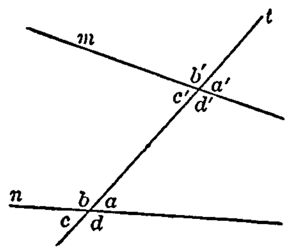
coincide with the line L_2, angle ϕ in the cut. The tangent of the angle from L_1 to L_2 is given by

$$\tan\phi=\tan(\theta_2-\theta_1)=\frac{m_2-m_1}{1+m_1m_2},$$

where $m_1=\tan\theta_1$ and $m_2=\tan\theta_2$. The angle between lines L_1 and L_2 is the least positive angle between the two lines (the angle between two parallel lines is defined to be of measure O). The angle between two lines in space (whether or not they intersect) is the angle between two intersecting lines which are parallel respectively to the two given lines. The cosine of this angle is equal to the sum of the products in pairs of the corresponding direction cosines of the lines (see DIRECTION—direction cosines). The angle between two intersecting curves is the angle between the tangents to the curves at their *point of intersection*. The angle between a line and a plane is the smaller (acute) angle which the line makes with its projection in the plane. The angle between two planes is the dihedral angle which they form (see DIHEDRAL): this is equal to the angle between the normals to the planes (see above, angle between two lines in space). When the equations of the planes are in normal form, the cosine of the angle between the planes is equal to the sum of the products of the corresponding coefficients (coefficients of the same variables) in their equations.

angle of a lune. See LUNE.

angles made by a transversal. The angles made by a line (the transversal) which cuts two or more other lines. In the figure, the transversal t cuts the lines m and n. The angles a, b, c', d' are **interior angles**; a', b',



c, d are **exterior angles**; a and c', and b and d' are the pairs of **alternate-interior angles**; b' and d, a' and c are the pairs of **alternate exterior angles**; a' and a, b' and b, c' and c, d' and d are the **exterior-interior** or **corresponding angles.**

angles of a polygon. The angles made by adjacent sides of the polygon and lying on the interior of the polygon. This definition suffices for any polygon, even if concave, provided no side (not extended) cuts more than two sides. If this condition does not hold, the sides must be directed in some order when defining the polygon in order to uniquely define the angles between them. See DIRECTED—directed line.

angle of reflection. See REFLECTION.

angle of refraction. See REFRACTION.

base angles of a triangle. The angles in the triangle having the base of the triangle for their common side.

central angle. See CENTRAL.

complementary angles. See COMPLEMENTARY.

conjugate angles. Two angles whose sum is a perigon; two angles whose sum is 360°. Such angles are also said to be *explements* of each other.

coterminal angles. See COTERMINAL.

dihedral angle. See DIHEDRAL.

direction angles. See DIRECTION—direction angles.

eccentric angle. See ELLIPSE—parametric equations of an ellipse.

Euler's angles. See EULER—Euler's angles.

explementary angles. See above, conjugate angles.

exterior angles. See EXTERIOR.

face angles. See below, polyhedral angle.

flat angle. Same as STRAIGHT ANGLE.

hour angle of a celestial point. The angle between the plane of the meridian of the observer and the plane of the hour circle of the star—measured westward from the plane of the meridian. See HOUR—hour angle and hour circle.

interior angle. See INTERIOR.

measure of an angle. See DEGREE, MIL, and RADIAN.

negative angle. An angle generated by revolving a line in the clockwise direction from the initial line. See ANGLE.

negatively oriented angle. An oriented angle for which the rotation is clockwise. Same as NEGATIVE ANGLE.

obtuse angle. An angle numerically greater than a right angle and less than a straight angle; sometimes used for all angles numerically greater than a right angle.

opposite angle. See OPPOSITE—opposite vertices (angles) of a polygon.

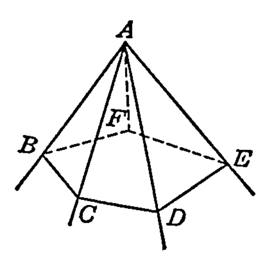
oriented angle. An angle with which the idea of directed rotation is associated.

perigon (angle). An angle containing 360°.

plane angle. See above, ANGLE.

polar angle. See POLAR—polar coordinates in the plane.

polyhedral angle. The configuration formed by the lateral faces of a polyhedron which have a common vertex (A-$BCDEF$ in the figure); the positional relation of a set of planes determined by a point and the



sides of some polygon whose plane does not contain the point. The planes (ABC, etc.) are called **faces** of the angle; the lines of intersection of the planes are called **edges** of the *polyhedral angle*. Their point of intersection (A) is called the **vertex.** The angles (BAC, CAD, etc.) between two successive edges are called **face angles.** A sec-

tion of a polyhedral angle is the polygon formed by cutting all the edges of the angle by a plane not passing through the vertex.

positive angle. An angle generated by revolving a line in the counterclockwise direction from the initial line. Also called a positively oriented angle. See ANGLE.

quadrant angles. See QUADRANT.

quadrantal angles. See QUADRANTAL.

reflex angle. An angle greater than a straight angle and less than two straight angles; an angle between 180° and 360°.

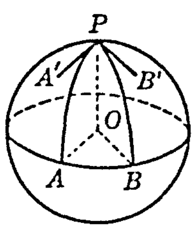
related angle. See RELATED.

right angle. Half of a straight angle; an angle containing 90° or $\frac{1}{2}\pi$ radians.

sides of an angle. The straight lines forming the angle.

solid angle. See SOLID.

spherical angle. The figure formed at the intersection of two great circles on a sphere; the difference in direction of the arcs of two great circles at their point of intersection. In the figure the spherical



angle is APB. It is equal to the plane angles $A'PB'$ and AOB. See DIRECTION—direction of a curve.

straight angle. An angle whose sides lie on the same straight line, but extend in opposite directions from the vertex; an angle of 180° or π radians. *Syn.* Flat angle.

supplementary angles. See SUPPLEMENTARY.

tetrahedral angle. A polyhedral angle having four faces.

trihedral angle. A polyhedral angle having three faces.

trisection of an angle. See TRISECTION.

vertex angle. The angle opposite the base of a triangle.

vertex of an angle. The point of intersection of the sides.

vertical angle. The angle at the vertex of a triangle. Usually called *vertex angle.*

vertical angles. Two angles such that each side of one is a prolongation, through the vertex, of a side of the other.

zero angle. The figure formed by two straight lines drawn from the same point in the same direction (so as to coincide); an angle whose measure in degrees is 0.

AN'GU-LAR, *adj.* Pertaining to an angle; circular; around a circle.

angular acceleration. See ACCELERATION—angular acceleration.

angular distance. See DISTANCE—angular distance between two points.

angular momentum. See MOMENTUM—moment of momentum.

angular speed. See SPEED.

angular velocity. See VELOCITY.

AN'HAR-MON'IC RATIO. See RATIO—anharmonic ratio.

AN-NI'HI-LA'TOR, *n.* The annihilator of a set S is the class of all functions of a certain type which *annihilate* S in the sense of being zero at each point of S. *E.g.*, if the functions are continuous linear functionals and S is a subset of a normed linear space N, then the annihilator of S is the linear subset S' of the first conjugate space N^* which consists of all continuous linear functionals which are zero at each point of S. Analogously the annihilator of a linear subset S of Hilbert space is the orthogonal complement of S.

AN'NU-AL, *adj.* Yearly.

annual premium (net annual premium). See PREMIUM.

annual rent. Rent, when the payment period is a year. See RENT.

AN-NU'I-TANT, *n.* The life (person) upon whose existence each payment of a life annuity is contingent, *i.e.*, the beneficiary of an annuity.

AN-NU'I-TY, *n.* A series of payments at regular intervals. An annuity contract is a written agreement setting forth the amount of the annuity, its cost, and the conditions under which it is to be paid (sometimes called an annuity policy, when the annuity is a temporary annuity). The payment interval of an annuity is the time between

successive payment dates; the **term** is the time from the beginning of the first payment interval to the end of the last one. An annuity is a **simple annuity,** or a **general annuity,** according as the payment intervals do, or do not, coincide with the interest conversion periods. A **deferred annuity** (or **intercepted annuity**) is an annuity in which the first payment period begins after a certain length of time has lapsed; it is an **immediate annuity** if the term begins immediately. An **annuity due** is an annuity in which the payments are made at the beginning of each period. If the payments are made at the end of the periods, the annuity is called an **ordinary annuity.** An **annuity certain** is an annuity that provides for a definite number of payments, as contrasted to a **life annuity,** which is a series of payments at regular intervals during the life of an individual (a **single life annuity**) or of a group of individuals (a **joint life annuity**). A **last survivor annuity** is an annuity payable until the last of two (or more) lives end. An annuity whose payments continue forever is called a **perpetuity.** A **temporary annuity** is an annuity extending over a given period of years, provided the recipient continues to live throughout that period, otherwise terminating at his death. A **reversionary annuity** is an annuity to be paid during the life of one person, beginning with the death of another. An annuity whose payments depend upon certain conditions, such as some person (not necessarily the beneficiary) being alive, is called a **contingent annuity.** A **forborne annuity** is a life annuity whose term began sometime in the past; *i.e.*, the payments have been allowed to accumulate with the insurance company for a stated period. In case a group contributes to a fund over a stated period and at the end of the period the accumulated fund is converted into annuities for each of the survivors, the annuity is also called a *forborne annuity.* A life annuity is **curtate,** or **complete,** according as a proportionate amount of a payment is not made, or is made, for the partial period from the last payment before death of the beneficiary to the time of death. A complete annuity is also called an **apportionate annuity** and a **whole-life annuity.** An annuity is **increasing** if each payment after the first is larger than the preceding payment; it is **decreasing** if each payment except the last is larger than the next payment. Also see TONTINE—tontine annuity.

accumulated value of an annuity. The accumulated value (or **amount**) of an annuity at a given date is the sum of the compound amounts of the annuity payments to that date. The **amount** of an annuity is the accumulated value at the end of the term of the annuity.

annuity bond. See BOND.

cash equivalent (or **present value**) **of an annuity.** See VALUE—present value.

consolidated annuities (consols). See CONSOLIDATED.

ANN'U-LUS, *n.* [*pl.* **annuli**]. The portion of a plane bounded by two concentric circles in the plane. The area of an annulus is the difference between the areas of the two circles, namely $\pi(R^2-r^2)$, where R is the radius of the larger circle and r is the radius of the smaller.

A-NOM'A-LY, *n.* **anomaly of a point.** See POLAR—polar coordinates in the plane.

AN'TE-CED'ENT, *n.* (1) The first term (or numerator) of a ratio; that term of a ratio which is compared with the other term. In the ratio 2/3, 2 is the *antecedent* and 3 is the *consequent.* (2) See IMPLICATION.

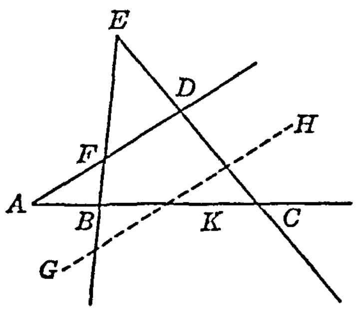
AN'TI-DE-RIV'A-TIVE, *n.* **antiderivative of a function.** Same as the PRIMITIVE or INDEFINITE INTEGRAL of the function. See INTEGRAL—indefinite integral.

AN'TI-HY-PER-BOL'IC **functions.** Same as INVERSE HYPERBOLIC functions. See HYPERBOLIC—inverse hyperbolic functions.

AN'TI-LOG'A-RITHM, *n.* **antilogarithm of a given number.** The number whose logarithm is the given number; *e.g.*, anti-$\log_{10} 2 = 100$. *Syn.* Inverse logarithm. To find an antilogarithm corresponding to a given logarithm that is not in the tables, subtract the next smaller mantissa from the given one and from the next larger one, divide the former difference by the latter and annex the quotient to the number

corresponding to the smaller mantissa. See CHARACTERISTIC.

AN'TI-PAR'AL-LEL, *adj.* **antiparallel lines.** Two lines which make, with two given lines, angles that are equal in opposite order. In the figure the lines AC and



AD are antiparallel with respect to the lines EB and EC, since $\angle EFD = \angle BCD$ and $\angle ADE = \angle EBC$. Two parallel lines have a similar property. The parallel lines AD and GH also make equal angles with the lines EB and EC, but in the same order; *i.e.*, $\angle EFD = \angle BGH$ and $\angle ADE = \angle GHD$.

AN'TI-SYM-MET'RIC, *adj.* **antisymmetric dyadic.** See DYAD.

AN'TI-TRIG'O-NO-MET'RIC, *adj.* **antitrigonometric functions.** Same as INVERSE TRIGONOMETRIC FUNCTIONS. See INVERSE.

A'PEX, *n.* [*pl.* apexes or apices]. A highest point relative to some line or plane. The apex of a triangle is the vertex opposite the side which is considered as the base; the apex of a cone is its vertex.

AP'-OL-LO'NI-US. problem of Apollonius. To construct a circle tangent to three given circles.

A POS-TE'RI-O'RI, *adj.* **a posteriori knowledge.** Knowledge from experience. *Syn.* Empirical knowledge.

a posteriori probability. See PROBABILITY—empirical or a posteriori probability.

A-POTH'E-CAR-Y, *n.* **apothecaries' weight.** The system of weights used by druggists. The pound and the ounce are the same as in troy weight, but the subdivisions are different. See DENOMINATE NUMBERS in the appendix.

AP'O-THEM, *n.* **apothem of a regular polygon.** The perpendicular distance from the center to a side. *Syn.* Short radius.

AP-PAR'ENT, *adj.* **apparent distance.** See DISTANCE—angular distance between two points.

apparent time. Same as APPARENT SOLAR TIME. See TIME.

APPLIED MATHEMATICS. See MATHEMATICS.

AP-POR'TION-A-BLE, *adj.* **apportionable annuity.** See ANNUITY.

AP-PROACH', *v.* **approach a limit.** See LIMIT—limit of a variable.

AP-PROX'I-MATE, *adj.*, *v.* To calculate nearer and nearer to a correct value; used mostly for numerical calculations. *E.g.*, one *approximates* the square root of 2 when he finds, in succession, the numbers 1.4, 1.41, 1.414, whose successive squares are nearer and nearer to 2. Finding any one of these decimals is also called approximating the root; that is, to *approximate* may mean either to secure one result near a desired result, or to secure a succession of results approaching a desired result.

approximate result, value, answer, root, etc. One that is nearly but not exactly correct. Sometimes used of results either nearly or exactly correct. See ROOT—root of an equation.

AP-PROX'I-MA'TION, *n.* (1) A result that is not exact, but is accurate enough for some specific purpose. (2) The process of obtaining such a result.

approximation by differentials. See DIFFERENTIAL. *E.g.*, to find the approximate change in the area of a circle of radius 2 feet when the radius increases .01 foot, we have $A = \pi r^2$, from which $dA = 2\pi r\,dr = 2\pi \times 2 \times \frac{1}{100} = \frac{1}{25}\pi$ square feet, which is the approximate increase in area.

successive approximations. The successive steps taken in working toward a desired result or calculation. See APPROXIMATE.

A PRI-O'RI, *adj.* **a priori fact.** Used in about the same sense as axiomatic or self-evident fact.

a priori knowledge. Knowledge obtained from pure reasoning from cause to effect, as contrasted to empirical knowledge (knowledge obtained from experience); knowledge which has its origin in the mind and is (supposed to be) quite independent of experience.

a priori reasoning. Reasoning which arrives at conclusions from definitions and assumed axioms or principles; deductive reasoning.

AR'A-BIC, *adj.* **Arabic numerals:** 1, 2, 3, 4, 5, 6, 7, 8, 9, 0; introduced into Europe from Arabia, probably originating in India, although the source is not definitely known.

AR'BI-TRAR'Y, *adj.* **arbitrary assumption.** An assumption constructed at the pleasure of the writer without regard to its being consistent either with the laws of nature or (sometimes) with accepted mathematical principles.

arbitrary constant. See CONSTANT.

arbitrary ϵ. A statement is true for *arbitrary* ϵ if it is true for any numerical value (usually restricted to be positive) which may be assigned to ϵ. This idiom usually occurs in situations where small values of ϵ are of the most interest.

arbitrary function in the solution of partial differential equations. A function which may take many forms and still satisfy the differential equation under consideration. *E.g.*, $z/x=f(y)$ or $F[(z/x), y]=0$ (where the last equation can be solved for z/x) are solutions of $x(\partial z/\partial x)-z=0$, in which f and F are arbitrary functions.

arbitrary parameter. Same as *parameter* in its most commonly used sense. The addition of the attribute *arbitrary* places emphasis upon the fact that this particular parameter is entirely subject to the values directly assigned by the thinker, rather than by the conditions of the discussion or problem at hand.

ARC, *n.* A segment, or piece, of a curve.

arc length. The length in linear units of an arc of a curve. See LENGTH—length of a curve.

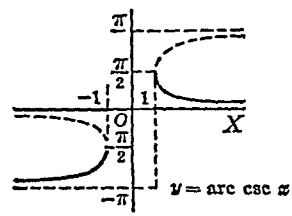
differential (or element) of arc. See ELEMENT—element of integration.

limit of the ratio of an arc to its chord. See LIMIT—limit of the ratio of an arc to its chord.

minor arc of a circle. See SECTOR—sector of a circle. *Syn.* Short arc.

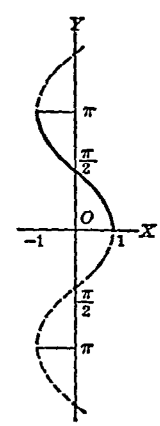
short arc. See SHORT.

ARC CO-SE'CANT, *n.* **arc cosecant of a number** x. Written $\csc^{-1} x$ or arc csc x; an angle whose cosecant is x. *E.g.*, $\csc^{-1} 2$



is equal to 30°, 150°, or in general, $n180°+(-1)^n 30°$. See VALUE—principal value of an inverse trigonometric function. *Syn.* Inverse cosecant, anticosecant. (In the figure, y is in radians.)

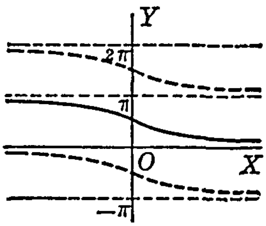
ARC CO'SINE, *n.* **arc cosine of a number** x. Written $\cos^{-1} x$ or arc cos x; an angle whose cosine is x. *E.g.*, arc cos $\frac{1}{2}$ is equal to 60°, 300°, or in general $n360° \pm 60°$. See VALUE—principal value of an inverse



trigonometric function. *Syn.* Inverse cosine, anticosine. The graph shows $y=\cos^{-1} x$ (y in radians).

ARC CO-TANGENT, *n.* **arc cotangent of a number** x. Written $\cot^{-1} x$, $\text{ctn}^{-1} x$ or

arc cot x; an angle whose cotangent is x. *E.g.*, arc cot 1 is equal to 45°, 225°, or in general $n180° + 45°$. See VALUE—principal value of an inverse trigonometric function. *Syn.* Inverse cotangent, anticotangent.



$y = \text{arc cot } x$

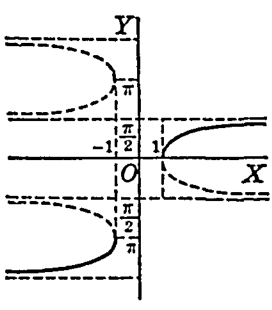
(In the figure, y is in radians.)

ARCHIMEDES. **spiral of Archimedes.** See SPIRAL.

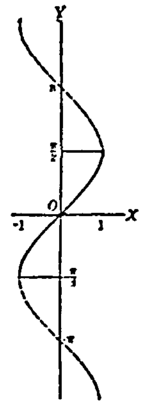
AR′CHI-ME′DE-AN. **Archimedean property.** The property of real numbers that for any positive numbers a and b there is a positive integer n such that $a < nb$.

ARC=HY-PER-BOL′IC, *adj.* **arc-hyperbolic sine, cosine, etc.** See HYPERBOLIC—inverse hyperbolic functions.

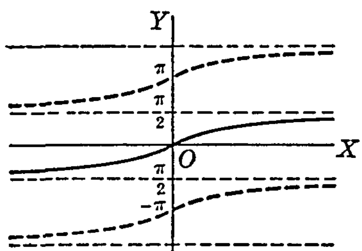
ARC SE-CANT′, *n.* **arc secant of a number** x. Written $\sec^{-1} x$ or arc sec x. An angle whose secant is x. *E.g.*, arc sec 2 is equal to 60°, 300°, or in general, $n360° \pm 60°$. See VALUE—principal value of an inverse trigonometric function. *Syn.* Inverse secant, antisecant. (In the figure, y is in radians.)



ARC SINE, *n.* **arc sine of a number** x. Written $\sin^{-1} x$ or arc sin x. An angle whose sine is x. *E.g.*, arc sin $\frac{1}{2}$ is equal to 30°, 150°, or in general $n180° + (-1)^n 30°$. See VALUE—principal value of an inverse trigonometric function. *Syn.* Inverse sine, antisine. The figure is the graph of $y = \sin^{-1} x$ (y in radians).



ARC TAN-GENT′, *n.* **arc tangent of a number** x. Written $\tan^{-1} x$ or arc tan x.



y arc tan x

An angle whose tangent is x; *e.g.*, arc tan 1 is equal to 45°, 225°, or in general, $n180° + 45°$. See VALUE—principal value of an inverse trigonometric function. *Syn.* Inverse tangents, antitangent. (In the figure, y is in radians.)

A′RE-A, *n.* **area of a curved surface** (a sphere, ellipsoid, etc.). The limit approached by the sum of the areas of the faces of a polyhedron whose faces are tangent to the surface as the lengths of the edges of the polyhedron approach zero. In case the surface is not of such a nature as to permit such a polyhedron (*e.g.*, if it is not closed) it can, in general, still be covered by a surface consisting of a set of polygons, each tangent to the surface, in such a way that each edge can be made to approach zero. The resulting limit of the sum of the areas of the polygons is the area of the surface (if the limit exists). See SURFACE—surface area.

area of a lune. See LUNE.

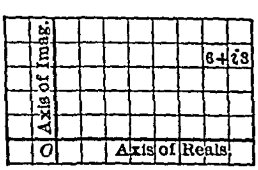
area of a plane region (a triangle, square, circle, etc.). The greatest lower bound of the sum of the areas of nonoverlapping squares which together completely cover the region, the area of a square being defined as the square of the length of its side. See MEASURABLE—measurable set, ELEMENT—element of integration, GEOMETRIC—method of geometric exhaustion, and the special configuration.

differential (or element) of area. See ELEMENT—element of integration, and SURFACE—surface area, surface of revolution.

lateral area of a cone, cylinder, parallelepiped, etc. See the specific configuration.

relations between areas of similar surfaces. Areas of similar surfaces have the same ratio (vary as) the squares of corresponding lines. *E.g.*, (1) the areas of two circles are in the same ratio as the squares of their radii, (2) the areas of two similar triangles are in the same ratio as the squares of corresponding sides or altitudes.

ARGAND DIAGRAM. Two perpendicular axes on one of which real numbers are represented and on the other pure imaginaries, thus providing a frame of reference for graphing complex numbers. These axes are called the **real axis** and the **imaginary axis** or the **axis of reals** and the **axis of imaginaries.**



AR'GU-MENT, *n.* **argument of a complex number.** Same as AMPLITUDE. See AMPLITUDE—amplitude of a complex number.

argument of a function. Same as the INDEPENDENT VARIABLE. See FUNCTION.

arguments in a table of values of a function. The values of the independent variable for which the values of the function are tabulated. The arguments in a trigonometric table are the angles for which the functions are tabulated; in a log-table, the numbers for which the logarithms are tabulated.

A-RITH'ME-TIC, *n.* The study of the integers 1, 2, 3, 4, 5, $\cdots$ under the operations of addition, subtraction, multiplication, division, raising to powers, and extracting roots, and the use of the results of these studies in everyday life.

four fundamental operations of arithmetic. Addition, subtraction, multiplication, and division.

AR-ITH-MET'IC or AR-ITH-MET'I-CAL, *adj.* Employing the principles and symbols of arithmetic.

arithmetic component. In a computing machine, any component that is used in performing arithmetic, logical, or other similar operations.

arithmetic mean (or average). See AVERAGE.

arithmetic mean, or means, between two given numbers. The other terms of an arithmetical progression of which the given numbers are the first and last terms; a single *mean* between two numbers is their average, that is, one half of their sum. When the number of terms, n, and the first and last are given, the other terms can be written out after finding the difference, d, from the formula, $l=a+(n-1)d$, where a and l are the first and last terms, respectively. See below, arithmetic progression.

arithmetic number. See NUMBER—arithmetic numbers.

arithmetic progression. Denoted by **A.P.** A sequence, each term of which is equal to the sum of the preceding term and a constant; written: $a, a+d, a+2d, \cdots, a+(n-1)d$, where a is called the **first term,** d the **common difference** or simply the **difference,** and $a+(n-1)d$ the **last** or **nth term.** The positive integers, 1, 2, 3, $\cdots$ form an arithmetic progression. See below, arithmetic series.

arithmetic series. The indicated sum of the terms of an *arithmetic progression.* This sum, to n terms, is denoted by S_n and is equal to

$$\tfrac{1}{2}n(a+l) \quad \text{or} \quad \tfrac{1}{2}n[2a+(n-1)d].$$

AR-ITH-MOM'E-TER, *n.* A computing machine.

ARM, *n.* **arm of an angle.** A side of the angle.

AR-RANGE'MENT, *n.* Same as PERMUTATION (1).

AR-RAY', *n.* (*Statistics.*) Arrangement of a series of items according to values of the items. Usually from largest to smallest, or the reverse.

AS-CEND'ING, *adj.* **ascending power series.** Same as POWER SERIES. See SERIES.

ascending powers of a variable in a polynomial. Powers of the variable that increase as the terms are counted from left to right, as in the polynomial

$$a+bx+cx^2+dx^3+\cdots.$$

ASCOLI'S THEOREM. From any set of uniformly bounded functions *equicontinuous* on a bounded closed (compact) set (such as a closed interval) it is possible to select an infinite sequence which converges uniformly to a limit function which is also continuous.

AS-SESSED', *adj.* **assessed value.** A value set upon property for the purpose of taxation.

AS-SES'SOR, *n.* One who estimates the value of (evaluates) property as a basis for taxation.

AS'SETS, *n.* **assets of an individual or firm.** All of his (or its) goods, money, collectable accounts, etc., which have value; the opposite of liabilities.

fixed assets. Assets represented by equipment for use but not for sale—such as factories, buildings, machinery, and tools.

wasting assets. See DEPRECIATION.

AS-SO'CIATE, *adj.* **associate matrix.** See HERMITIAN—Hermitian conjugate of a matrix.

AS-SO'CI-AT'ED, *adj.* **associated radius of convergence.** If the power series

$$\sum a_{i_1, i_2, \cdots, i_n} z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}$$

converges for $|z_j| < r_j$, $j=1, 2, \cdots n$, and diverges for $|z_j| > r_j$, $j=1, 2, \cdots n$, where r_j is positive, then the set $r_1, r_2, \cdots r_n$ is called a set of associated radii of convergence for the series. *E.g.*, for the series

$$1+z_1z_2+z_1^2z_2^2+\cdots \equiv \frac{1}{1-z_1z_2}$$

associated radii clearly are any positive numbers r_1, r_2 with $r_1r_2=1$.

AS-SO'CI-A'TIVE, *adj.* A method of combining objects two at a time is *associative* if the result of the combination of three objects (order being preserved) does not depend on the way in which the objects are grouped. If the operation is denoted by $\circ$ and the result of combining x and y by $x \circ y$, then

$$(x \circ y) \circ z = x \circ (y \circ z)$$

for any x, y and z for which the "products" are defined. For ordinary addition of numbers, the associative law states that $a+(b+c)=(a+b)+c$ for any numbers a, b, c. This law can be extended to state that in any sum of several terms any method of grouping may be used (*i.e.*, at any stage of the addition one may add any two adjacent terms). The associative law for multiplication states that

$$(ab)c=a(bc)$$

for any numbers a, b, c. This law can be extended to state that in any product of several factors any method of grouping may be used (*i.e.*, at any stage of the multiplication one may multiply any two adjacent factors). See GROUP.

AS-SUMP'TION, *n.* See AXIOM, and below, fundamental assumptions of a subject.

empirical assumption. See EMPIRICAL—empirical formula, assumption, or rule.

fundamental assumptions of a subject. A set of assumptions upon which the subject is built. For instance, in algebra the commutative and associative laws are *fundamental assumptions*. Sets of fundamental assumptions for the same subject vary more or less with different writers.

AS-SUR'ANCE, *n.* Same as INSURANCE.

AS'TROID, *n.* The hypocycloid of four cusps.

AS-TRO-NOM'I-CAL, *adj.* **astronomical frame of reference.** See FRAME

A'SYM-MET'RIC, *adj.* **asymmetric relation.** See SYMMETRIC—symmetric relation.

AS'YMP'TOTE, *n.* A line such that a point, tracing a given curve and simultaneously receding to an infinite distance from the origin, approaches indefinitely near to the line; a line such that the perpendicular distance from a moving point on a curve to the line approaches zero as the point moves off an infinite distance from the origin. *Tech.* An asymptote is a tangent at infinity, *i.e.*, a line tangent to (touching) the curve at an ideal point. See figure under HYPERBOLA.

asymptote to the hyperbola. When the equation of the hyperbola is in the standard form $x^2/a^2-y^2/b^2=1$, the lines $y=bx/a$ and $y=-bx/a$ are its asymptotes. This can be sensed by writing the above equation in the form $y=\pm(bx/a)\sqrt{1-a^2/x^2}$ and noting that a^2/x^2 approaches zero as x increases without limit. *Tech.* The numerical difference between the corresponding ordinates of the lines and the hyperbola is

$$|bx/a|1-\sqrt{1-a^2/x^2}) = |bx/a|/(1+\sqrt{1-a^2/x^2}),$$

which approaches zero as x increases, and the distances from the hyperbola to the lines are the product of this infinitesimal by the cosines of the angles the lines make with the x-axis; hence the distances between the lines and the hyperbola each approach zero as x increases. See above, ASYMPTOTE.

AS'YMP-TOT'IC, *adj.* **asymptotic directions on a surface at a point.** Directions at a point P on a surface S for which $D\,du^2+2D'\,du\,dv+D''\,dv^2=0$. See FUNDAMENTAL—fundamental coefficients of a surface. Asymptotic directions at P on S are the directions at P on S in which the tangent plane at P has contact of at least the third order. See DISTANCE—distance from a surface to a tangent plane. Asymptotic directions are also the directions in which the *normal curvature* vanishes. At a planar point all directions are asymptotic directions; otherwise there are exactly two asymptotic directions, which are real and distinct, real and coincident, or conjugate imaginary, according as the point on the real surface S is hyperbolic, parabolic, or elliptic.

asymptotic cone of an hyperboloid. If either of the hyperboloids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

and

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

is cut by the plane $y=mx$, hyperbolas are formed whose asymptotes pass through the origin. The cone described by these lines as m varies is called the *asymptotic cone of the hyperboloid* under consideration.

asymptotic distribution. If a distribution $F(x)$ of a random variable x is a function of a parameter n (*e.g.*, n may be the size of a sample and x the mean), the limit of $F(x)$ as $n\to\infty$ is the asymptotic distribution function of x. In particular, if two quantities u and σ can be obtained so that the distribution function of $\left(\frac{x-u}{\sigma}\right)=y_n$ will be, in the limit as $n\to\infty$, equal to

$$\lim_{n\to\infty} p(y_n<t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-x^2/2}\,dx,$$

then $F(x)$ is *asymptotically* normally distributed. This means that x is asymptotically normally distributed in the sense that the limit, as $n\to\infty$, of the probability of $\left(\frac{x-u}{\sigma}=y_n\right)<t$, is given by the normal distribution regardless of whether or not x has a mean and variance of u and σ. Whatever the distribution of x the probability of the variable y is given in the limit by the normal distribution, if x can be so transformed as to be asymptotically normal.

asymptotic expansion. A divergent series of the form $a_0+(a_1/z)+(a_2/z^2)+(a_3/z^3)$ $\cdots+(a_n/z^n)+\cdots$, where the numbers a_i are constants, is an *asymptotic expansion* of a function $f(z)$ if $\lim_{z\to\infty} z^n[f(z)-S_n(z)]=0$ for any fixed value of n, where $S_n(z)$ is the sum of the first n terms of the series. *E.g.*,

$$\int_x^\infty t^{-1}e^{x-t}\,dt=(1/x)-(1/x^2)+(2!/x^3) -\cdots+[(-1)^{n-1}(n-1)!]/[x^n+(-1)^n n!] \int_t^\infty \frac{e^{x-t}}{t^{n+1}}\,dt.$$

For n fixed and x sufficiently large, the product of x^n and the last term (or remainder) can be made less than any preassigned

tive quantity. Hence the series with general (nth) term $(-1)^{n-1}(n-1)!/x^n$ is the asymptotic expansion of the function of x given by the integral. This fact is written:

$$\int_x^{\infty} t^{-1}e^{x-t}\,dt \sim (1/x)-(1/x^2) + (2!/x^3)-(3!/x^4)+\cdots.$$

asymptotic line on a surface. A curve C on a surface S such that the direction of C at each of its points is an asymptotic direction on S; curves defined by the differential equation $D\,du^2+2D'\,du\,dv+D''\,dv^2=0$. In general there are two such curves through each point of S. See above, asymptotic directions on a surface at a point.

AT-TEN'U-A'TION, *n.* In correlation, the reduction of correlation between two variables because of independent errors of measurement in at least one of the variables is attenuation of correlation.

AT-TRAC'TION, *n.* **center of attraction.** See CENTER—center of mass.

gravitational attraction. See GRAVITATION.

AT'TRI-BUTE, *n.* A qualitative characteristic, the presence or absence of which may assign a quantitative value to the variable. Thus, a defective item in a production process may be valued at 0 and a nondefective at 1. Qualitative characteristics may be fundamentally quantitative; thus, if the quantitative value exceeds a critical value, the item possesses the *attribute*, *e.g.*, blemishes on a fruit.

AUG-MEN'TED, *adj.* **augmented matrix.** See MATRIX.

AU-TO-MOR'PHIC, *adj.* **automorphic function.** A single-valued function $f(z)$, analytic except for poles in a domain D of the complex plane, is said to be automorphic with respect to a group of linear transformations provided for each transformation T of the group it is true that, if z lies in D, then also $T(z)$ lies in D, and $f[T(z)]\equiv f(z)$.

AU-TO-MOR'PHISM, *n.* See ISOMORPHISM.

AU'TO-RE-GRES'SIVE, *n.* **autoregressive series.** If the variable $y=f(t)$, written y_t, is of the form

$$y_t=ay_{t-1}+by_{t-2}+\cdots+my_{t-m}+k,$$

the variable y forms an *autoregressive series.* Specifically, a difference equation in the variable y forms an *autoregressive series.*

AUX-IL'IA-RY, *adj.* **auxiliary circle of the ellipse.** The larger of the two eccentric circles of the ellipse. See ECCENTRIC—eccentric circles of the ellipse.

auxiliary circle of the hyperbola. See HYPERBOLA—auxiliary circle of the hyperbola.

auxiliary equation. See DIFFERENTIAL—linear differential equations.

AV'ER-AGE, *n.* (1) A single number typifying or representing a set of numbers of which it is a function. (2) A single number computed such that it is not less than the smallest nor larger than the largest of a set of numbers. A generalized formulation is

$$A=\left(\frac{\sum_{i=1}^{n} q_i x_i^{y}}{\sum_{i=1}^{n} q_i}\right)^{1/y},$$

where q_i are weights, y is arbitrary, and x_i are the n numbers, not necessarily different, to be averaged. The **weighted arithmetic average** or **weighted mean** is obtained by setting $y=1$; thus

$$M=\frac{\sum_{i=1}^{n} q_i x_i}{\sum_{i=1}^{n} q_i}.$$

The numbers q_i are called the weights; if they are all equal, the weighted mean reduces to the arithmetic mean or arithmetic average (usually called simply the mean or average). *E.g.*, the average of the numbers 60, 70, 80, 90, is their sum divided by 4, or 75. If one desired to give more preference to the grades a student makes as the semester advances, he could do so by using a weighted average. If the grades were 60, 70, 80, 90, the *average* would be 75, but if 1, 2, 3, 4, were used as

weights, the *weighted average* would be $(60+140+240+360)/10$, or 80. The **harmonic average** is the reciprocal of the arithmetic average of reciprocals of the numbers:

$$H_m = \frac{\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n} q_i\left(\frac{1}{x_i}\right)}.$$

In the generalized formula, the harmonic average is obtained by setting $y=-1$. The nth root of the product of a set of n positive numbers is the **geometric average,** or **geometric mean,**

$$G_m = \sqrt[n]{x_1 x_2 \cdots x_n}$$

The antilogarithm of the arithmetic average of the logarithms of a set of values is the geometric mean. In the generalized formula, the geometric mean is equal to the limit of A as $y \to 0$.

average curvature. See CURVATURE—average curvature of a curve in a plane.

average deviation. Same as MEAN DEVIATION. See DEVIATION—mean or average deviation.

average ordinate. See MEAN—mean value of a function.

average speed and velocity. See SPEED, and VELOCITY.

average value of a function. See MEAN—mean value of a function.

moving average. The k-period *moving average* is the series of arithmetic averages obtained by averaging subsets of k successive equal-intervaled terms in a time series. Thus the average of the first k terms is usually identified with the midpoint of that interval. The second average is obtained from the second subset of k items counting from the second term in the series. Weighted averages may be used.

AV'ER-AG-ING, *n.* **averaging an account.** Finding the *average date.* See EQUATED—equated date.

AV'OIR-DU-POIS', *adj.* **avoirdupois weight.** A system of weights using the pound as its basic unit, the pound being equal to 16 ounces. See DENOMINATE NUMBERS in the appendix.

AX'I-AL, *adj.* **axial symmetry.** Symmetry with respect to a line. The line is called the **axis of symmetry.**

AX'I-OM, *n.* (1) A self-evident and generally accepted principle. (2) An assumption or postulate. The distinction between postulate and axiom is not very sharp. *Axiom* refers more to the *a priori* truth of a theorem than *postulate.* One may postulate something that could be proved but would hardly call it an axiom.

axiom of choice. See CHOICE.

axiom of continuity. To every point on the real axis there corresponds a real number (rational or irrational); the assumption that there exist numbers such as those indicated by the *Cauchy necessary and sufficient conditions for convergence,* and the *Dedekind cut postulate.* *Syn.* Principle of continuity.

axiom of countability. See SEPARABLE—separable space.

axiom of superposition. Any figure may be moved about in space without changing either its shape or size.

Euclid's axioms or "common notions." (1) Things equal to the same thing are equal to each other. (2) If equals are added to equals, the results are equal. (3) If equals are subtracted from equals the remainders are equal. (4) Things which coincide with one another are equal. (5) The whole is greater than any of its parts. Axioms (4) and (5) are not universally attributed to Euclid.

AX'IS, *n.* [*pl.* **axes**]. See CONE, CYLINDER, ELLIPSE, ELLIPSOID, HYPERBOLA, PARABOLA, PENCIL—pencil of planes, RADICAL—radical axis, REFERENCE—axis of reference, SYMMETRIC—symmetric geometric configurations.

axis of a curve or surface. Same as an AXIS OF SYMMETRY. See SYMMETRIC—symmetric geometric configurations.

axis of perspectivity. See PERSPECTIVE—perspective position.

axis of revolution. See SOLID—solid of revolution.

coordinate axis. A line along which (or parallel to which) a coordinate is measured. See CARTESIAN—Cartesian coordinates.

major and minor axis of an ellipse. See ELLIPSE.

polar axis. See POLAR—polar coordinates in the plane.

principal axes of inertia. See MOMENT—moment of inertia.

real and imaginary axes. The straight lines upon which the scales of reals and imaginaries have been laid off when plotting complex numbers in rectangular coordinates. See ARGAND DIAGRAM.

transverse and conjugate axis of the hyperbola. See HYPERBOLA.

AZ'I-MUTH, *n.* **azimuth of a point in a plane.** See POLAR—polar coordinates in the plane.

azimuth of a celestial point. See HOUR—hour angle and hour circle.

AZ'I-MUTH'AL, *adj.* **azimuthal map.** A map of a spherical surface S in which the points of S are projected onto a tangent plane from a point on that diameter of S that is perpendicular to the plane. An azimuthal map is said to be a **gnomic** map if the point of projection is the center of the sphere; it is an **orthographic map** if the point of projection is at infinite distance. See PROJECTION—stereographic projection of a sphere on a plane.

B

B.T.U. See BRITISH—British thermal unit.

BAC-TE'RI-AL, *adj.* **law of bacterial growth.** The increase per second of bacteria growing freely in the presence of unlimited food is proportional to the number present. It is defined by the equation $dN/dt=kN$, where k is a constant, t the time, N the number of bacteria present, and Nk the rate of increase. The solution of this equation is $N=ce^{kt}$. This is also called the **law of organic growth.** See DERIVATIVE.

BAIRE. Baire's category theorem. See CATEGORY.

Baire function. A real-valued function f which has the property that for any real number a the set of all x for which $f(x)>a$ is a *Borel set.* Equivalent definitions result if the set of all x satisfying $f(x)\geqq a$, or the set of all x satisfying $a\leqq f(x)\leqq b$ for arbitrary a and b, are required to be Borel sets (and either or both of the signs $\leqq$ could be replaced by $<$). Any Baire function is *measurable.* The Baire functions can be classified as follows. The set of continuous functions are of the **first Baire class.** In general, a function is of Baire class α if it is not of Baire class β for any $\beta<\alpha$ and is a point-wise limit of functions which belong to Baire classes corresponding to numbers preceding α. By transfinite induction, these classes are defined for all ordinal numbers corresponding to denumerable well-ordered sets. No additional functions are obtained by further extensions. *Syn.* Borel measurable functions. To every measurable function there corresponds a Borel measurable function which differs from f only on a set of *measure zero.*

property of Baire. A set S contained in a set T has the property of Baire if each nonempty open set U contains a point where either S or the complement of S is of *first category.* A set has the property of Baire if and only if it can be made into an open (or a closed) set by adjoining and taking away suitable sets of the first category, or if and only if it can be represented as a G_δ set plus a set of first category, or if and only if it can be represented as an F_σ set minus a set of first category. The class of all sets having the property of Baire is the σ-algebra generated by the open sets together with the sets of first category. See BOREL—Borel set, MEASURABLE—measurable set.

BANACH. Banach algebra. See ALGEBRA—Banach algebra.

Banach space. A vector space whose scalar multipliers are the real numbers (or the complex numbers) and which has associated with each element x a real number $\|x\|$, called the **norm** of x, satisfying the postulates: (1) $\|x\|>0$ if $x\neq 0$; (2) $\|ax\|=|a|\cdot\|x\|$ for all real numbers a; (3) $\|x+y\|\leq\|x\|+\|y\|$ for all x and y; (4) the space is complete, a *neighborhood* of an element x being the set of all y satisfying $\|x-y\|<\epsilon$ for some fixed ϵ. Without postulate (4), the space is a **normed linear space** or **normed vector space.** The Banach space is a **real Banach space** or a **complex Banach space** according as the scalar multipliers

are real numbers or complex numbers. Examples of Banach spaces are *Hilbert space*, the spaces $l^{(r)}(r \geqq 1)$ of all sequences $x=(x_1, x_2, \cdots)$ for which $\sum_{i=1}^{\infty} |x_i|^r$ is finite and $\|x\|=\left[\sum_{i=1}^{n} |x_i|^r\right]^{1/r}$, and the space (C) of all continuous functions f defined on the interval [0, 1] with $\|f\|=max|f(x)|$ for $0 \leq x \leq 1$.

Banach-Steinhaus theorem. Let X and Y be Banach spaces and let $T_1, T_2, \cdots$ be a sequence of bounded linear transformations from X to Y. If the set $\|T_1(x)\|, \|T_2(x)\|, \cdots$ is bounded for each x of X, then there is a number M such that $\|T_n(x)\| \leqq M\|x\|$ for all x of X and each n.

Hahn-Banach theorem. See HAHN-BANACH THEOREM.

BANK, *n.* **bank discount.** See DISCOUNT.

bank note. A note given by a bank and used for currency. It usually has the shape and general appearance of government paper money.

mutual saving bank. See MUTUAL.

BAR, *n.* See AGGREGATION.

bar graph. See GRAPH.

BAR'Y-CEN'TER, *n.* Same as CENTER OF MASS. See CENTER—center of mass.

BAR'Y-CEN'TRIC, *adj.* **barycentric coordinates.** Let $p_0, p_1, \cdots, p_n$ be n linearly independent points of n-dimensional Euclidean (or vector) space E_n. Then for each point x of E_n there is one and only one set $\lambda_0, \cdots, \lambda_n$ of real numbers for which

$$x=\lambda_0 p_0+\lambda_1 p_1+ \cdots \lambda_n p_n$$

and $\lambda_0+\lambda_1+\cdots+\lambda_n=1$. The point x is (by definition) the center of mass of point masses $\lambda_0, \lambda_1, \cdots, \lambda_n$ at the points $p_0, \cdots, p_n$, respectively, and the numbers $\lambda_0, \lambda_1, \cdots, \lambda_n$ are said to be *barycentric coordinates* of the point x. The motivation for this definition is that if three objects have weights $\lambda_0, \lambda_1, \lambda_2$ with $\lambda_0+\lambda_1+\lambda_2=1$, and their centers of mass are at the points $p_0=(x_0, y_0, z_0)$, $p_1=(x_1, y_1, z_1)$, $p_2=(x_2, y_2, z_2)$, then the center of mass of the three objects together is the point

$$\begin{aligned}\bar{p}&=\lambda_0 p_0+\lambda_1 p_1+\lambda_2 p_2\\ &=(\lambda_0 x_0+\lambda_1 x_1+\lambda_2 x_2, \lambda_0 y_0+\lambda_1 y_1+\lambda_2 y_2,\\ &\qquad \lambda_0 z_0+\lambda_1 z_1+\lambda_2 z_2).\end{aligned}$$

BASE, *n.* A base of a geometric configuration is usually a side (or face) upon which (perpendicular to which) an altitude is constructed, or is thought of as being constructed. See the particular geometric configuration. For an expression such as a^n, the quantity a is called the *base* and n the exponent. Also see the various headings below.

base angles of a triangle. The two angles which have the base of the triangle for a common side.

base for a topological space. A class B of open sets is a **base for the topology** of a topological space T if each open set is the union of some of the members of B. A **subbase for a topology** is a class S of open sets such that the class of all finite intersections of members of S is a base for the topology. A class N of open sets is a **base for the neighborhood system** of a point x (or a **local base** at X) if x belongs to each member of N and any open set which contains x also contains a member of N. A **subbase** for the neighborhood system of a point x (or a **local subbase** at x) is a class S of sets such that the class of all finite intersections of members of S is a base for the neighbourhood system of x. A topological space is said to satisfy the **first axiom of countability** if each point has a countable base for its neighborhood system; it satisfies the **second axiom of countability** if its topology has a countable base. A metric space satisfies the second axiom of countability if and only if it is *separable*.

base in mathematics of finance. A number, usually a sum of money, of which some per cent is to be taken; a sum of money upon which interest is to be calculated.

base of a logarithmic system. See LOGARITHM.

base of a system of numbers. The number of units, in a given digit's place or decimal place, which must be taken to denote 1 in the next higher place. *E.g.*, if the base is ten, ten units in units place are denoted by 1 in the next higher place, which is ten's place; if the base is twelve, twelve units in units place are denoted by 1 in the next higher place, which is twelve's place—that is, when the base is twelve, 23 means $2 \times$ twelve$+3$. *Tech.*, an integer to any base is of form $d_0+d_1(\text{base}) +d_2(\text{base})^2+d_3(\text{base})^3+\cdots$, where d_0, d_1,

d_2, d_3, etc., are each nonnegative integers less than the base. A number between 0 and 1 can be written as

$$.d_1d_2d_3\ldots=\frac{d_1}{base}+\frac{d_2}{(\text{base})^2}+\frac{d_3}{(\text{base})^3}+\cdots.$$

BA'SIS, *n.* **basis of a vector space.** A set of linearly independent vectors such that every vector of the space is equal to some linear combination of vectors of the basis. If the vectors of the basis are mutually orthogonal, the basis is an **orthogonal basis**; if they are also all of unit length, the basis is a **normal** (or **normalized**) **orthogonal basis,** or an **orthonormal basis.** If there is a finite number of vectors in the basis, the space is said to be **finite dimensional** and its dimension is equal to the number of vectors in its basis. Otherwise, it is **infinite dimensional.** For an infinite dimensional (and separable) vector space with a vector length (or *norm*) defined, a basis usually means a sequence of elements $x_1, x_2, \cdots$ such that every x is uniquely expressible in the form $x=\sum_{i=1}^{\infty} a_ix_i$ (meaning that the limit as n becomes infinite of the length of $x-\sum_{i=1}^{n} a_ix_i$ is zero). The examples given of *Banach spaces* possess such a basis, while for *Hilbert space* the sequence $x_1, x_2, \cdots$ is a normal orthogonal basis if x_p is an element for each p having all components zero except the pth, which is unity.

Hamel basis. See HAMEL.

BAYES' THEOREM. (1) If an event A can occur only when one of the B_1, B_2, $\cdots$, B_n exhaustive and incompatible events occurs; and (2) if the *a priori* probabilities of the events B_i, denoted by $P(B_i)$, are known when nothing is known about the occurrence of the event A; and (3) if the conditional probability of the event A to occur when B_i has been known to occur is $P(A, B_i)$ and is known for all i, then the **a posteriori probability** of B_i, $P(B_i, A)$, when it is known that A has occurred, is given by

$$P(B_i, A)=\frac{P(B_i)P(A, B_i)}{\sum_{j=1}^{n} P(B_j)P(A, B_j)}.$$

$P(B_i, A)$ is also known as the **inverse probability** of the event B_i. *E.g.*, 4 urns are equally likely to be sampled. Number 1 contains 1 white and 2 red balls, number 2 has 1 white and 3 red, number 3 has 1 white and 4 red, and number 4 has 1 white and 5 red. The *a priori* probability of an urn being sampled is $\frac{1}{4}=P(B_i)$; $P(A, B_i)$ equals $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$ respectively for $i=1, \cdots, 4$, where A is the draw of a white ball. Application of Bayes' formula yields $P(B_2,A)=\frac{15}{47}$. $P(B_i)$ need not all be equal. Bayes' theorem may be applied to confidence intervals and tests of hypotheses if the *a priori* probabilities are known. The usual Neyman and Pearson confidence interval and test of hypotheses do not rely on *a priori* probabilities. See PROBABILITY—mathematical or *a priori* probability.

BEAR'ING, *n.* **bearing of a line.** (*Surveying.*) The angle which the line makes with the north and south line; its direction relative to the north-south line.

bearing of a point, with reference to another point. The angle that the line through the two points makes with the north and south line.

BEHREN'S-FISHER PROBLEM. The problem of determining the probability of drawing two random samples, whose means differ by k (k may equal zero), from normal populations the difference of whose means is known but the ratio of whose variances is not known.

BEI, *adj.* **bei function.** See BER—ber function.

BEND, *adj.* **bend point.** A point on a plane curve where the ordinate is a maximum or minimum.

BENDING MOMENT. See MOMENT.

BEN'E-FI'CI-ARY, *n.* (*Insurance.*) The one to whom the amount guaranteed by the policy is to be paid.

BEN'E-FIT, *n.* **benefits of an insurance policy.** The sum or sums which the company promises to pay provided a specified event occurs, such as the death of the insured or his attainment of a certain age.

BER, *adj.* **ber function.** The **ber, bei, her, hei, ker,** and **kei** functions are defined by the relations:

$$\mathrm{ber}_n(z) \pm i\,\mathrm{bei}_n(z) = J_n(z\,e^{\pm 3\pi i/4}),$$
$$\mathrm{her}_n(z) + i\,\mathrm{hei}_n(z) = H_n^{(1)}(z\,e^{3\pi i/4}),$$
$$\mathrm{her}_n(z) - i\,\mathrm{hei}_n(z) = H_n^{(2)}(z\,e^{-3\pi i/4}),$$
$$\mathrm{ker}_n(z) \pm i\,\mathrm{kei}_n(z) = i^{\mp n} K_n(z\,e^{\pm \pi i/4}),$$

where J_n is a *Bessel function,* $H_n^{(1)}$ and $H_n^{(2)}$ are *Hankel functions,* and K_n is a *modified Bessel function* of the second kind. The following conventions are also used: $ber_0(z) = ber(z)$, $\mathrm{bei}_0(z) = \mathrm{bei}(z)$, etc. It follows that:

$$2\,\mathrm{ker}_n(z) = -\pi\,\mathrm{hei}_n(z);$$
$$2\,\mathrm{kei}_n(z) = \pi\,\mathrm{her}_n(z).$$

These six functions are real when n is real and z is real and positive. In particular,

$$\mathrm{ber}\,x = 1 - \frac{x^4}{2^2 \cdot 4^2} + \frac{x^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \cdots,$$

$$\mathrm{bei}\,x = \frac{x^2}{2^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \frac{x^{10}}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2} - \cdots.$$

Also, $\int_0^x t\,\mathrm{ber}(t)\,dt = x\,\mathrm{bei}'(x)$, $\int_0^x t\,\mathrm{bei}(t)\,dt = -x\,\mathrm{ber}'(x)$, these still being valid if *ber* is replaced by *ker* and *bei* by *kei.*

BERNOULLI. Bernoulli's equation. A linear differential equation of the form

$$\frac{dy}{dx} + yf(x) = y^n g(x).$$

Bernoulli's numbers. (1) The numerical values of the coefficients of $x^2/2!$, $x^4/4!$, $\cdots$, $x^{2n}/(2n)!, \cdots$ in the expansion of $x/(1-e^{-x})$, or $xe^x/(e^x-1)$. Substituting the exponential series for e^x and starting the division by the expansion of (e^x-1) one obtains, for the first four terms of this quotient,

$$1 + (\tfrac{1}{2})x + (\tfrac{1}{6})x^2/2! - (\tfrac{1}{30})x^4/4!.$$

The odd terms all drop out after the term $(\frac{1}{2})x$. Some authors denote the Bernoulli numbers by B_1, B_2, etc. Others use B_2, B_4, etc. With the first notation: $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$, $B_6 = \frac{691}{2730}$, $B_7 = \frac{7}{6}$, $B_8 = \frac{3617}{510}$. In general,

$$B_n = \frac{(2n)!}{2^{2n-1}\pi^{2n}} \sum_{i=1}^{\infty} (1/i)^{2n}.$$

(2) The numbers defined by the relation $\dfrac{t}{e^t-1} = \sum_{i=1}^{\infty} B_n' \dfrac{t^n}{n!}$. It follows that $B_{2n}' = B_n$ except possibly for sign, that $B_{2n+1}' = 0$, for all $n > 1 (B_1' = -\frac{1}{2})$, and that $n!B_n' = B_n(0)$, where $B_n(z)$ is the nth Bernoullian polynomial. See below, Bernoulli's polynomials (1). Various trivial variations of these definitions are sometimes given.

Bernoulli's polynomials. (1) The polynomials $B_n(z)$ defined by

$$\frac{te^{zt}}{e^t-1} = \sum_1^{\infty} B_n(z)t^n.$$

The first four Bernoulli polynomials are $B_1(z) = z - \frac{1}{2}$, $B_2(z) = (z^2/2) - (z/2) + \frac{1}{12}$, $B_3(z) = (z^3/3!) - (z^2/4) + (z/12)$, $B_4(z) = (z^4/4!) - (z^3/12) + (z^2/24) - \frac{1}{720}$. It follows that $B'_{n+1}(z) = B_n(z)$, $B_n(z+1) - B_n(z) = nz^{n-1}$ $(n > 1)$;

$$B_{2n}(z) = (-1)^{n-1} \sum_{r=1}^{\infty} \frac{2\cos 2r\pi z}{(2r\pi)^{2n}}$$

and

$$B_{2n+1}(z) = (-1)^{n-1} \sum_{r=1}^{\infty} \frac{2\sin 2r\pi z}{(2r\pi)^{2n+1}} \qquad (n \geq 1)$$

(2) The polynomials $\phi_n(z)$ defined by

$$t\,\frac{e^{zt}-1}{e^t-1} = \sum_{n=1}^{\infty} \frac{\phi_n(z)t^n}{n!}.$$

It follows that $\phi_n(z) = n!\,[B_n(z) - B_n']$, and that $\phi(0) = 0$. See above, Bernoulli's polynomials (2). Trivial variations of these definitions are sometimes given.

Bernoulli's theorem. (*Statistics.*) Let: (1) p be the probability of the event A on a trial, and (2) m/n be the observed proportion of the event A in n trials. Then the probability that $\left|\frac{m}{n} - p\right| < \epsilon$ has a limit of one as $n \to \infty$, for any arbitrary ϵ. *Syn.* Law of large numbers.

lemniscate of Bernoulli. See LEMNISCATE.

BERTRAND. Bertrand curve. A curve whose principal normals are the principal normals of a second curve. *Syn.* Conjugate curve.

Bertrand's postulate. There is always at least one prime number between n and $(2n-2)$, provided n is greater than 3.

E.g., if n is 4, $2n-2=6$ and the prime 5 is between 4 and 6. Bertrand's "postulate" is a true theorem [P. L. Tchebycheff (1852)].

BESSEL. Bessel functions. For n a positive or negative integer, the nth Bessel function, $J_n(z)$, is the coefficient of t^n in the expansion of $e^{z[t-1/t]/2}$ in powers of t and $1/t$. In general,

$$J_n(z)=\frac{1}{\pi}\int_0^\pi \cos(nt-z\sin t)\,dt$$
$$=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{z}{2}\right)^{n+2r},$$

the second form being valid if $n\neq -1, -2, \cdots$. $J_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\sin z$, and for all n,

$$2[dJ_n(z)/dz]=J_{n-1}(z)-J_{n+1}(z),$$
$$(2n/z)J_n(z)=J_{n-1}(z)+J_{n+1}(z),$$

and $J_n(z)$ is a solution of Bessel's differential equation. Sometimes called *Bessel functions of the first kind.* See HANKEL—Hankel function, NEUMANN—Neumann function, and below, modified Bessel functions.

Bessel's differential equation. The differential equation

$$z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}+(z^2-n^2)y=0.$$

Bessel's inequality. (1) For any real function $F(x)$ and an *orthogonal normalized* system of real functions $f_1, f_2, \cdots$ on an interval (a, b), Bessel's inequality is

$$\int_a^b [F(x)]^2\,dx \geq \sum_{n=1}^{p}\left[\int_a^b F(x)f_n(x)\,dx\right]^2,$$

or for complex valued functions,

$$\int_a^b |F(x)|^2\,dx \geq \sum_{n=1}^{p}\left|\int_a^b F(x)\overline{f_n(x)}\,dx\right|^2.$$

These are valid for all p if the functions $F, f_1, f_2, \cdots$ are assumed to be Riemann integrable (or, more generally, if they are Lebesgue measurable and their squares are Lebesgue integrable). For the Fourier coefficients of any (measurable) real function whose square is Riemann (or Lebesgue) integrable, Bessel's inequality becomes:

$$\frac{1}{\pi}\int_0^{2\pi}[F(x)]^2\,dx \geq (a_0/2)^2+\sum_{k=1}^{n}(a_k^2+b_k^2),$$

for all n, where $a_k=\frac{1}{\pi}\int_0^{2\pi}F(x)\cos kx\,dx$, $b_k=\frac{1}{\pi}\int_0^{2\pi}F(x)\sin kx\,dx$ $(k=0, 1, 2, \cdots)$.

(2) For a vector space with an inner product $(\mathbf{x}, \mathbf{y})$ and an orthogonal normalized set of vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$, Bessel's inequality is

$$(\mathbf{u}, \mathbf{u})=|\mathbf{u}|^2 \geq \sum_{k=1}^{n}|(\mathbf{u}, \mathbf{x}_k)|^2.$$

See RIESZ-FISCHER THEOREM, VECTOR—vector space, and PARSEVAL—Parseval's theorem.

modified Bessel functions. The *modified Bessel functions of the first kind* and *of the second kind* are the functions $I_n(z)=i^{-n}J_n(iz)$ and

$$K_n(z)=\tfrac{1}{2}\pi(\sin n\pi)^{-1}[I_{-n}(z)-I_n(z)];$$

$K_n(z)$ is the limit of this expression if n is an integer. These functions are real when n is real and z is positive. Also, $I_n(z)$ is a solution of the **modified Bessel's differential equation,**

$$z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}-(z^2+n^2)y=0,$$

and $I_n(z)=\sum_{r=0}^{\infty}\frac{1}{n!\Gamma(n+r+1)}\left(\frac{z}{2}\right)^{n+2r}$. The functions I_n and I_{-n} are independent solutions of this differential equation when n is not an integer, while the limit of K_n is a second solution when n is an integer. These functions satisfy various recurrence relations, such as $I_{n-1}(z)-I_{n+1}(z)=(2n/z)I_n(z)$ and $K_{n-1}(z)-K_{n+1}(z)=-(2n/z)K_n(z)$. The definition of $K_n(z)$ is sometimes taken as the product of $\cos n\pi$ and the above value (I_n and K_n then satisfy the same recurrence formulae). See BER—ber function.

BE'TA, *n.* The second letter in the Greek alphabet, written β.

beta coefficient. See CORRELATION—multiple correlation.

beta function. The integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ m>0,\ n>0.$$

It is denoted by $\beta(m, n)$. In terms of the Γ function,

$$\beta(m, n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

See GAMMA—gamma function. The **incomplete beta function** is defined by

$$\beta_x(m, n) = \int_0^x t^{m-1}(1-t)^{n-1}\,dt,$$

which is equal to $m^{-1}x^m F(m, 1-n;\ m+1;\ x)$, where F is the hypergeometric function.

beta weight. See CORRELATION—normal correlation.

BETTI, Betti group. See HOMOLOGY—homology group.

Betti number. Let H_r be the r-dimensional *homology group* (of a *simplicial complex K*) formed by using the group G. If G is the group of integers modulo π, where π is a prime, then G is a field, H_r is a linear (vector) space, and the dimension of H_r is the **r-dimensional Betti number (modulo π)** of K. If G is the group of integers, then H_r is a commutative group with a finite number of generators and is the *Cartesian product* of infinite cyclic groups $E_1, \cdots, E_m$ and cyclic groups $F_1, \cdots, F_n$ of finite orders $r_1, \cdots, r_n$ (see TORSION—torsion coefficients of a group). The number m is the **r-dimensional Betti number** and $r_1, \cdots, r_n$ are the r-dimensional **torsion coefficients** of K. The Betti numbers (especially the 1-dimensional Betti number modulo 2, or 1 plus this number) are sometimes called *connectivity numbers* (see CONNECTIVITY). For an ordinary closed surface, $\chi = 2 - B_2^1$, where χ is the Euler characteristic and B_2^1 is the 1-dimensional Betti number modulo 2. If the surface is not closed (has boundary curves), then $\chi = 1 - B_2^1$. If the surface is orientable, then the *genus* of the surface is equal to $\frac{1}{2}B_2^1$.

BI-AN'NU-AL, *adj.* Twice a year. *Syn.* Semiannual.

BI'ASED, or BI'ASSED, *adj.* **biased statistic.** If the expected value E of a statistic obtained from random sampling is not equal to the parameter or quantity being estimated, the statistic is *biased*. More precisely, if from a population with a frequency function, $f(x, t_1, t_2, \cdots t_n)$, where x is the variable and t_i are the parameters of the function, random samples are drawn each of size n, and if for each of all the possible random samples of size n, a statistic $T_i(n)$ is obtained as an estimate of t_i, the statistic $T_i(n)$ is biased if $E(T_i(n)) \neq t_i$. If the equality holds the estimate is unbiased. *E.g.*, the expression $\frac{\Sigma(X-\bar{x})^2}{n}$, where n is the size of a random sample from a normal distribution and $\bar{x}$ is the mean of n items, gives a biased estimate of the variance, but if n is replaced by $n-1$, it is **unbiased.**

BI-COM-PACT', *adj.* See COMPACT—compact set.

BI-COM-PAC'TUM, *n.* See COMPACTUM.

BI'CON-DI'TION-AL, *adj.* See EQUIVALENCE—equivalence of propositions.

BIENAYME - TCHEBYCHEFF INEQUALITY. (*Statistics.*) Let $\bar{x}_n$ be the mean of the sample values $(x_1, x_2, \cdots x_n)$ of the random variable x whose mean is u and whose standard deviation is σ. Then the probability of $(|\bar{x}_n - u| \leq \sigma t)$ is equal to or greater than $1-(1/t^2)$. σt may be replaced by a constant ϵ, whence $1-(1/t^2)$ is replaced by $1-(\sigma^2/\epsilon^2)$. Also known as **Tchebycheff's inequality.**

BI-EN'NI-AL, *adj.* Once in two years; every two years.

BI'HAR-MON'IC, *adj.* **biharmonic boundary value problem.** See BOUNDARY.

biharmonic function. A solution of the fourth order partial differential equation $\Delta\Delta u = 0$, where Δ is the Laplace operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$; thus, a solution $u(x, y, z)$ of the equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{dy^4} + \frac{\partial^4 u}{dz^4} + 2\,\frac{\partial^4 u}{\partial x^2 \partial y^2} + 2\,\frac{\partial^4 u}{\partial y^2 \partial z^2} + 2\,\frac{\partial^4 u}{dz^2 \partial x^2} = 0.$$

The definition applies equally well to functions of two, four, or any other number of independent variables. Biharmonic functions occur in the study of electrostatic boundary value problems and elsewhere in mathematical physics.

BI-LIN'E-AR, *adj.* A mathematical expression is *bilinear* if it is linear with respect to each of two variables or positions. *E.g.*:

The function $f(x, y)=3xy$ is linear in x and y, since $f(x_1+x_2, y)=3(x_1+x_2)y=3x_1y+3x_2y=f(x_1, y)+f(x_2, y)$ and $f(x, y_1+y_2)=f(x, y_1)+f(x, y_2)$. The *scalar product* of vectors $\mathbf{x}=x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}$ and $\mathbf{y}=y_1\mathbf{i}+y_2\mathbf{j}+y_3\mathbf{k}$ is $\mathbf{x}\cdot\mathbf{y}=x_1y_1+x_2y_2+x_3y_3$, which is bilinear since $(\mathbf{u}+\mathbf{v})\cdot\mathbf{y}=\mathbf{u}\cdot\mathbf{y}+\mathbf{v}\cdot\mathbf{y}$ and $\mathbf{x}\cdot(\mathbf{u}+\mathbf{v})=\mathbf{x}\cdot\mathbf{u}+\mathbf{x}\cdot\mathbf{v}$. The scalar product and the function $3xy$ are **bilinear forms** (see FORM). The function $F(u, v)$ whose value at x is

$$\int_0^1 t^2u(t, x)v(t, x)\,dt$$

is a bilinear function of u and v, where u and v are functions of two variables.

bilinear concomitant. See ADJOINT—adjoint of a differential equation.

BILL, *n.* A statement of money due, usually containing an itemized statement of the goods or services for which payment is asked.

BIL'LION, *n.* (1) In the U.S. and France, a thousand millions (1,000,000,000). (2) In England and Germany, a million millions (1,000,000,000,000).

BI-MO'DAL, *adj.* **bimodal distribution.** A distribution with two modes. *I.e.*, there are two different values which are conspiculously more frequent than neighboring values.

BI'NA-RY, *adj.* **binary representation of numbers.** Writing numbers with the base 2 (see BASE—base of a system of numbers). *E.g.*, the number $45\frac{5}{8}$ in the decimal system would be 101101.101 when written with base 2.

BI-NO'MI-AL, *n.* A polynomial of two terms, such as $2x+5y$ or $2-(a+b)$.

binomial coefficients. The coefficients of the variables in the expansion of $(x+y)^n$. The $(r+1)$th binomial coefficient of order n (n a positive integer) is $n!/[r!(n-r)!]$, the number of combinations of n things r at a time, and is denoted by $\binom{n}{r}$, ${}_nC_r$, $C(n, r)$, or $C_r{}^n$. The sum of the binomial coefficients is equal to 2^n, obtained by putting unity for each of x and y in $(x+y)^n$. See below, binomial theorem.

binomial differential. A differential of the form $x^m(a+bx^n)^p\,dx$, where a and b are any constants and the exponents m, n, and p are rational numbers.

binomial distribution (binomial frequency distribution). The distribution of the various possible number of successes in a given number of trials; the distribution of probabilities of successes exhibited by the quotients of the coefficients in the binomial expansion and their sum. *E.g.*, if two coins be thrown, the probability that both will be heads is $\frac{1}{4}$, that one will be heads and the other tails is $\frac{2}{4}$, and that both will be tails is $\frac{1}{4}$. If x represents heads only, y tails only, and xy head and tails, then in the expression $(x^2+2xy+y^2)/4$, $x^2/4$ denotes that the probability of getting 2 heads is $\frac{1}{4}$, $2xy/4$ that the probability of getting a head and a tail is $\frac{2}{4}$, and $y^2/4$ that the probability of getting 2 tails is $\frac{1}{4}$. Again if three coins be thrown, the probability that all will be heads, two heads and one tail, etc., is well represented by $(x^3+3x^2y+3xy^2+y^3)/8$. *Tech.* If the frequency function of the binomial distribution is $f(x)=(p+q)^n$, where x is the number of observations of a particular event in n trials and the probability of this event is p and $p+q=1$, then the value of the function for $x=i$ is given by the ith term, $C_n{}^ip^ix^{n-i}$, in the binomial expansion, $C_n{}^i$ being the number of combinations of n items taken i at a time. *E.g.*, consider the proportion of heads in n tosses of a coin: the probability of 1 head in 4 tosses of a coin is given by $C_4{}^1(.5)^1(.5)^3=4(.5)(.125)=.25$. As n increases, the binomial distribution tends toward a normal distribution (unless p is very small so that np is a constant, in which case the Poisson distribution is the limiting form). The *mathematical expectation* (mean) of the binomial distribution is np, and the *variance* is npq.

binomial equation. An equation of the form $x^n-a=0$.

binomial expansion. The expansion given by the binomial theorem.

binomial formula. The formula given by the binomial theorem.

binomial series. A binomial expansion which contains infinitely many terms. That is, the expansion of $(x+y)^n$, where n is not a positive integer or zero. Such an expansion converges and represents

the function for all powers provided $|y| < |x|$. *E.g.*,

$$\sqrt{3} = (2+1)^{1/2} = 2^{1/2} + \tfrac{1}{2}(2)^{-1/2} - (\tfrac{1}{2})^3(2)^{-3/2} + \cdots.$$

binomial surd. See SURD.

binomial theorem. A theorem (or rule) for the expansion of a power of a binomial. The theorem can be stated thus: The first term in the expansion of $(x+y)^n$ is x^n; the second term has n for its coefficient, and the other factors are x^{n-1} and y; in subsequent terms the powers of x decrease by unity for each term and those of y increase by unity, while any coefficient can be obtained from the previous coefficient by multiplying the latter by the exponent of x in the previous term and dividing by the number of terms to and including the previous term. *E.g.*, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$. In general,

$$(x+y)^n = x^n + nx^{n-1}y + [n(n-1)/2!]x^{n-2}y^2 + \cdots + y^n$$

if n is a positive integer. The general term, say the rth term, is

$$[n(n-1)\cdots(n-r+2)/(r-1)!]x^{n-r+1}y^{r-1}.$$

This coefficient is also written

$$\frac{n!}{(r-1)!(n-r+1)!}.$$

The $(r+1)$th term is often used since it is simpler. See above, binomial coefficients. The binomial theorem holds for any exponent whatever under certain restrictions on x and y. See above, binomial series.

BI-NOR'MAL, *n.* See NORMAL—normal to a curve or surface.

BI-PAR'TITE, *n.* **bipartite cubic.** The locus of the equation

$$y^2 = x(x-a)(x-b),\ 0 < a < b.$$

The curve is symmetric about the x-axis and intersects the x-axis at the origin and at the points $(a, 0)$ and $(b, 0)$. It is said to be bipartite because it has two entirely separate branches.

BI-QUAD-RAT'IC, *adj.* **biquadratic equation.** An algebraic equation of the fourth degree. *Syn.* Quartic.

BI-REC-TANG'U-LAR, *adj.* **birectangular triangle.** A spherical triangle, two of whose angles are right angles.

BIRKHOFF. Poincaré-Birkhoff fixed point theorem. See POINCARÉ.

BI-SECT', *v.* To divide in half.

bisect an angle. To draw a line through the vertex dividing the angle into two equal angles.

bisect a line segment. To find the point on the line segment and equally distant from the ends. *Analytically*, the Cartesian coordinates of the midpoint can be found as the arithmetic means or averages of the corresponding coordinates of the two end points. See POINT—point of division. If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are the end points of a line segment, the coordinates of the midpoint are

$$x = (x_1 + x_2)/2, \quad y = (y_1 + y_2)/2.$$

BI-SECT'ING, *adj.* **bisecting point of a line segment.** Same as MIDPOINT of a line segment.

BI-SEC'TOR, *n.* **bisector of an angle.** The straight line which divides the angle into two equal angles.

bisector of the angle between two intersecting planes. A plane containing all the points equidistant from the two planes. There are two such bisectors for any two such planes. Their equations are obtained by equating the distances of a variable point from the two planes—first giving these distances like signs, and then unlike signs. See DISTANCE—distance from a plane to a point.

equations of the bisectors of the angles between two lines. The equations obtained by equating the distances from a variable point to each of the lines (the distances being taken first with the same sign and then with opposite signs). See DISTANCE—distance from a line to a point.

BI-SE'RI-AL, *adj.* **biserial correlation coefficient.** A correlation coefficient adapted to the bivariate case in which, although both variables are continuous, one is recorded in dichotomy form. It is assumed that the dichotomized variable is normally distributed. Then

$$r_{bis} = \frac{(M_p - M_p)pq}{\sigma \cdot z},$$

where M_p and M_q are means of the upper and lower sections of the dichotomized variable, p and q are the proportions of cases in each section, z is the ordinate of a normal distribution at the point which divides a normal distribution in the proportions p and q, and σ is the standard deviation in the sample of the continuously measured variable.

BLOCK, *n.* **randomized blocks.** A method of assigning an experiment to obtain sample observations for the analysis of variance, whereby two factors contributing to variation in the variables under study may be controlled. *E.g.*, in a study of quality of product, within the class of one of the factors under control (say identity of machine producing the product), another factor (say operator of machine) is experimentally controlled by assigning the operators at random one to each machine. Here blocks are identified with machines and the operator is randomized in the block. As a result, variation attributable to machines and operators may be estimated and tested with the interaction as the error estimate. Repeated observations within each block-operator matching will yield an error estimate against which the interaction may be tested. The term *blocks* was first employed in agricultural field plots experiments. See VARIANCE—analysis of variance.

BOARD MEASURE. The system of measuring used in measuring lumber. See MEASURE—board measure.

BOD'Y, *n.* **convex body.** See CONVEX—convex set.

BOLZA. problem of Bolza. In the calculus of variations, the general problem of determining, in a class of curves subject to constraints of the form $Q_j(x, y, y')=0$ and

$$g_k[x_1, y(x_1), x_2, y(x_2)] + \int_{x_1}^{x_2} f_k(x, y, y')\,dx=0,$$

an arc that minimizes a function of the form

$$I=g[x_1, y(x_1), x_2, y(x_2)]+\int_{x_1}^{x_2} f(x, y, y')\,dx.$$

BOLZANO. Bolzano's theorem. A single-valued, real-valued function $f(x)$ of a real variable x is zero for at least one value of x on an interval $[a, b]$ if it is continuous on the closed interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs.

Bolzano-Weierstrass theorem. If E is a bounded set containing infinitely many points, there is a point x which is a limit point of E. The set E may be a set of real numbers, a set in a plane, or a set in n-dimensional Euclidean space. An equivalent statement of the theorem is that for any (finite dimensional) Euclidean space the concepts of bounded closed sets and compact sets are equivalent. This theorem is frequently credited to Weierstrass, but was proved by Bolzano in 1817 and seems to have been known to Cauchy.

BOND, *n.* A written agreement to pay interest (dividends) on a certain sum of money and to pay the sum in some specified manner, unless it be a **perpetual bond** (which draws interest, but whose principal need never be paid). **Callable** (or **optional**) **bonds** are redeemable prior to maturity at the option of the issuing corporation, usually under certain specified conditions and at certain specified times. An **annuity bond** is redeemed in equal installments which include the interest on the unpaid balance and sufficient payment of the face of the bond to redeem it by the end of a specified time. **Coupon bonds** are bonds for which the interest is paid by means of coupons (in effect, the coupons are post-dated checks, attached to the bond, which may be detached and used at the specified date); **registered bonds** are bonds whose ownership is registered with the debtor, the interest being paid by check directly to the registered owner. If an issue of bonds is such that part of the bonds mature on a certain date and part of the bonds mature at each of certain dates thereafter (usually each year), the bonds are said to be **serial bonds. Collateral trust bonds** are bonds issued by corporations whose assets consist primarily of securities of subsidiaries and of other corporations (the securities are deposited with a trust company as trustee); **guaranteed bonds** are bonds for which some corporation (in addition to the one which issues the bonds) guarantees payment of

principal or interest or both; **debenture bonds** are unsecured and usually protected only by the credit and earning power of the issuer; **mortgage bonds** have the highest priority in case of liquidation of the corporation (they are called *first mortgage bonds, second mortgage bonds*, etc.).

"and interest price", purchase price, and **redemption price** of a bond. See PRICE.

bond rate. See DIVIDEND—dividend on a bond:

bond table. A table showing the values of a bond at a given bond rate for various investment rates, and for various periods. Most tables are based on interest computed semiannually (the usual practice) and on the assumption that the bonds will be redeemed at par.

book value of a bond. See VALUE.

dividend on a bond. See DIVIDEND—dividend on a bond.

par value of a bond. The principal named in the bond. *Syn.* Face value.

premium bonds. See PREMIUM.

valuation of bonds. Computing the *present value*, at the investor's rate of interest, of the face value of the bond and of the interest payments (an annuity whose rental is equal to the dividend payments on the bond).

$$P = C(1+i)^{-n} + R[1-(1+i)^{-n}]/i,$$

where P denotes the value of the bond, C its redemption value, R the interest payments (coupon value if a coupon bond), n the number of periods before redemption, and i the investor's (purchaser's) rate per period.

yield of a bond. See YIELD.

BONNET. Bonnet's mean value theorem. See MEAN—mean value theorems (or laws of the mean) for integrals.

BO'NUS, *n.* A sum paid in addition to a sum that is paid periodically, as bonuses added to dividends, wages, etc. See INSURANCE—participating insurance policy.

BOOK, *n.* **book value.** See VALUE.

BOOL'E-AN, *adj.* **Boolean algebra.** A set which is a *ring* with the added properties that $x \cdot x = x$, for each x, and there is an element I such that $x \cdot I = x$ for each x. If the Boolean algebra is a class of sets, then *addition* and *multiplication* for the ring correspond to *symmetric difference* and *intersection* of sets and I is a set which contains each set of the class of sets. If a class of subsets of a set S contains the complement of each of its members and the union of any two of its members, then it is a Boolean algebra if the ring operations of *addition* and *multiplication* are taken to be *symmetric difference* and *intersection.* Conversely, any Boolean algebra is an algebra of subsets for some class of subsets of some set. If, for any Boolean algebra, the operations $\cup$ and $\cap$, and the concept of inclusion, are defined by

$$A \cup B = (A+B)+(A \cdot B),$$
$$A \cap B = A \cdot B,$$
$$A \subset B \quad \text{if and only if} \quad A \cap B = A,$$

then these correspond to the union, intersection, and inclusion concepts for sets and the following statements can easily be proved ($A+A$ can be proved to have the same value for all elements A of the Boolean algebra and this common value is denoted by θ):

$$A \cup (B \cup C) = (A \cup B) \cup C,$$
$$A \cap (B \cap C) = (A \cap B) \cap C,$$
$$A \cup B = B \cup A,$$
$$A \cap B = B \cap A,$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$A \cup A = A \cap A = A, \quad \theta \cup A = I \cap A = A,$$
$$\theta \subset A \subset I,$$
$$A = B \quad \text{if} \quad A \subset B \quad \text{and} \quad B \subset A,$$
$$A \subset C \quad \text{if} \quad A \subset B \quad \text{and} \quad B \subset C.$$

If the complement A' of A is defined to be $A+I$, then

$$(A \cap B)' = A' \cup B', \quad (A \cup B)' = A' \cap B',$$
$$A \cup A' = I, \quad A \cap A' = \theta,$$
$$(A')' = A, \quad I' = \theta, \quad \theta' = I.$$

The simplest Boolean algebra is the one whose elements are the empty set and the set of one point, θ and I. Then $A \cup B = I$ if and only if one (or both) of A and B is I, and $A \cap B = \theta$ if and only if one (or both) of A and B is θ. As well as being interpreted as an algebra of sets, a Boolean algebra can also be interpreted as an algebra of elementary logical properties of statements (propositions). The statement $p = q$ means that the statements denoted by "p" and "q"

are logically equivalent; $p \cup q$ denotes the statement "p or q"; $p \cap q$ denotes the statement "p and q". If p is the statement (or propositional function) "triangle x is isosceles", and "q" is the statement "triangle x is equilateral", then $p \cup q$ is the statement "triangle x is isosceles or triangle x is equilateral"; $p \cap q$ is the statement "triangle x is isosceles and triangle x is equilateral", and $q \subset p$ is the statement "$q \cap p = q$" (*i.e.*, "for any triangle x, x is isosceles if x is equilateral").

BOR′DER-ING, *v.* **bordering a determinant.** Annexing a column and a row. Usually refers to annexing a column and a row which have unity as a common element—all the other elements of either the column or the row being zero. This increases the order of the determinant by unity but does not change its value.

BOREL. Borel covering theorem. Same as the HEINE-BOREL THEOREM.

Borel's first definition of the sum of a divergent series. If Σa_n is the series to be summed, the sum by this definition is

$$S = \lim_{\alpha\to\infty} \lim_{n\to\infty} \frac{s_0 + s_1\alpha + s_2\alpha^2/2! + \ldots + s_n\alpha^n/n!}{1 + \alpha + \alpha^2/2! + \cdots + \alpha^n/n!}$$

$$= \lim_{\alpha\to\infty} \left(e^{-\alpha} \sum_{n=0}^{\infty} \frac{s_n}{n!} \alpha^n\right),$$

where $s_i = \sum_{j=0}^{i} a_j$. This definition is *regular.* See SUMMATION — summation of divergent series.

Borel's integral definition of the sum of a divergent series. The sum of Σa_n is defined as $\int_0^\infty e^{-x} \sum_0^\infty a_n \frac{x^n}{n!}\, dx$, where x is real, if this limit exists. This definition is *regular.* See SUMMATION—summation of divergent series.

Borel measurable function. See BAIRE—Baire function.

Borel set. Any set which can be obtained from the closed and open sets on the real line by repeated applications of operations of union and intersection to denumerable numbers of sets. The class of all Borel sets is the *σ-algebra* generated by the class of all open sets (or by the class of all closed sets, or by the class of all intervals). Examples of Borel sets are F_σ sets, which are countable unions of closed sets, and G_δ sets, which are countable intersections of open sets. Any Borel set is a *measurable set.* A Borel set is sometimes called a **Borel measurable set.**

BOUND, *n.* **class bound.** See LIMIT—limits of a class interval.

greatest lower bound of a set of numbers having a lower bound. Either the least number in the set (if this exists) or the greatest number less than all the numbers in the set. In the latter case the greatest lower bound is also an accumulation point. The set of numbers $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$, has a greatest lower bound, zero, which is also an accumulation point.

least upper bound of a set of numbers having an upper bound. Either the largest of the set (if this exists) or the least number greater than every number in the set. In the latter case the least upper bound is also an accumulation point. *E.g.*, the set of numbers, .3, .33, .333, $\cdots$, has the least upper bound $\frac{1}{3}$.

lower bound of a set of numbers. A number which is less than (or equal to) every number in the set.

upper bound of a set of numbers. A number which is greater than or equal to every number in the set.

BOUND′A-RY, *n.* **biharmonic boundary value problem.** For a region R with boundary surface S, the biharmonic boundary value problem is the problem of determining a function $U(x, y, z)$ that is biharmonic in R and is such that its first-order partial derivatives coincide with prescribed boundary value functions on S. This problem, along with the Dirichlet problem, arises in particular problems concerning elastic bodies.

boundary of a set. See INTERIOR—interior of a set.

boundary of a simplex and a chain and boundary operator. See CHAIN—chain of simplexes.

boundary value problem. (*Differential Equations.*) The problem of finding a solution to a given differential equation or set of equations which will meet certain specified requirements for a given set of

values of the independent variables—the boundary points. Many of the problems of mathematical physics are of this type.

first boundary value problem of potential theory (the Dirichlet problem). Given a region R, its boundary surface S, and a function f defined and continuous over S, to determine a solution U of Laplace's equation $\nabla^2 U=0$ which is regular in R, continuous in $R+S$, and which satisfies the equation $U=f$ on the boundary. This problem occurs in electrostatics and heat flow. It has at most one solution. See GREEN—Green's function.

second boundary value problem of potential theory (the Neumann problem). Given a region R, its boundary surface S, and a function f defined and continuous over S and such that $\iint f\,dS$ over S vanishes, to find a solution of Laplace's equation $\nabla^2 U=0$ which is regular in R, which together with its normal derivative is continuous in $R+S$, and such that its normal derivative is equal to f on the boundary S. This problem occurs in fluid dynamics. Any two of its solutions differ at most by a constant. See NEUMANN—Neumann's function (potential theory).

third boundary value problem of potential theory. As in the two above problems, except the function U is required to satisfy the equation $k\,\partial U/\partial n+hU=f$ on the boundary, where k, h, and f are prescribed functions that are continuous on S. This problem includes the other two and is of importance in heat flow and fluid mechanics. If $h/k>0$, it has at most one solution. See ROBIN—Robin's function.

BOUND'ED, *adj.* **bounded linear transformation.** See LINEAR—linear transformation.

bounded quantity, or function. A quantity whose numerical value is always less than or equal to some properly chosen constant. The ratio of a leg of a right triangle to the hypotenuse is a bounded quantity since it is always less than or equal to 1; that is, the functions $\sin x$ and $\cos x$ are bounded functions since they are always equal to or less than 1. The function $\tan x$ in the interval $(0, \frac{1}{2}\pi)$ is not bounded.

bounded sequence. See SEQUENCE—bound to a sequence.

bounded set of numbers. A set of numbers all of which are between two definite numbers; a set of numbers for which there are numbers A and B such that $A\leqq x\leqq B$ for each number x of the set.

bounded set of points. A set of points for which the set of distances between pairs of points is a bounded set. The least upper bound of such distances is called the **diameter** of the set. A set T is **totally bounded** if, for any $\epsilon>0$, there is a finite set of points in T such that each point of T is at distance less than ϵ from at least one of these points. A metric space is *compact* if and only if it is *complete* and *totally bounded.*

bounded variation. See VARIATION—total variation of a function.

essentially bounded function. A function f for which there is a number K such that the set of all x for which $|f(x)|>K$ is of measure zero. The greatest lower bound of such numbers K is the **essential supremum** of $|f(x)|$.

BOX, *n.* **three boxes game.** A game in which there are three boxes marked 1, 2, and 3. For a given play of the game, player A removes the bottom of one of the boxes, but player B does not know which one it is. Player B then puts an amount of money equal to the number marked on the box in each of two of the three boxes. He loses the money put in the box with no bottom and wins the money put in the others. This is a *zero-sum* game with *imperfect information.* The *payoff matrix* does not have *a saddle point* and the solutions are *mixed strategies.* The solutions are $(0, \frac{1}{2}, \frac{1}{2})$ for A and $(\frac{3}{5}, \frac{2}{5}, 0)$ for B, meaning that A removes the bottoms of boxes 2 and 3, each with probability $\frac{1}{2}$, player B puts money into boxes 1 and 2, or 1 and 3, with respective probabilities $\frac{3}{5}$ and $\frac{2}{5}$ (never in 2 and 3). The *value* of this game is 1 (with B the *maximizing player*).

BOYLE'S LAW. At a given temperature, the product of the volume of a gas and the pressure (pv) is constant. Also called **Boyle and Mariott's** law. Approximately true for moderate pressures.

BRACE, *n.* See AGGREGATION.

BRA-CHIS'TO-CHRONE, *adj.* **brachistochrone problem.** The calculus of variations problem of finding the equation of the path down which a particle will fall from one point to another in the shortest time. Proposed by John Bernoulli in 1696. It is easily shown that the time required for a particle with the initial velocity v_0 to fall along a path $y=f(x)$ from a point (x_1, y_1) to a point (x_2, y_2) is

$$t=\frac{1}{\sqrt{2g}}\int_{x_1}^{x_2}\sqrt{\frac{1+y_1{}^2}{y-a}}\,dx,$$

where $a=(y_1-v_0{}^2)/2g$. The solution of the problem then requires the determination of a $y(x)$ which minimizes the integral. See CALCULUS—calculus of variations. Newton, Leibnitz, l'Hospital, and James and John Bernoulli all found the correct solution, which is a *cycloid* through the two points.

BRACK'ET, *n.* See AGGREGATION.

BRANCH, *n.* **branch of a curve.** Any section of a curve separated from the other sections of the curve by discontinuities or special points such as vertices, maximum or minimum points, cusps, nodes, etc. One would speak of the two *branches* of an hyperbola, or even of four *branches* of an hyperbola; or of two *branches* of the semicubical parabola, or of the *branch* of a curve above (or below) the x-axis.

infinite branch. See INFINITE.

branch cut of a Riemann surface. A line or curve C on a Riemann surface such that on crossing C a variable point is considered as passing from one sheet to another.

branch of a multiple-valued analytic function. The single-valued analytic function $W=f(z)$ corresponding to values of z on a single sheet of the Riemann surface of definition.

branch-point of a Riemann surface. A point of the Riemann surface at which two or more sheets of the surface hang together.

BREADTH, *n* **breadth of a plane figure.** The length of a cross section of a plane figure all of whose cross sections are equal. If not all cross sections are equal, *breadth* is sometimes understood to mean the *longest cross section. Syn.* Width.

BRIDG'ING, *v.* **bridging in addition.** In adding a one-place number to a second number, *bridging* is said to occur if the sum is in a decade different from that in which the second number lies. Thus bridging occurs in $14+9=23$ but not in $14+3=17$. See DECADE.

bridging in subtraction. If the difference obtained by subtracting a number from a second number (the minuend) is in a decade different from that in which the minuend lies, bridging is said to have occurred. Thus bridging occurs in the examples $64-9=55$, $34-27=7$, but not in $64-3=61$.

BRIGGS' LOGARITHMS. Logarithms using 10 as a base. *Syn.* Common logarithms. See LOGARITHM.

BRITISH, *adj.* **British thermal unit** or **B.T.U.** The amount of heat required to raise the temperature of 1 lb. of water 1°F., when the water is at its maximum density, which is at 4°C. or 39.2°F.

BRO'KEN, *adj.* **broken line.** A line consisting of segments of lines joined end to end and not forming a continuous straight line.

BRO'KER, *n.* One who buys and sells stocks and bonds on commission, that is, for pay equal to a given percentage of the value of the paper. *Broker* is sometimes applied to those who sell any kind of goods on commission, but commission merchant, or commission man, is more commonly applied to those who deal in staple goods.

BRO'KER-AGE, *n.* A commission charged for selling or buying stocks, bonds, notes, mortgages, and other financial contracts. See BROKER.

BROUWER. Brouwer's fixed-point theorem. Let C be a circular disk consisting of a circle and the region within the circle. Then, for any continuous transformation which transforms each point of C into a point of C, there is some point which the transformation leaves fixed. The transformation is not assumed to be one-to-one. This theorem is also true for *closed n-cells* ($n\geqq 1$), *e.g.*, for a closed interval or for a sphere with its interior.

BUDAN. Budan's theorem. The number of real roots of $f(x)=0$ between a and b $(a<b)$, where $f(x)$ is a polynomial of degree n, is $V(a)-V(b)$, or less by an even number, $V(a)$ and $V(b)$ being the numbers of variations in sign of the sequence

$$f(x), f'(x), f''(x), \cdots, f^{(n)}(x),$$

when $x=a$ and $x=b$, respectively. (Vanishing terms in the sequence are not counted and m-tuple roots are counted as m roots.) *E.g.*, to find the number of roots of $x^3-5x+1=0$ between 0 and 1, we form the sequence x^3-5x+1, $3x^2-5$, $6x$, 6, then substitute 0 and 1 for x, successively. This give the sequences $1, -5, 0, 6$ and $-3, -2, 6, 6$, whence $V(0)-V(1)=2-1=1$. Thus there is one root between 0 and 1. Similarly the other roots can be located between 2 and 3 and between -3 and -2.

BUFF'ER, *n.* In a computing machine, a switch that transmits a signal if any one of several signals is received by the switch; thus a buffer is the machine equivalent of the logical "or". See DISJUNCTION, and GATE. *Syn.* Inverse gate.

BUILD'ING, *n.* **building and loan association.** A financial organization whose objective is to loan money for building homes. One plan, called the *individual account plan*, is essentially as follows: Members may buy shares purely as an investment, usually paying for them in monthly installments at an annual nominal rate; or they may borrow money (shares) from the company with which to build, securing (guaranteeing) this money with mortgages on their homes. In both cases the monthly payments are called dues. Failure to meet monthly payments on time is sometimes subjected to a fine which goes into the profits of the company. The profits of the company are distributed to the share purchasers, thus helping to mature (complete the payments on) their shares. In practice, the interest rate is usually figured so that it returns all profits automatically. A *serial plan* is a plan under which shares are issued at different times to accommodate new members. Monthly dues are paid and profits distributed to all shareholders. This plan naturally resolves into the INDIVIDUAL ACCOUNT PLAN. A *guaranteed stock plan* is a plan in which certain investors provide certain funds and guarantee the payment of certain dividends on all shares, any surplus over this guarantee being divided among these basic stockholders. A *terminating plan* is a plan under which the members pay dues for a certain number of years to facilitate their building homes, the highest bidder getting the use of the money, since there is not enough to go around. New members coming in have to pay back-dues and back-earnings. This is the earliest plan of building and loan association and is not usually practiced now.

BULK, *n.* **bulk modulus.** See MODULUS.

BUNIAKOWSKI. Buniakowski's inequality. See SCHWARZ—Schwarz's inequality.

BURALI-FORTI PARADOX. The "set of all ordinal numbers" (each of which is an *order type* of a well-ordered set) is a well-ordered set. However, the order type Y of this set is then a largest ordinal number. This is impossible, since $Y+1$ is a larger ordinal number (Y is the order type of a certain well-ordered set and $Y+1$ is the order type of the well-ordered set obtained by introducing a single new element to follow every member of this set).

C

C.G.S. UNITS. Units of the centimeter—gram—second system. Centimeter measures the distance (length); gram, mass; and second, time. See ERG, and FORCE—unit of force.

CABLE, *n.* **parabolic cable.** See PARABOLIC.

CAL'CU-LATE, *v.* To carry out some mathematical process; to supply theory or formula and secure the results (numerical or otherwise) that are required; a looser and less technical term than compute. One may say, "Calculate the volume of a cylinder with radius 4′ and altitude 5′"; he may also say, "Calculate the derivative of $\sin(2x+6)$." *Syn.* Compute.

CALCULATING MACHINE. Same as COMPUTING MACHINE, but see COMPUTER.

CAL′CU-LUS, *n.* **calculus of variations.** The study of the theory of maxima and minima of definite integrals whose integrand is a known function of one or more independent variables and of one or more dependent variables and their derivatives, the problem being to determine the dependent variables so that the integral will be a maximum or a minimum. The simplest such integral is of the form

$$I = \int_a^b f(x, y, dy/dx)\, dx,$$

where $y(x)$ is to be determined to make I a maximum or a minimum (whichever is desired). The name *calculus of variations* originated as a result of notations introduced by Lagrange in about 1760 (see VARIATION). Other integrals studied are of the form

$$I = \int_a^b f(x, y_1, \cdots y_n, y_1', \cdots y_n')\, dx,$$

where $y_1, \cdots, y_n$ are unknown functions of x, or multiple integrals such as

$$I = \int_a^b \int_a^b f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) dx\, dy,$$

where z is an unknown function of x and y, or multiple integrals of higher order or of various numbers of dependent variables (the integrand may also be a function of derivatives of higher order than the first). See BRACHISTOCHRONE—brachistochrone problem, ISOPERIMETRIC—isoperimetric problem in the calculus of variations, and EULER—Euler's equation.

differential calculus. The study of the variation of a function with respect to changes in the independent variable, or variables, by means of the concepts of derivative and differential; in particular the study of slopes of curves, nonuniform velocities, accelerations, forces, approximations to the values of a function, maximum and minimum values of quantities, etc. See DERIVATIVE.

fundamental lemma of the calculus of variations. See FUNDAMENTAL.

fundamental theorem of the integral calculus. See FUNDAMENTAL—fundamental theorem of the integral calculus.

infinitesimal calculus. Ordinary calculus; so called because of its use of infinitesimal quantities. Sometimes refers only to that part of the calculus which deals with differentials and sums of infinitesimals.

integral calculus. The study of integration as such and its application to finding areas, volumes, centroids, equations of curves, solutions of differential equations, etc.

CALL′A-BLE, *adj.* **callable bonds.** See BOND.

CAL′O-RIE (or **CAL′O-RY**), *n.* The amount of heat required to raise one gram of water one degree Centigrade. The calorie thus defined varies slightly for different temperatures. A standard calorie is usually defined as the amount of heat required to raise one gram of water from 14.5° to 15.5°C. This unit is about the average amount required to raise one gram of water one degree at any point between 0° and 100°C. A more exact definition (generally accepted in the U.S.) is that one calorie equals 4.1840 absolute joules.

CAN′CEL, *v.* (1) To divide numbers (or factors) out of the numerator and denominator of a fraction;

$$\frac{6}{8} = \frac{2\times 3}{2\times 4} = \frac{3}{4},$$

the number 2 having been *canceled out.* (2) Two quantities of opposite sign but numerically equal are said to *cancel* when added; $2x+3y-2x$ reduces to $3y$, the terms $2x$ and $-2x$ having *canceled* out.

CAN′CEL-LA′TION, *n.* The act of dividing like factors out of numerator and denominator of a fraction; sometimes used of two quantities of different signs which cancel each other in addition. Also used for the process of eliminating z when replacing $x+z=y+z$ by $x=y$, or $xz=yz$ by $x=y$ (if $z\neq 0$). See DOMAIN—integral domain, and SEMI—semigroup.

CA-NON′I-CAL, *adj.* **canonical correlation.** Consider two sets of random variables. Let L_1 and L_2 each be linear functions of the two sets. The maximum correlation between L_1 and L_2 relative to the linear functions is the *canonical correlation*

between the two sets of variables, subject to certain restrictions on the coefficients in the linear functions of the two sets of variables.

canonical form of a matrix. That which has been considered the simplest and most convenient form to which square matrices of a certain class can be reduced by a certain type of transformation. *Syn.* Normal form. *E.g.*: (1) Any square matrix can be reduced by *elementary operations* or an *equivalent transformation* to the canonical form having nonzero elements only in the principal diagonal; or when the elements are polynomials (or integers, etc.) to **Smith's canonical form** having zeros except in the principal diagonal and each diagonal element being a factor of the next lower (if not zero). (2) Any matrix can be reduced by a *collineatory transformation* to the **Jacobi canonical form** having zeros below the principal diagonal and characteristic roots as elements of the principal diagonal, or to the **classical canonical form** having zeros except for a sequence of *Jordan matrices* situated along the principal diagonal. The exact type of the classical canonical matrix is specified by its **Segre characteristic**—a set of integers which are the orders of the Jordan submatrices, those integers which correspond to submatrices containing the same characteristic root being bracketed together. When the characteristic roots are distinct, the classical canonical form is a diagonal matrix. (3) A *symmetric matrix* can be reduced to a diagonal matrix by a *congruent transformation.* (4) A *normal matrix* (and hence a *Hermitian* or a *unitary matrix*) can be reduced by a *unitary transformation* to a diagonal matrix having characteristic roots along the principal diagonal.

canonical representation of a space curve in the neighborhood of a point. Representation of the curve in the neighborhood of the point P_0, with the arc length from the point as parameter and the axes of the moving trighedral as coordinate axes. The representation has the form $x=s-\frac{1}{6}\frac{1}{\rho_0{}^2}s^3+\cdots$, $y=\frac{1}{2\rho_0}s^2+\frac{1}{6}\frac{d}{ds}\left(\frac{1}{\rho}\right)_0 s^3+\cdots$, $z=-\frac{1}{6}\frac{1}{\rho_0\tau_0}s^3+\cdots$, where ρ_0 and τ_0 are the radii of curvature and torsion, respectively, at P_0.

CAN'TI-LE'VER, *adj.* **cantilever beam.** A projecting beam supported at one end only.

CANTOR. **Cantor set.** The set of points formed from the closed interval $[0, 1]$ by removing first the middle third of the interval, then the middle third of each remaining interval, and so on indefinitely, the intervals removed being open intervals. The Cantor set is *perfect* and *nondense* and all its points are *frontier points.* Also called the **Cantor discontinuum,** and the **Cantor ternary set.**

CAP'I-TAL, *adj., n.* **capital stock.** The money invested by a corporation to carry on its business; wealth used in production, manufacturing, or business of any sort, which having been so used is available for use again. Capital stock may be disseminated by losses but is not consumed in the routine process of a business.

circulating capital. Capital consumed, or changed in form, in the process of production or of operating a business—such as that used to purchase raw materials.

fixed capital. Capital invested permanently—such as that invested in buildings, machinery, etc.

CAP'I-TAL-IZED, *adj.* **capitalized cost.** The sum of the first cost of an asset and the present value of replacements to be made perpetually at the ends of given periods.

CARATHEODORY. **Caratheodory measure.** See MEASURE.

CARDAN. **Cardan's solution of the cubic.** A solution of the *reduced cubic* (see REDUCED—reduced cubic),

$$x^3+ax+b=0,$$

by the substitution $x=u+v$ [$x=u+v$ will be a root of the equation if $u^3+v^3=-b$ and $uv=-\frac{1}{3}a$, or if u^3 is a root of the quadratic equation in u^3,

$$(u^3)^2+b(u^3)-a^3/27=0, \text{ and } uv=-\tfrac{1}{3}a].$$

If u_1 is a cube root of $\frac{1}{2}(-b+\sqrt{b^2-4a^3/27})$, and $v_1=-\frac{1}{3}a/u_1$, then the three roots of the reduced cubic are:

$$z_1=u_1+v_1,\ z_2=\omega u_1+\omega^2 v_1,$$
$$z_3=\omega^2 u_1+\omega v_1,$$

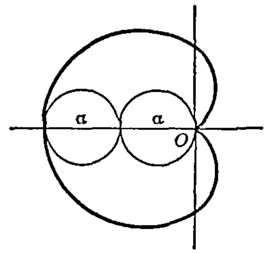
where $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$ is a cube root of unity. This is equivalent to the formula

$$x = [-\tfrac{1}{2}b + \sqrt{R}]^{1/3} + [-\tfrac{1}{2}b - \sqrt{R}]^{1/3},$$

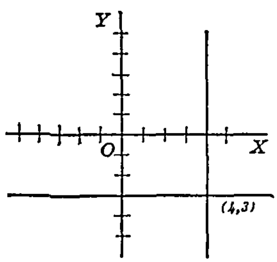
where $R = (\frac{1}{2}b)^2 + a^3/27$ and the cube roots are to be chosen so that their product is $-\frac{1}{3}a$. The number R is negative if and only if the three roots of the cubic are real and distinct, which is called the *irreducible case*, since the formulas (although still correct) involve the cube roots of complex numbers. Ferro discovered this solution, but Cardan first published it.

CAR'DI-NAL, *adj.* **cardinal number.** A number which designates the manyness of a set of things; the number of units, but *not the order in which they are arranged*; used in distinction to signed numbers. *E.g.*, when one says 3 dolls, the 3 is a cardinal number. *Tech.* Two sets are said to have the same cardinal number if their elements can be put into one-to-one correspondence with each other. Thus a symbol or *cardinal number* can be associated with any set. The cardinal number of a set is also called the **potency** of the set and the **power** of the set (*e.g.*, a set whose elements can be put into 1—1 correspondence with the real numbers is said to have the **power of the continuum**). The cardinal number of the set $a_1, a_2, \cdots a_n$ is denoted by n. The cardinal number of all countably infinite sets is called **Aleph-null** or **Aleph-zero** and is designated by $\aleph_0$, and the cardinal number of all real numbers is designated by c. The cardinal number 2^c is the cardinal number of the set of all subsets of the real numbers (*i.e.*, the set of all real-valued functions defined for all real numbers) and is greater than c in the sense that the real numbers can be put into one-to-one correspondence with a subset of the real functions but not conversely. See ORDINAL—ordinal number.

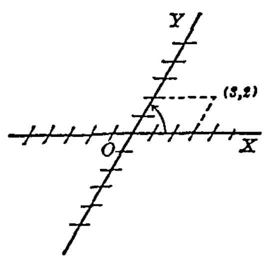
CAR'DI-OID, *n.* The locus (in a plane) of a fixed point on a given circle which rolls on an equal but fixed circle. If a is the radius of the fixed circle, ϕ the vectorial angle, and r the radius vector—where the pole is on the fixed circle and the polar axis is on a diameter of the fixed circle—the polar equation of the cardioid is $r = 2a \sin^2 \frac{1}{2}\phi = a(1 - \cos \phi)$. A *cardioid* is an epicycloid of one loop and a special case of the limaçon.



CAR-TE'SIAN, *adj.* **Cartesian coordinates. In the plane,** a point can be located by its distances from two intersecting straight lines, the distance from one line being measured along a parallel to the other line. The two intersecting lines are called **axes** (x-axis and y-axis), **oblique axes** when they are not perpendicular, and **rectangular axes** when they are perpendicular. The coordinates are then called **oblique coordinates** and **rectangular coordinates,** respectively. The coordinate measured from the y-axis parallel to the x-axis is called the **abscissa** and the other coordinate is called the **ordinate.**



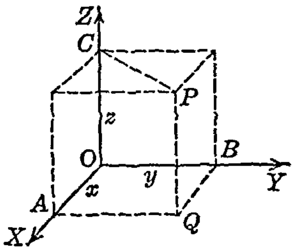
Rectangular Axis



Oblique Axis

In space, three planes (XOY, XOZ, and YOZ in the figure) can be used to locate points by giving their distance from each of the planes along a line parallel to the intersection of the other two. If the planes are mutually perpendicular, these distances are called the **rectangular Cartesian coordinates** of the point in space, or the **rectangular** or **Cartesian coordinates.** The three intersections of these three planes are called the **axes of coordinates** and are usually labeled the x-axis, y-axis, and z-axis. Their common point is called the origin. The three axes are called a **coordinate trihedral** (see TRIHEDRAL). The coordinate planes divide space into eight compartments, called

octants. The octant containing the three positive axes as edges is called the **1st octant** (or **coordinate trihedral).** The other octants are usually numbered 2, 3, 4, 5, 6, 7, 8; 2, 3, and 4 are reckoned counterclockwise around the positive z-axis (or clockwise if the coordinate system is left-handed), then the quadrant vertically beneath the first quadrant is labeled 5, and the remaining quadrants 6, 7, and 8, taken in counterclockwise (or clockwise) order as before.



A rectangular space coordinate is quite commonly thought of as the projection of the line from the origin to the point upon the axis perpendicular to the plane from which the coordinate is measured; *i.e.*, $x=OA$, $y=OB$, and $z=OC$ in the figure.

Cartesian product. See PRODUCT—Cartesian product.

Cartesian space. Same as EUCLIDEAN SPACE.

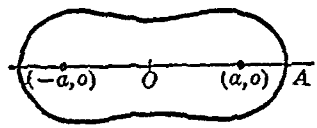
CASH, *n.* Money of any kind; usually coin or paper money, but frequently includes checks, drafts, notes, and other sorts of commercial paper, which are immediately convertible into currency.

cash equivalent of an annuity. Same as PRESENT VALUE. See SURRENDER.

CAS-SIN'I. ovals of Cassini. The locus of the vertex of a triangle when the product of the sides adjacent to the vertex is a constant and the length of the opposite side is fixed. When the constant is equal to one-fourth the square of the fixed side, the curve is called a **lemniscate.** If k^2 denotes the constant and a one-half the length of the fixed side, the Cartesian equation takes the form

$$[(x+a)^2+y^2][(x-a)^2+y^2]=k^4.$$

If k^2 is less than a^2, the curve consists of two distinct ovals; if k^2 is greater than a^2, it consists of one, and if k^2 is equal to a^2, it reduces to the lemniscate. The figure illustrates the case in which $k^2>a^2$.



CAST'ING, *n.* **casting out nines.** A method used to check multiplication (and sometimes division); based on the fact that the *excess of nines* in the product equals the excess in the product of the excesses in the multiplier and multiplicand. See EXCESS. *E.g.*, to check the multiplication $832\times 736=612{,}352$ add the digits in 612,352, subtracting 9 as the sum reaches or exceeds 9. This gives 1. Do the same for 832, and for 736; the results are 4 and 7. Now multiply 4 by 7, getting 28. Then add 2 and 8 and subtract 9. This leaves 1—which is the same excess that was gotten for the product. This method can also be used to check addition (or subtraction), since the *excess of nines* in a sum is equal to the excess in the sum of the excesses of the addends.

CAT'E-GO-RY, *n.* **Baire's category theorem.** The theorem which states that a *complete metric space* is of second category in itself. An equivalent statement is that the intersection of any sequence of dense open sets in a complete metric space is dense. *E.g.*, the space C of all functions which are continuous on the closed interval $[0, 1]$ is a complete metric space if the distance $d(f, g)$ is defined to be the least upper bound of $|f(x)-g(x)|$. The set of all members of C which are differentiable at one or more points of $[0, 1]$ can be shown to be of *first category* in C, so that the set of continuous functions not differentiable at any point of $[0, 1]$ is of *second category.*

Banach's category theorem. The theorem which states that if a set S contained in a topological space T (of type T_1) is of second category in T, then there is a *nonempty open set* U in T such that S is of second category at every point of U. It follows from this theorem that a subset of T is of first category in T if it is of first category at each point of T.

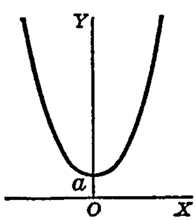
category of sets. A set S is of **first category** in a set T if it can be represented as

a countable union of sets each of which is *nowhere dense* in T. Any set which is not of the first category is said to be of **second category.** A set S is of **first category at the point** x if there is a neighborhood U of x such that the intersection of U and S is of first category. The complement of a set of first category in T is called a **residual set** of T (sometimes residual set is used only for complements of sets of first category in sets T which have the property that every non-empty open set in T is of second category). A subset S of the real line is of the first category if and only if there is a one-to-one transformation of the line onto itself for which S corresponds to a set of *measure zero* which is also an F_σ set (see BOREL—Borel set).

CAT′E-NA-RY, *n.* The plane curve in which a uniform cable hangs when suspended from two points. Its equation in rectangular coordinates is

$$y=(a/2)(e^{x/a}+e^{-x/a}),$$

where a is the y-intercept.



CAT′E-NOID, *n.* The surface of revolution generated by the rotation of a catenary about its axis. The only minimal surface of revolution is the catenoid. See CATENARY.

CAUCHY. Cauchy distribution. $C(x; L, u) = \dfrac{L}{\pi L^2+(x-u)^2}$ is the **Cauchy frequency function.** It is a unimodal, symmetric distribution around the value $x=u$, which is the mode and median, but not the mean since the distribution has no positive finite moments whatsoever. The means of random samples of a Cauchy distribution have the same distribution as the population. Student's t—distribution with one degree of freedom is a Cauchy distribution with $L=1$ and $u=0$.

Cauchy-Hadamard theorem. The theorem states that the radius of convergence of the Taylor series $a_0+a_1z+a_2z^2+\cdots$ in the complex variable z is given by

$$r=\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}}.$$

Cauchy-Riemann partial differential equations. For functions $u=u(x, y)$, $v=v(x, y)$, the **Cauchy-Riemann equations** are $\partial u/\partial x=\partial v/\partial y$ and $\partial u/\partial y=-\partial v/\partial x$. The equations characterize analytic functions $u+iv$ of the complex variable $z=x+iy$, and are satisfied if, and only if, the map is directly *conformal* except at points where all four partial derivatives vanish.

Cauchy sequence. See SEQUENCE—Cauchy sequence.

Cauchy's condensation test for convergence. If Σa_n is a series of positive monotonic decreasing terms and p is any positive integer, then the series $a_1+a_2+a_3+\cdots$ and $pa_p+p^2a_{p^2}+p^3a_{p^3}+\cdots$ are either both convergent or both divergent.

Cauchy's condition for convergence of a sequence. An infinite sequence converges if, and only if, the numerical difference between every two of its terms is as small as desired, provided both terms are sufficiently far out in the sequence. *Tech.* The infinite sequence $s_1, s_2, s_3, \cdots, s_n, \cdots$ converges if, and only if, for every $\epsilon>0$ there exists an N such that

$$|s_{n+h}-s_n|<\epsilon$$

for all $n>N$ and all $h>0$. See SEQUENCE—Cauchy sequence. Same as CAUCHY'S CONDITION FOR CONVERGENCE of a series when s_n is looked upon as the sum to n terms of the series

$$s_1+(s_2-s_1)+(s_3-s_2)+\cdots+(s_n-s_{n-1})+\cdots.$$

Cauchy's condition for convergence of a series. The sum of any number of terms can be made as small as desired by starting sufficiently far out in the series. *Tech.* A necessary and sufficient condition for convergence of a series is that given any $\epsilon>0$, there exists an N, dependent on ϵ, such that

$$|S_{n+h}-S_n|<\epsilon \quad \text{for all} \quad n>N$$

and all $h>0$, where S_n denotes the sum of the first n terms and S_{n+h} the sum of the first $n+h$ terms of the series.

Cauchy's form of the remainder for Taylor's theorem. See TAYLOR'S THEOREM.

Cauchy's inequality. The inequality

$$\left|\sum_1^n a_i b_i\right|^2 \leqq \sum_1^n |a_i|^2 \cdot \sum_1^n |b_i|^2.$$

Also see SCHWARZ—Schwarz's inequality.

Cauchy's integral formula. The formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta,$$

where $f(z)$ is an analytic function of the complex variable z in a finite simply connected domain D, C is a *simple closed rectifiable curve* in D, and z is a point in the finite domain bounded by C. This formula can be extended to the following, for n any positive integer:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta.$$

Cauchy's integral test for convergence of an infinite series. Let $f(x)$ be a function which, for x greater than some positive number, is a monotonically decreasing positive function of x and is such that $f(n)=a_n$ (the nth term of the series) for all n sufficiently large. Then a necessary and sufficient condition for convergence of the series, Σa_n, is that there exists a number a such that

$$\int_a^\infty f(x)\,dx \quad \text{converges.}$$

In the case of the p series,

$$\Sigma 1/n^p, \quad f(x) = 1/x^p,$$

$$\int_1^\infty x^{-p}\,dx = x^{1-p}/(1-p)]_1^\infty \quad \text{if} \quad p \neq 1,$$

$$= \log x]_1^\infty \quad \text{if} \quad p = 1,$$

$$\lim_{x\to\infty} \frac{x^{1-p}}{1-p} = 0 \quad \text{if} \quad p > 1,$$

$$= \infty \quad \text{if} \quad p < 1$$

and

$$\lim_{x\to\infty} \log x = \infty.$$

Hence the p series converges for $p > 1$ and diverges for $p \leqq 1$.

Cauchy's integral theorem. *If* $f(z)$ is analytic in a finite simple connected domain D of the complex plane, and C is a closed rectifiable curve in D, then

$$\int_C f(z)\,dz = 0.$$

Cauchy's mean value formula. See MEAN—second mean value theorem.

Cauchy's radical test for convergence. If the nth root of the nth term of a series of positive terms ultimately becomes and remains less than some number less than unity, the series converges; if it becomes and remains equal to or greater than unity, the series diverges. Consider the series,

$$1 + x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots.$$

Here the nth root of the nth term is $n^{1/n}x$. Since $\lim_{n\to\infty} n^{1/n} = 1$, for any x_0 numerically less than 1, it is possible to choose an N such that $n^{1/n}x_0$ is less than 1 for all $n > N$. Hence the series converges if $|x| < 1$. This radical test serves whenever the ratio test does, but the converse is not true. It follows from the above test that if $\lim_{n\to\infty}(a_n)^{1/n} = r$ for a series Σa_n, then the series converges if $r < 1$ and diverges if $r > 1$. If $r = 1$ no conclusion can be drawn unless $a_n^{1/n} \geqq 1$ for n greater than some properly chosen N, in which case the series diverges.

Cauchy's ratio test. See RATIO—ratio test.

CAUSE, *n.* **chance cause.** See CHANCE.

CAVALIERI'S THEOREM. If two solids have equal altitudes and all plane sections parallel to their bases and at equal distances from their bases are equal, the solids have the same volume.

CE-LES'TIAL, *adj.* Of or pertaining to the skies or heavens.

altitude of a celestial point. See ALTITUDE.

celestial equator. See HOUR—hour angle and hour circle, and EQUATOR.

celestial horizon, meridian, and pole. See HOUR—hour angle and hour circle.

celestial sphere. The conceptual sphere on which all the celestial objects are seen in projection and appear to move.

CELL, *n.* An n-dimensional cell (n-cell) is a set which is homeomorphic either with the set of points $(x_1, \cdots, x_n)$ of n-dimensional Euclidean space for which $\Sigma x_i^2 < 1$, or with the set for which $\Sigma x_i^2 \leqq 1$ (it is an **open n-cell** in the first case, a **closed n-cell** in the other). A 0-cell is a point; a 1-cell is an open or a closed interval or a con-

tinuous deformation of an open or a closed interval. Circles (or simple polygons) and their interiors are examples of closed 2-cells; spheres (or simple polyhedrons) and their interiors are closed 3-cells. A closed n-cell is sometimes called a *solid n-sphere*, or an *n-disk.*

CEN′TER, *n.* **center of attraction.** Same as CENTER OF MASS.

center of a circle. See CIRCLE.

center of curvature of a plane curve. See CURVATURE—curvature of a plane curve.

center of curvature (first curvature) of a space curve at a point. The center of the osculating circle of the curve at the point.

center of a curve. The point (if it exists) about which the curve is symmetrical. Curves such as the hyperbola, which are not closed, but are symmetrical about a given point, are said to have this point as a center, but the term center commonly refers to closed curves such as circles and ellipses. *Syn.* Center of symmetry. See SYMMETRY—symmetry of a geometric configuration.

center of an ellipse. See ELLIPSE.

center of geodesic curvature. See GEODESIC—center of geodesic curvature.

center of gravity. Same as CENTER OF MASS.

center of mass. The point at which a mass (body) can be considered as being concentrated without altering the effect of the attraction that the earth has upon it; the point in a body through which the resultant of the gravitational forces, acting on all its particles, passes regardless of the orientation of the body; the point about which the body is in equilibrium; the point such that the moment about any line is the same as it would be if the body were concentrated at that point. See MOMENT—moment of mass. The center of mass is the point of the body which has the same motion that a particle having the mass of the whole body would have if the resultant of all the forces acting on the body were applied to it. If the body consists of a **set of particles,** the center of mass is the point determined by the vector

$\bar{\mathbf{r}}=\dfrac{\sum_i \mathbf{r}_i m_i}{\sum_i m_i}$, where $\mathbf{r}_i$ is the position vector of the mass m_i in the system of particles $m_1, m_2, \cdots, m_n$. In case of a continuous distribution of mass, the vector $\bar{\mathbf{r}}$ locating the center of mass of a body is given by

$\bar{\mathbf{r}}=\dfrac{\int_s \mathbf{r}\, dm}{\int_s dm}$, where the integration is carried out throughout the space s occupied by the body. The coordinates $\bar{x}, \bar{y}$, and $\bar{z}$ of the center of mass are given by

$$\bar{x}=(1/m)\int_s x\, dm, \quad \bar{y}=(1/m)\int_s y\, dm,$$
$$\bar{z}=(1/m)\int_s z\, dm,$$

where m is the total mass of the body, x, y, z are the coordinates of some point in the element of mass, dm, and $\int_s$ indicates that the integration is to be taken over the entire body, the integration being single, double, or triple depending upon the form of dm; dm may, for instance, be one of the forms: $\rho\, ds$, $\rho x\, dy$, $\rho\, dy\, dx$, $\rho\, dz\, dy\, dx$, where ρ is density. If elements such as $y\, dx$ or $x\, dy$ are used, one must take for the point in the element of mass the approximate center of mass of these strips (elements). Same as CENTER OF ATTRACTION and CENTER OF GRAVITY. See CENTROID.

center of normal curvature of a surface for a given point and direction. The center of curvature of the normal section of the surface S through the point P of S and in the given direction. The coordinates of the center of normal curvature are given by $(x+RX,\ y+RY,\ z+RZ)$, where (x, y, z) are the coordinates of P, (X, Y, Z) are the direction cosines of the normal to S at P, and R is the radius of normal curvature of S at P in the given direction. See RADIUS—radius of normal curvature of a surface, and MEUSNIER'S THEOREM.

center of pressure of a surface submerged in a liquid. That point at which all the force could be applied and produce the same effect as when the force is distributed.

center of a regular polygon. The center of its inscribed and circumscribed circles.

center of a sheaf. See SHEAF—sheaf of planes.

center of similarity (or similitude) of two configurations. See RADIALLY—radially related figures.

centers of principal curvature of a surface at a point. The centers of normal curvature at the point and in the two principal directions. See above, center of normal curvature of a surface, and DIRECTION—principal directions on a surface at a point. *Syn.* Principal centers of curvature of a surface at a point.

center of a quadric surface. The point of symmetry of the surface. See ELLIPSOID, and HYPERBOLOID.

radical center. See RADICAL.

CEN-TES'I-MAL, *adj.* **centesimal system of measuring angles.** The system in which the right angle is divided into 100 equal parts, called *degrees*, a *degree* into 100 *minutes* and a *minute* into 100 *seconds*. Not in common use.

CEN'TI-GRADE, *adj.* **centigrade thermometer.** A thermometer on which 0° and 100°, respectively, indicate the freezing and boiling points of water. See CONVERSION—conversion from centigrade to Fahrenheit.

CEN'TI-GRAM, *n.* One hundredth of a gram. See DENOMINATE NUMBERS in the appendix.

CEN'TI-ME'TER, *n.* One hundredth part of a meter. See DENOMINATE NUMBERS in the appendix.

CEN'TRAL, *adj., n.* **central angle in a circle.** An angle whose sides are radii. An angle with its vertex at the center. See figure under CIRCLE.

central conics. Ellipses and hyperbolas.

central death rate. See DEATH.

central of a group. The set of all elements of the group which commute with every element of the group. The central is an *invariant subgroup*, but may be contained properly in an invariant subgroup. See GROUP.

central limit theorem. (*Statistics.*) Let $x_1, x_2, \cdots, x_n$ be independent random variables. Then whatever the form of their distribution—subject to certain very general conditions—the sum $X=\Sigma x_i$ is asymptotically normally distributed with mean $M=\Sigma m_i$ and variance $V=\Sigma\sigma_i^2$, where m_i and σ_i^2 are the means and variances of the n random variables. If the random variables all have the same distribution function, then the sufficient condition is that the variances be finite, and it follows that the arithmetic mean of the several variables is asymptotically normally distributed with mean equal to the uniform mean of the several distributions and variance equal to σ^2/n. Other extensions as well as the most general conditions are known.

central plane and point of a ruling on a ruled surface. See RULING.

central projection. See PROJECTION.

central quadrics. Quadrics having centers—ellipsoids and hyperboloids.

measures of central tendency. See MEASURE—measures of central tendency.

CEN-TRIF'U-GAL, *adj.* **centrifugal force.** (1) The force which a mass, constrained to move in a path, exerts on the constraint in a direction along the radius of curvature. (2) A particle of mass m, rotating with the angular velocity ω about a point O at a distance r from the particle, is subjected to a force, called **centrifugal force,** of magnitude $m\omega^2 r$ (or mv^2/r, where v is the speed of the particle relative to O). The direction of this force on the particle is away from the center of rotation. The equal and oppositely directed force is called **centripetal force.**

CEN-TRIP'E-TAL, *adj.* **centripetal acceleration.** See ACCELERATION.

centripetal force. The force which restrains a body, in motion, from going in a straight line. It is directed toward the center of curvature. A force equal, but opposite in sign, to the **centrifugal force.**

CEN'TROID, *n.* **centroid of a configuration.** The point whose coordinates are the *mean values* of the coordinates of the points in the configuration. This means, for configurations over which integration can be performed, that the coordinates of the centroid, $\bar{x}$, $\bar{y}$, and $\bar{z}$, are given by

$$\bar{x}=\left[\int_s x\,ds\right]/s,$$

$$\bar{y}=\left[\int_s y\,ds\right]/s$$

and

$$\bar{z}=\left[\int_s z\,ds\right]/s;$$

where $\int_s$ denotes the integral over the configuration, ds denotes an element of area, arc length, or volume, and s denotes the area, arc length, or volume of the configuration. The centroid is the same as the center of mass if the object is regarded as having constant density (constant mass per unit area, length, or volume). Also, $\bar{x}$ may be looked upon as the mean of the x coordinates of all points in the configuration, and similarly for $\bar{y}$ and $\bar{z}$. See CENTER—center of mass, INTEGRAL—definite integral, and MEAN—mean value of a function over a given range.

CER'TAIN, *adj.* **annuity certain.** See ANNUITY.

CESÀRO'S SUMMATION FORMULA. A specific method of attributing a sum to certain divergent series. A sequence of partial sums, (S_n), where $S_n = \sum_{i=0}^{n} a_i$, is replaced by the sequence $(S_n^{(k)}/A_n^{(k)})$, where

$$S_n^{(k)} = \binom{n+k-1}{n} S_0 + \binom{n+k-2}{n-1} S_1 + \cdots + S_n$$

and

$$A_n^{(k)} = \binom{k+n}{n} = \sum_{i=0}^{n} \binom{n+k-1-i}{n-i}$$

$\binom{n}{r}$ being the rth binomial coefficient of order n. If the sequence, $(S_n^{(k)}/A_n^{(k)})$, has a limit, the series Σa_n is said to be summable C_k, or (C, k), to this limit. In terms of the a_i of the original series,

$$S_n^{(k)}/A_n^{(k)} = a_0 + \frac{n}{k+n} a_1 + \frac{n(n-1)}{(k+n-1)(k+n)} a_2 + \cdots + \frac{n!}{(k+1)(k+2)\cdots(k+n)} a_n$$

Cesàro's summation formula is *regular.* See SUMMATION—summation of divergent series.

CHAIN, *adj., n.* (1) A simply ordered set. See ORDERED, and NESTED—nested sets. (2) See below, chain of simplexes, etc.

chain discounts. See DISCOUNT—discount series.

chain of simplexes. Let G be a commutative group with the group operation indicated as addition. Let $S_1^r, S_2^r, \cdots, S_n^r$ be oriented *r-dimensional simplexes* of a *simplicial complex K.* Then

$$x = g_1 S_1^r + g_2 S_2^r + \cdots + g_n S_n^r$$

is an **r-dimensional chain,** or an **r-chain.** It is understood that if $^*S^r$ is the simplex S^r with its orientation changed, then $g(^*S^r) = (-g)S^r$ for any g of G. The set of all r-chains is a group if chains are added in the natural way, *i.e.,* by adding coefficients of each oriented simplex. The group G is usually taken as either the group I of integers or one of the finite groups I_n of integers modulo an integer n. Of the latter, the group I_2 of integers modulo 2 is especially useful. If G is one of these groups of integers, then the **boundary of an r-simplex** S^r is defined to be the $(r-1)$-chain

$$\Delta(S^r) = \epsilon_0 B_0^{r-1} + \epsilon_1 B_1^{r-1} + \cdots + \epsilon_n B_n^{r-1},$$

where $B_0^{r-1}, \cdots, B_n^{r-1}$ is the set of all $(r-1)$-dimensional faces of S^r and ϵ_k is $+1$ or -1 according as S^r and B_k^r are *coherently oriented* or *noncoherently oriented.* If $r=0$, the boundary ΔS^0 is defined to be 0. The **boundary of the chain x** is defined to be

$$\Delta(x) = g_1 \Delta S_1^r + g_2 \Delta S_2^r + \cdots + g_n \Delta S_n^r.$$

It follows that the boundary of a boundary is 0, *i.e.,* $\Delta(\Delta x) = 0$ for x any chain. A chain whose boundary is 0 is called a **cycle** (any boundary is a cycle). *E.g.,* a chain of "edges" $S_1^1, S_2^1, \cdots, S_n^1$ is a cycle if the "edges" are joined so as to form a closed oriented path. See HOMOLOGY—homology group.

chain rule. For **ordinary differentiation,** the rule of differentiation which states that, if $F(u)$ is a function of u and u is a function of x, then

$$D_x F(u) = D_u F(u) \cdot D_x u.$$

This rule can be used repeatedly, *e.g.,* as $D_x u[v(w)] = D_v u \cdot D_w v \cdot D_x w$, or along with other differentiation formulas (see DIFFERENTIATION FORMULAS in the appendix, and DERIVATIVE) in an explicit differentiation, such as

$$D_x[(x^2+1)^3+3]^2 = 2[(x^2+1)^3+3] \cdot 3(x^2+1)^2 \cdot 2x.$$

The chain rule can also be used to change variables; *e.g.*, if y is replaced by $z=1/y$ in the differential equation $D_x y+y^2=0$, one uses the formula

$$D_x z=D_y z\cdot D_x y=(-1/y^2)D_x y$$

to obtain $-y^2D_x z+y^2=0$, or $D_x z=1$. Now let F be a function of one or more variables $u_1, u_2, \cdots, u_n$ and each of these variables be a function of one or more variables $x_1, x_2, \cdots$. The **chain rule for partial differentiation** is

$$\frac{\partial F}{\partial x_p}=\sum_{i=1}^{n}\frac{\partial F}{\partial u_i}\frac{\partial u_i}{\partial x_p}.$$

If each of the variables $u_1, u_2, \cdots, u_n$ is a function of one variable x, then this formula becomes

$$\frac{dF}{dx}=\sum_{i=1}^{n}\frac{\partial F}{\partial u_i}\frac{du_i}{dx}.$$

This is called the **total derivative** of F with respect to x. *E.g.*, if $z=f(x, y)$, $x=\phi(t)$ and $y=\theta(t)$, the total derivative of z with reference to t is given by

$$\frac{dz}{dt}=f_x(x, y)\phi'(t)+f_y(x, y)\theta'(t).$$

ϵ-chain. See EPSILON—epsilon chain.

surveyor's chain. A chain 66 feet long containing 100 links, each link 7.92 inches long. Ten square chains equal one acre. See DENOMINATE NUMBERS in the appendix.

CHANCE, *adj., n.* Same as PROBABILITY. Has considerable popular, but little technical, usage.

chance cause. If, in a given set of causes, each cause is associated with probability less than one and greater than zero of being able to operate, then the set of causes is a set of **chance causes.**

chance variable. Any variable that may assume each of its possible values x_i with definite probability (not necessarily the same for all values). In general, any variable which may have a probability function is a chance variable, even though the frequency function is not known. Same as RANDOM VARIABLE or STOCHASTIC VARIABLE.

CHANGE, *n.* **change of base in logarithms.** See BASE—change of base in logarithms.

change of coordinates. See TRANSFORMATION—transformation of coordinates.

change of variable in integration. See INTEGRATION—integration by substitution.

cyclic change of variables. Same as CYCLIC PERMUTATION. See PERMUTATION.

CHAR'AC-TER, *n.* **finite character.** A **collection A of sets is of finite character** if A contains any set all of whose finite subsets belong to A and each finite subset of a member of A belongs to A. **A property of subsets of a set is of finite character** if a subset S has the property if and only if each nonempty finite subset of S has the property. *E.g.*, the property of being *simply ordered* is of finite character, while the property of being *well-ordered* is not. If a property is of finite character, then the collection of all sets with this property is of finite character. If a collection A of sets is of finite character, then the property of belonging to A is of finite character. See ZORN–Zorn's lemma.

group character. A *character* of a group G is a homomorphism of G into the group of complex numbers of absolute value 1; *i.e.*, it is a continuous function $f(x)$ defined on G for which $f(x)$ is a complex number with $|f(x)|=1$ and $f(x)f(y)=f(x\cdot y)$ for all x and y of G (the group operation of G is indicated here by multiplication). The set of all characters of G is called a **character group,** the "product" of characters f and g being defined to be the character h defined by $h(x)=f(x)g(x)$ for each x of G. If G is commutative and locally (bi)compact, then G is algebraically isomorphic with the character group of its character group. The character group can be given a topology by defining neighborhoods of a point so that U is a neighborhood of a character f if there are elements $x_1, \cdots, x_n$ of G and a positive number ϵ such that U is the set of all characters G for which

$$|f(x_k)-g(x_k)|<\epsilon \quad \text{for} \quad k=1, \cdots, n.$$

It then follows that the character group is a topological group and is locally compact if G is locally compact; it is discrete if G is compact. If G is the group of translations of the real line, then the character group of G is isomorphic with G.

CHAR'AC-TER-IS'TIC, *adj., n.* **characteristic curves of a surface.** That conjugate

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system of curves on a surface S such that the directions of the tangents of the two curves of the system through any point P of S are the characteristic directions at P on S See CONJUGATE—conjugate system of curves on a surface, and CHARACTERISTIC—characteristic directions on a surface. The characteristic curves are parametric if and only if $D:D''=E:G$ and $D'=0$. See FUNDAMENTAL—fundamental coefficients of a surface.

characteristic directions on a surface. The pair of conjugate directions on a surface S at a point P of S which are symmetric with respect to the directions of the lines of curvature on S through P. The characteristic directions on S at P are unique except at umbilical points, and are the directions which minimize the angle between pairs of conjugate directions on S at P.

characteristic equation of a matrix. Let I be the unit matrix of the same order as the square matrix A. If $d(xI-A)$ is the determinant of the matrix $xI-A$, then $d(xI-A)=0$ is the characteristic equation of A and $d(xI-A)$ is the **characteristic function** of A. Thus the characteristic equation of the matrix $M=\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}$ is the equation $\begin{vmatrix}x-2 & -1\\ -2 & x-3\end{vmatrix}=0$, or $x^2-5x+4=0$. It is known that every matrix satisfies its characteristic equation (the Hamilton-Cayley Theorem). *E.g.*, the matrix $M\cdot M-5M+4I$ is zero. The **reduced characteristic equation** of a matrix is the equation of lowest degree which is satisfied by the matrix. If A is a matrix of order n and I is the unit matrix of order n, then the reduced characteristic equation is $f(x)/g(x)=0$, where $f(x)$ is the determinant of the matrix $xI-A$ and $g(x)$ is the greatest common divisor of the $(n-1)$-rowed minor determinants of $xI-A$. The **reduced characteristic function** is $f(x)/g(x)$. The reduced characteristic equation is also called the **minimal** (or **minimum**) **equation,** and the matrix **derogatory** if its order is greater than that of its reduced characteristic equation. See EIGENVALUE—eigenvalue of a matrix.

characteristic function. (*Statistics.*) The characteristic function of a random variable with frequency function $f(x)$ is $\phi(t)\equiv\int_{-\infty}^{\infty} e^{itx}f(x)\,dx$, where t is a real number. Let $\phi^n(t)$ be the nth derivative of $\phi(t)$. $i^{-n}\phi^n(0)$ will give the nth moment of x if the nth moment exists. If x is replaced by cx, where c is a constant, the characteristic function of x, $\phi(t)$, becomes $\phi(ct)$. If the x_i are independently distributed with characteristic functions $f_i(x_i)$, $i=1,\cdots n$, then the characteristic function of $x_1+x_2+\cdots+x_n$ is $\prod^n f_i(x_i)$. From the characteristic functions of individual variables, it is possible to obtain the distribution of the sum of the individual component variables. The usefulness of this follows from the fact that the logarithm of the characteristic functions may be set equal to an infinite series, the terms of which involve powers of ("*it*") and the semi-invariants of the distribution. Simply adding the various infinite series, one for each distribution, gives the logarithm of the characteristic function of the composite variable—and thence the moments and distribution of the composite variables. See CUMULANTS and SEMI—semi-invariants.

characteristic function of a matrix. See above, characteristic equation of a matrix.

characteristic function of a set. The function $f(x)$ which is such that $f(x)=1$ for each point x in the set, and $f(x)=0$ if x is not in the set.

characteristic of the logarithm of a number. See LOGARITHM—characteristic and mantissa of a logarithm.

characteristic number of a matrix. Same as CHARACTERISTIC ROOT. See below.

characteristic numbers and functions in the study of integral equations. Same as EIGENVALUES and EIGENFUNCTIONS.

characteristic of a one-parameter family of surfaces. The limiting curve of intersection of two neighboring members of the family as they approach coincidence—*i.e.*, as the two values of the parameter determining the two members of the family of surfaces approach a common value. The equations of a given characteristic curve are the equation of the family taken with the partial derivative of this equation with respect to the parameter, each equation being evaluated for a particular value of the parameter. The locus of the characteristic curves, as the parameter varies,

is the envelope of the family of surfaces. *E.g.*, if the family of surfaces consists of all spheres of a given, fixed radius with their centers on a given line, the characteristic curves are circles having their centers on the line, and the envelope is the cylinder generated by these circles.

characteristic root of a matrix. A root of the characteristic equation of the matrix. *Syn.* Characteristic number, latent root. See EIGENVALUE—eigenvalue of a matrix.

Euler characteristic. See EULER—Euler characteristic.

Segre characteristic of a matrix. See CANONICAL—canonical form of a matrix.

CHARGE, *n.* **Coulomb's law for point-charges.** See COULOMB.

density of charge. See various headings under DENSITY.

electrostatic unit of charge. See ELECTROSTATIC.

point-charge. A point endowed with electrical charge. The electrical counterpart of point-mass or particle, *i.e.*, electrical charge considered as concentrated at a point.

set (or complex) of point-charges. A collection of charges located at definite points of space. Sometimes the term *complex* carries the connotation that the maximum distance between the various pairs of charges is small in comparison to the distance to the field-points at which the electrical effects are to be determined.

surrender charge. See SURRENDER.

CHARLIER CHECK. A check on computational accuracy where powers of observed values are involved. Rests on the following type of relationship:

$$\sum_{i=1}^{k} n_i(x_i+1)^2 = \sum_{i=1}^{k} n_i x_i^2 + 2\sum_{i=1}^{k} n_i x_i + \sum_{i=1}^{k} n_i,$$

where n_i is the frequency of observed values, x_i. May be used for higher powers by appropriate expansions.

CHART, *n.* **flow chart.** In machine computation, a diagram with labeled boxes, arrows, etc., showing the logical pattern of a problem, but not ordinarily including machine-language instructors and commands. See CODING, and PROGRAMMING—programming for a computing machine.

CHECK, *n.* A draft upon a bank, usually drawn by an individual. *v.* To verify by repetition or some other device.

Charlier check. See CHARLIER CHECK.

check on a solution of an equation. Any process used to increase the probability of correctness of a solution. One such method is the substitution of the calculated root in the original equation (the equation before any changes have been made in it, such as squaring, transposing terms, etc.). If the root is correct, the result of this substitution should be an identity which takes the form $0=0$ after all terms have been transposed to the same side and combined.

CHINESE. Chinese remainder theorem. The theorem states that numbers x exist which satisfy the congruences $x=b_i$ (mod m_i), $i=1, 2, 3, \cdots n$, if the m_i are relatively prime in pairs and constitute a single number class modulo $\prod_1^n m_i$.

CHI-SQUARE. $\chi^2=\sum_{i=1}^{k} x_i^2$, where the x_i are independently and normally distributed with mean of zero and variance of 1. The sampling frequency function of χ^2 is

$$f(\chi^2)=\frac{(\chi^2)^{(n-2)/2}e^{-(x^2/2)}}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}$$

with n degrees of freedom. Discovered by Helmert in 1876. For $n>30$, $\sqrt{2\chi^2}$ is distributed approximately normally with mean $\sqrt{2n-1}$, and with unit variance. If χ_i^2 $(i=1, \cdots, k)$ are independently distributed with $n_1, \cdots, n_k$ degrees of freedom, $\Sigma\chi_i^2$ is distributed as χ^2 with $\sum_{i=1}^{k} n_i$ degrees of freedom. For any independently and normally distributed variables with means u_i and variance σ_i^2,

$$\sum_{j=1}^{n_i}\sum_{i=1}^{k}\frac{(x_{ij}-u_i)^2}{\sigma_i^2}=\chi^2$$

with $\sum_{i=1}^{k} n_i$ degrees of freedom, if the u_i and σ_i^2 are known. If these latter are not known, and instead are estimated from the $\sum_{i=1}^{k} n_i$ observations, one degree of freedom

is lost for each parameter estimated. Widely used for testing statistical hypotheses about (1) independence between an observed frequency and a hypothesis, or between two or more observed frequencies and the hypothetical frequencies, as in contingency tables or in problems of the goodness of fit of fitted frequency distributions; (2) the variance of a normal distribution on the basis of observed sample variances; (3) the ratio of two variances. Also used in combining probabilities drawn from a number of independent random samples. See COCHRAN'S THEOREM, and DISTRIBUTION —F distribution

chi-square test. A test of compatibility of observed and expected frequencies, based on the quantity

$$\chi^2=\sum_{i=1}^{k}\frac{(n_i-e_i)^2}{e_i}$$

where k is the number of frequencies, n_i and e_i are the *ith* pair of observed and expected frequencies, and $\Sigma n_i=\Sigma e_i=N$. If N is sufficiently large, the frequency function of χ^2 is approximately that of the chi-square frequency function discussed above, with $n=k-1$. See CHI-SQUARE.

CHOICE, *n.* An alternative elected by one of the players, or determined by a random device, for a move in the play of a game. See GAME, MOVE, and PLAY.

axiom of choice. Given any collection of sets, there exists a "method" of designating a particular element of each set as a "special" element of that set; for any collection A of sets there exists a single-valued function f such that $f(S)$ is an element of S for each set S of the collection A. See ORDERED—well-ordered set, ZORN—Zorn's lemma. *Syn.* Zermelo's axiom.

finite axiom of choice. The *axiom of choice* for the special case that the collection of sets is a finite collection.

CHORD, *n.* A chord of any curve (or surface) is a segment of a straight line between two designated points of intersection of the line and the curve (surface). See CIRCLE, SPHERE, etc.

chord of a circle, sphere, etc. See the specific configuration.

chord of contact with reference to a point outside of a circle. The chord joining the points of contact of the tangents to the circle from the given point.

focal chords of conics. See FOCAL.

supplemental or supplementary chords in a circle. See SUPPLEMENTAL.

CHRISTOFFEL. Christoffel symbols. Certain symbols representing particular functions of the coefficients, and of the first-order derivatives of the coefficients, of a quadratic differential form. The differential form is usually the first fundamental quadratic differential form of a surface. For the quadratic differential form $g_{11}\,dx_1^2+2g_{12}\,dx_1\,dx_2+g_{22}\,dx_2^2$, the **Christoffel symbols of the first kind** are $\begin{bmatrix} i\,j \\ k \end{bmatrix}=\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_j}+\frac{\partial g_{jk}}{\partial x_i}-\frac{\partial g_{ij}}{\partial x_k}\right)$, where i, j, $k=1$, 2, separately. For a quadratic form in n variables, we have $i, j, k=1, 2, \cdots, n$, separately. The symbol $\begin{bmatrix} i\,j \\ k \end{bmatrix}$ is sometimes replaced by $[ij, k]$, $C_{ij}{}^k$, or Γ_{ijk}. The symbols are symmetric in i and j. For the quadratic differential form

$$g_{11}\,dx_1^2+2g_{12}\,dx_1\,dx_2+g_{22}\,dx_2^2,$$

the **Christoffel symbols of the second kind** are $\begin{Bmatrix} i\,j \\ k \end{Bmatrix}=g^{k1}\begin{bmatrix} i\,j \\ 1 \end{bmatrix}+g^{k2}\begin{bmatrix} i\,j \\ 2 \end{bmatrix}$, where i, j, $k=1$, 2, separately, g^{ki} is the cofactor of g_{ki} in the determinant $\Delta=\begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix}$ divided by Δ, and $\begin{bmatrix} i\,j \\ k \end{bmatrix}$ are the Christoffel symbols of the first kind. The symbol $\begin{Bmatrix} i\,j \\ k \end{Bmatrix}$ is sometimes replaced by $\begin{Bmatrix} k \\ i\,j \end{Bmatrix}$ in keeping with the summation convention, or by $\Gamma_{ij}{}^k$. The symbols are symmetric in i and j. For the quadratic differential form $g_{ij}\,dx^i\,dx^j$ in n variables $x^1, x^2, \cdots x^n$ (where the *summation convention* applies), the **Christoffel symbols of the first kind** are $\begin{bmatrix} i\,j \\ k \end{bmatrix}=\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j}+\frac{\partial g_{ji}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^i}\right)$, it being assumed that $g_{ij}=g_{ji}$ for all i and j. The **Christoffel symbols of the second kind** are $\begin{Bmatrix} k \\ i\,j \end{Bmatrix}=g^{k\sigma}\begin{bmatrix} i\,j \\ \sigma \end{bmatrix}$, where g is the determinant having g_{ji} in the ith row and jth column and $g^{ij}=[\text{cofactor of } g_{ji}]/g$. Neither type of Christoffel symbols are *tensors*. The law connecting Christoffel

symbols of the second kind in two systems of coordinates x^i and $\bar{x}^i$ is

$$\overline{\left\{ {i \atop j\,k} \right\}} = \left\{ {\lambda \atop \mu\,\nu} \right\} \frac{\partial x^\mu}{\partial \bar{x}^j}\frac{\partial x^\nu}{\partial \bar{x}^k}\frac{\partial \bar{x}^i}{\partial x^\lambda} + \frac{\partial^2 x^\lambda}{\partial \bar{x}^j \partial \bar{x}^k}\frac{\partial \bar{x}^i}{\partial x^\lambda},$$

where $\left\{ {i \atop j\,k} \right\}$ and $\overline{\left\{ {i \atop j\,k} \right\}}$ are the Christoffel symbols of the second kind in the x^i and $\bar{x}^i$ coordinate systems, respectively. See below, Euclidean Christoffel symbols.

Euclidean Christoffel symbols. Christoffel symbols in an Euclidean space (*i.e.*, where rectangular Cartesian coordinates $y_1, y_2, \cdots, y_n$ exist such that the element of arc length ds is given by $ds^2 = \Sigma\, dy_i^2$). The Euclidean symbols of the second kind are all identically zero in rectangular Cartesian coordinates. However, the Euclidean Christoffel symbols are not all zero in general coordinates and are given by the alternative expression

$$\left\{ {i \atop j\,k} \right\} = \frac{\partial^2 y^\lambda}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial y^\lambda},$$

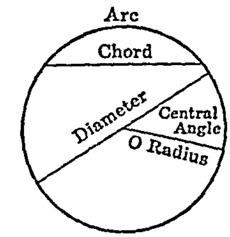
in terms of the transformation functions and their inverses taking the rectangular Cartesian coordinates y^i into the general coordinates x^i. Since the Euclidean Christoffel symbols of the second kind are all identically zero in rectangular Cartesian coordinates, it follows that *covariant differentiation* is ordinary partial differentiation in rectangular Cartesian coordinates. Hence successive covariant differentiation in Euclidean space is a commutative operation even in general coordinates so long as partial differentiation is commutative.

Riemann-Christoffel curvature tensor. See RIEMANN.

CI'PHER (or **CYPHER**), *n.* The symbol 0, denoting *zero.* *Syn.* Zero, naught.

CI'PHER, *v.* To compute with numbers; to carry out one or more of the fundamental operations of arithmetic.

CIR'CLE, *n.* (1) A plane curve consisting of all points at a given distance (called the **radius**) from a fixed point in the plane, called the **center.** (2) A region of a plane all points of whose boundary are at a given distance (called the **radius**) from a fixed point in the plane, called the **center.** See below, equation of a circle in the plane.



arc of circle. A curve which is one of the two pieces of the circle which are bounded by two given points.

area of a circle. Pi times the square of the radius, πr^2, or in terms of the diameter, $\frac{1}{4}\pi d^2$. See PI.

chord of a circle. A segment cut off on a secant by the circumference. See figure under CIRCLE.

circle of convergence. See CONVERGENCE—circle of convergence of a power series.

circle of curvature of a plane curve. See CURVATURE—curvature of a plane curve.

circle of curvature of a space curve. Same as the OSCULATING CIRCLE. See OSCULATING.

circumference of a circle. A term used to emphasize the fact that one is concerned with the curve itself and not with its radius, diameter, or what not. Used also for the length of a circle. The circumference of a circle is equal to $2\pi r$, where r is the radius and π is $3.1416-$. See PI.

circumscribed circle. See CIRCUMSCRIBED.

diameter of a circle. Twice the radius; the segment, intercepted by the circle, on any straight line passing through the center of the circle. See figure under CIRCLE.

eccentric circles of an ellipse. See ELLIPSE—parametric equations of an ellipse.

equation of a circle in the plane. *In rectangular Cartesian coordinates*, $(x-h)^2 + (y-k)^2 = r^2$, where r is the radius of the circle and the center is at the point (h, k). When the center is at the origin, this becomes: $x^2 + y^2 = r^2$. (See DISTANCE—distance between two points.) *In polar coordinates*, the equation is

$$\rho^2 + \rho_1^2 - 2\rho\rho_1 \cos(\phi - \phi_1) = r^2,$$

where ρ is the radius vector, ϕ the vectorial angle, (ρ_1, ϕ_1) the polar coordinates of the center, and r the radius. When the center of the circle is at the origin this equation becomes $\rho = r$. **The parametric equations**

of a circle are: $x=a \cos \theta$, $y=a \sin \theta$, where θ is the angle between the positive x-axis and the radius from the origin to the given point, and a is the radius of the circle.

equations of a circle in space. The equations of any two surfaces whose intersection is the circle; a sphere and a plane, each containing the circle, would suffice.

escribed circle. See ESCRIBED.

family of circles. All the circles whose equation can be obtained by assigning particular values to an essential constant in the equation of a circle. *E.g.*, $x^2+y^2=r^2$ is the family of circles with their centers at the origin, r being the essential constant in this case. See PENCIL—pencil of circles.

great circle. A section of a sphere by a plane which passes through its center; a circle (on a sphere) which has its diameter equal to that of the sphere.

hour circle of a celestial point. The great circle on the celestial sphere that passes through the point and the north and south celestial poles. See HOUR—hour angle and hour circle.

imaginary circle. The name given to the set of points which satisfy the equation $x^2+y^2=-r^2$, or $(x-h)^2+(y-k)^2=-r^2$, $r \neq 0$. Both coordinates of such a point can not be real. Although no points in the real plane have such coordinates, this terminology is desirable because of the algebraic properties common to these imaginary coordinates and the real coordinates of points on real circles.

inscribed circle. See INSCRIBED.

nine-point circle. The circle through the midpoints of the sides of a triangle, the feet of the perpendiculars from the vertices upon the sides, and the midpoints of the line segments between the vertices and the point of intersection of the altitudes.

null circle. A circle with radius zero; the point (0, 0), which is the locus of (the only point that satisfies) $x^2+y^2=0$; or, in general, (h, k), which is the only point that satisfies $(x-h)^2+(y-k)^2=0$.

osculating circle. See OSCULATING.

parallel circle. See SURFACE—surface of revolution.

pencil of circles. See PENCIL.

quadrature of a circle. See QUADRATURE—quadrature of a circle.

radius of a circle. The distance from the center to the circumference (see figure under CIRCLE). Also, any line segment from the center to the circumference.

secant of a circle. A line of unlimited length cutting the circumference.

small circle. A section of a sphere by a plane that does not pass through the center of the sphere.

squaring a circle. See QUADRATURE.

unit circle. See UNIT—unit circle.

CIR′CUIT, *n.* **flip-flop circuit.** In a computing machine, any bistable circuit that remains in either of its two stable states until receiving a signal changing it to the other state. Such a circuit often involves a characteristic configuration of vacuum tubes.

CIR′CU-LANT, *n.* A determinant in which the elements of each row are the elements of the previous row slid one place to the right (and the last element put first). The elements of the main diagonal are all identical.

CIR′CU-LAR, *adj.* **circular cone and cylinder.** See CONE and CYLINDER.

circular functions. The trigonometric functions.

circular (or cyclic) permutation. See PERMUTATION.

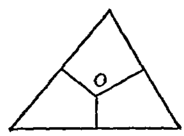
circular point of a surface. An elliptic point of the surface at which $D=kE$, $D'=kF$, $D''=kG$, $k \neq 0$. See SURFACE—fundamental coefficients of a surface, and ELLIPTIC—elliptic point of a surface. For a circular point, the principal radii of normal curvature are equal, and the *Dupin indicatrix* is a circle. A surface is a sphere if and only if all its points are circular points. The points where an ellipsoid of revolution cuts its axis of revolution are circular points. See PLANAR—planar point of a surface, and UMBILICAL—umbilical point of a surface.

uniform circular motion. See UNIFORM.

CIR′CU-LAT′ING, *adj.* **circulating decimal.** See DECIMAL—repeating decimal.

CIR′CUM-CEN′TER, *n.* **circumcenter of a triangle.** The center of the circumscribed circle (the circle passing through the three

vertices of the triangle); the point of intersection of the perpendicular bisectors of the sides; point O in the figure.



CIR′CUM-CIR′CLE, *n.* Same as CIRCUMSCRIBED CIRCLE.

CIR-CUM′FER-ENCE, *n.* (1) The boundary line of a circle; (2) the boundary line of any closed curvilinear figure. *Syn.* Periphery, perimeter.

circumference of a sphere. The circumference of any great circle on the sphere.

CIR′CUM-SCRIBED′, *adj.* A configuration composed of lines, curves, or surfaces, is said to be **circumscribed** about a polygon (or polyhedron) if every vertex of the latter is incident upon the former and the polygon (or polyhedron) is contained in the configuration. A polygon (or polyhedron) is circumscribed about a configuration if every side of the polygon (or face of the polyhedron) is tangent to the configuration and the configuration is contained in the polygon (or polyhedron). If one figure is circumscribed about another, the latter is said to be **inscribed** in the former. In particular, the **circumscribed circle of a polygon** is a circle which passes through the vertices of the polygon. The polygon is then an **inscribed polygon of the circle.** If the polygon is regular and s is the length of a side and n the number of sides, the radius of the circle is

$$r=\frac{s}{2}\csc\frac{180°}{n}.$$

If the polygon is a triangle with sides a, b, c and $s=\frac{1}{2}(a+b+c)$, then

$$r=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

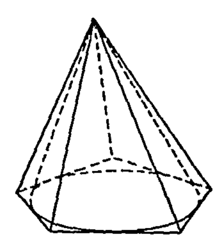
If the polygon is *regular* and has n sides, its area is $\frac{1}{2}r^2n\sin 360°/n$ and its perimeter is $2rn\sin 180°/n$, where r is the radius of the circumscribed circle. A **circumscribed polygon of a circle** is a polygon which has its sides tangent to the circle. The circle is then an **inscribed circle of the polygon** (see INSCRIBED—inscribed circle of a triangle). If the polygon is regular, its area is

$$nr^2\tan\frac{180°}{n},$$

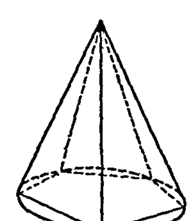
and its perimeter is

$$2nr\tan\frac{180°}{n},$$

where r is the radius of the inscribed circle and n the number of sides of the polygon. If s is the length of a side of the polygon and n the number of its sides, the radius is $\frac{1}{2}s\cot 180°/n$. A **circumscribed sphere** of a polyhedron is a sphere which passes through all the vertices of the polyhedron (the polyhedron is then said to be *inscribed* in the sphere). An **inscribed sphere** of a polyhedron is a sphere which is tangent to all the faces of the polyhedron (the polyhedron is then said to be *circumscribed* about the sphere). A **circumscribed pyramid of a cone** is a pyramid having its base circumscribed about the base of the cone and its vertex coincident with that of the cone. The cone is then an **inscribed cone of the pyramid.**



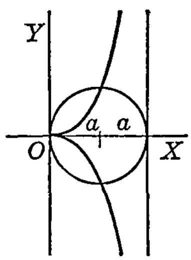
A circumscribed cone of a pyramid is a cone whose base is circumscribed about the base of the pyramid and whose vertex coincides with the vertex of the cone. The pyramid is then an inscribed pyramid of the cone.



A **circumscribed prism of a cylinder** is a prism whose bases are coplanar with, and

circumscribed about, the bases of the cylinder. The lateral faces of the prism are then tangent to the cylindrical surface and the cylinder is an **inscribed cylinder of the prism.** A **circumscribed cylinder of a prism** is a cylinder whose bases are coplanar with, and circumscribed about, the bases of the cylinder. The lateral edges of the prism are then elements of the cylinder and the prism is an **inscribed prism of the cylinder.**

CIS′SOID (cissoid of Diocles), *n.* The plane locus of a variable point on a variable line passing through a fixed point on a circle, where the distance of the variable point from the fixed point is equal to the distance from the line's intersection with the circle to its intersection with a fixed tangent to the circle at the extremity of the diameter through the fixed point; the locus of the foot of the perpendicular from the vertex of a parabola to a variable tangent.



If a is taken as the radius of the circle in the first definition, the polar equation of the cissoid is $r=2a \tan \phi \sin \phi$; its Cartesian equation is $y^2(2a-x)=x^3$. The curve has a cusp of the first kind at the origin, the x-axis being the double tangent. The cissoid was first studied by Diocles about 200 B.C., who gave it the name "Cissoid" (meaning *ivy*).

CIV′IL, *adj.* **civil year.** See YEAR.

CLAIRAUT. Clairaut's differential equation. A differential equation which is of the form $y=xy'+f(y')$ for some function $f(x)$. The *general solution* is $y=cx+f(c)$, and a *singular solution* is given by the parametric equations $y=-pf'(p)+f(p)$ and $x=-f'(p)$.

CLASS, *n.* **class frequency.** The frequency with which a variable assumes the set of values included in a given class interval.

class interval. (*Statistics.*) A grouping of the possible values of a variable. Thus the variables, which may be continuous from 0 to 100, may be grouped arbitrarily into (class) intervals ten units wide from 0 up to 10, 10 and up to 20, etc. The width of the class is sometimes called the *class interval.*

class limits. The upper and lower limits of the values of a class interval. Also called *class bounds.*

class mark. The value or name given to a particular class interval. Often the midvalue, or the integral value nearest the midpoint.

class of a plane algebraic curve. The greatest number of tangents that can be drawn to it from any point in the plane not on the curve.

equivalence class. See EQUIVALENCE.

subclass. Same as SUBSET. See SET and NUMBER.

CLOCK′WISE, *adj.* In the same direction of rotation as that in which the hands of a clock move around the dial.

counterclockwise. In the direction of rotation opposite to that in which the hands of a clock move around the dial.

CLOSED, *adj.* **closed curve.** A curve which has no end points. A section of the curve which completely incloses a section of a plane or surface is called a loop of the curve. *Tech.* A set of points which is the image of a circle under a continuous transformation. See CONTINUOUS—continuous correspondence of points, and SIMPLE—simple closed curve.

closed interval. See INTERVAL.

closed mapping or transformation. (1) See OPEN—open mapping. (2) A **linear transformation** T **is said to be closed** if it has the property that, if $\lim x_n=x_0$ and $\lim T(x_n)=y_0$ exist, where x_n is in the domain D of T for each n, then x_0 is in D and $T(x_0)=y_0$. This is equivalent to stating that the set of points of type $[x, T(x)]$ is closed in the *Cartesian product* $\bar{D}\times\bar{R}$ of the closure of the domain D and the closure of R, the range of T. If D and R are contained in Banach spaces B_1 and B_2 and R is of the *second category* in B_2, it follows that $R=B_2$

and there is a number M such that $\|x\| \leqq M\|T(x)\|$ for each x of T (if T is one-to-one, its inverse exists and is bounded). If $D = B_1$, then T itself is bounded.

closed set. A set of points U such that every limit point of U is a point of U; the complement of an open set. The set of points on and within a circle is a closed set.

closed surface. A surface with no boundary curves; a space such that each point has a neighborhood topologically equivalent to the interior of a circle. See SURFACE.

CLO'SURE, *n.* **closure of a set of points.** The set which contains the given set and all *accumulation points* of the given set. The closure of a closed set is the set itself, while the closure of any set is closed. The set of all limit points of a given set is called the **derived set.** The closure of a set U is usually denoted by $\bar{U}$ and the derived set by U'.

CLUS'TER, *adj.* **cluster point.** Same as ACCUMULATION POINT.

CO'A-LI'TION, *n.* In an n-person game, a set of players, more than one in number, who coordinate their strategies, presumably for mutual benefit. See GAME—cooperative game.

CO-AL'TI-TUDE, *n.* **coaltitude of a celestial point.** Same as ZENITH DISTANCE.

CO-AX'I-AL, *adj.* **coaxial circles.** Circles such that all pairs of the circles have the same radical axis.

coaxial planes. Planes which pass through the same straight line. The line is called the axis.

CO-BOUND'A-RY, *n.* See COHOMOLOGY—cohomology group.

CO-CY'CLE, *n.* See COHOMOLOGY—cohomology group.

COCHRAN'S THEOREM. If x_i $(i = 1, \cdots, n)$ are independently and normally distributed variables with zero mean and unit variance and if $q_1, q_2, \cdots, q_k$ are k quadratic forms in the variables x_i with rank $n_1, n_2, \cdots, n_k$, with $\sum_{j=1}^{k} q_j = \sum_{i=1}^{n} x_i^2$, then a necessary and sufficient condition that q_j be each independently distributed with the χ^2 distribution with n_j degrees of freedom is that $\sum_{j=1}^{k} n_j = n$. On the basis of this theorem it follows that with $x_1, \cdots, x_n$, a random sample from a normal distribution with mean u and variance σ^2, $\sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2}$ is distributed as χ^2 with $n-1$ degrees of freedom, where $\bar{x}$ is the mean of the sample. This theorem is useful, for example, in establishing the independence of the mean and sum of squares of deviations around the mean in random samples from a normal population.

CODAZZI EQUATIONS. The equations

$$\frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} - \left\{ \begin{matrix} 1 & 2 \\ & 1 \end{matrix} \right\} D + \left(\left\{ \begin{matrix} 1 & 1 \\ & 1 \end{matrix} \right\} - \left\{ \begin{matrix} 1 & 2 \\ & 2 \end{matrix} \right\} \right) D' + \left\{ \begin{matrix} 1 & 1 \\ & 2 \end{matrix} \right\} D'' = 0$$

and

$$\frac{\partial D''}{\partial u} - \frac{\partial D'}{\partial v} + \left\{ \begin{matrix} 2 & 2 \\ & 1 \end{matrix} \right\} D + \left(\left\{ \begin{matrix} 2 & 2 \\ & 2 \end{matrix} \right\} - \left\{ \begin{matrix} 1 & 2 \\ & 1 \end{matrix} \right\} \right) D' - \left\{ \begin{matrix} 1 & 2 \\ & 2 \end{matrix} \right\} D'' = 0,$$

involving the fundamental coefficients of the first and second orders of a surface. In tensor notation: $d_{\alpha\alpha,\beta} - d_{\alpha\beta,\alpha} = 0$, $\alpha \neq \beta$. There are no relations between these fundamental coefficients and their derivatives which cannot be derived from the Gauss equation and the Codazzi equations, for these three equations uniquely determine a surface to within its position in space. See CHRISTOFFEL—Christoffel symbols.

CO-DEC'LI-NA'TION, *n.* **codeclination of a celestial point.** Ninety degrees minus the declination; the complementary angle of the declination. See HOUR—hour angle and hour circle. Also called POLAR DISTANCE.

COD'ING, *n.* In machine computation, the detailed preparation from the programmer's instructions or flow charts, of machine commands that will lead to the

solution of the problem at hand. See PROBLEM—problem formulation, and PROGRAMMING—programming for a computing machine.

CO'EF-FI'CIENT, *n.* In elementary algebra, the numerical part of a term, usually written before the literal part, as 2 in $2x$ or $2(x+y)$. (See PARENTHESIS.) In general, the product of all the factors of a term except a certain one (or a certain set), of which the product is said to be the *coefficient*. *E.g.*, in $2axyz$, $2axy$ is the coefficient of z, $2ayz$ the coefficient of x, $2ax$ the coefficient of yz, etc. Most commonly used in algebra for the constant factors, as distinguished from the variables.

binomial coefficients. See BINOMIAL.

coefficient of alienation. See CORRELATION—normal correlation.

coefficient of friction. See FRICTION.

coefficient of linear expansion. (1) The quotient of the change in length of a rod, due to one degree change of temperature, and the original length (not the same at all temperatures). (2) The change in length of a unit rod when the temperature changes one degree centigrade beginning at 0°C.

coefficient of strain. See ONE—one-dimensional strains.

coefficient of thermal expansion. A term used to designate both the *coefficient of linear expansion* and the *coefficient of volume expansion.*

coefficient of variation. See VARIATION.

coefficient of volume (or cubical) expansion. (1) The change in volume of a unit cube when the temperature changes one degree. (The coefficient thus defined is different at different temperatures.) (2) The change in unit volume due to a change of 1°C. beginning at 0°C., *i.e.*, $(v-v_0)/(v_0t)$, where v_0 is the volume at 0°C., v the volume after the change of temperature, and t the change of temperature.

coefficients in an equation. (1) The coefficients of the variables. (2) The constant term and the coefficients of all the terms containing variables. If the constant term is not included, the phrase *coefficients of the variables* in the equation is often used.

confidence coefficient. See CONFIDENCE.

correlation coefficient between two sets of data (or numbers). A number between -1 and 1 which indicates the degree of linear relationship between the two sets of numbers. If the two sets of numbers are $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$, the coefficient of correlation r measures how near the points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ are to lying on a straight line. If $r=1$, the points lie on a line and the two sets of data are said to be in **perfect correlation.** *Tech.* The coefficient of correlation r is defined to be the quotient of the sum of the products of the algebraic deviations of the corresponding numbers of the two sets and the square root of the product of the sum of the squares of the deviations of each set; *i.e.*,

$$r=\frac{\sum_{i=1}^{n}(x_i-x)(y_i-y)}{\sqrt{\sum_{i=1}^{n}(x_i-x)^2\sum_{i=1}^{n}(y_i-y)^2}},$$

where x and y are the corresponding *means.* Also called **Pearson's coefficient.**

detached coefficients, multiplication and division by means of. Abbreviations of the ordinary multiplication and division processes used in algebra. The coefficients alone (with their signs) are used, the powers of the variable occurring in the various terms being understood from the order in which the coefficients are written, missing powers being assumed to be present with zero coefficients. *E.g.*, (x^3+2x+1) is multiplied by $(3x-1)$ by using the expressions $(1+0+2+1)$ and $(3-1)$. See SYNTHETIC—synthetic division.

determinant of the coefficients. See DETERMINANT—determinant of the coefficients.

differential coefficient. Same as DERIVATIVE.

leading coeffi cient. See LEADING.

Legendre's coefficients. See LEGENDRE—Legendre's polynomials.

matrix of the coefficients. See MATRIX.

phi coefficient. (*Statistics.*) A coefficient obtained from a four-fold table, in which the two variables are essentially dichotomous. The phi coefficient, ϕ, is defined by

$$\phi=\sqrt{\chi^2/n},$$

where χ^2 is computed from the cell entries

and the marginal totals, which are the basis of the expected values. See CHI-SQUARE.

regression coefficient. See REGRESSION.

relation between the roots and coefficients of a polynomial equation. See ROOT—root of an equation.

undetermined coefficients. See UNDETERMINED.

CO-FAC'TOR, *n.* **cofactor of an element of a determinant.** See MINOR—minor of an element of a determinant.

cofactor of an element of a matrix. This is defined only for square matrices, and is the same as the cofactor of the same element in the determinant of the matrix.

CO-FIN'AL, *adj.* **cofinal subset.** See MOORE-SMITH CONVERGENCE.

CO-FUNC'TION, *n.* See TRIGONOMETRIC—trigonometric cofunctions.

CO-HER'ENT-LY, *adv.* **coherently oriented.** See MANIFOLD, and SIMPLEX.

CO'HO-MOL'O-GY, *adj.* **cohomology group.** Let K be an n-dimensional simplicial complex and let Δ be the boundary operator, so that the boundary of a p-chain $x=\Sigma g_i S_i^p$ is $\Delta x=\Sigma g_i \Delta S_i^p$. Then Δx is a $(p-1)$-chain and the boundary operator maps the group of p-chains into the group of $(p-1)$-chains. In particular,

$$\Delta\sigma_i^p=\sum_j g_i^j \sigma_j^{p-1}$$

for each p-simplex σ_i^p, where $\{\sigma_j^{p-1}\}$ are $(p-1)$-simplexes and g_i^j are elements of the group G whose elements are used as coefficients in forming chains. If the matrix (g_i^j) is $r\times s$, its transpose is $s\times r$ and can be used to define a mapping

$$\nabla\sigma_j^{p-1}=\sum_i g_i^j \sigma_i^p,$$

the chain $\nabla\sigma_j^{p-1}$ being called the **coboundary** of σ_j^{p-1}. This operator can be extended to all $(p-1)$-chains by the definition

$$\nabla x=\Sigma g_i \nabla S_i^{p-1} \quad \text{if} \quad x=\Sigma g_i S_i^{p-1}.$$

A chain is called a **cocycle** if its coboundary is zero. The **r-dimensional cohomology group** is the quotient group T^r/H^r, where T^r is the group of all r-dimensional cocycles of K and H^r is the group of all cycles which are 0 or are coboundaries of an $(r-1)$-chain of K. The concepts of homology and cohomology can be defined for certain generalizations of simplicial complexes (called *complexes*) for which each *complex* has a *dual complex* such that the homology group of one is the cohomology group of the other.

COIN, *n.* **coin-matching game.** See GAME.

CO'IN-CIDE', *v.* To be coincident.

CO-IN'CI-DENT, *adj.* **coincident configurations.** Two configurations which are such that any point of either one lies on the other. Two lines (or curves or surfaces) which have the same equation are coincident. The locus of an equation of the form $[f(x, y)]^2=0$ is two coincident loci.

CO-LAT'I-TUDE, *n.* **colatitude of a point on the earth.** Ninety degrees minus the latitude; the complementary angle of the latitude.

COL-LAT'E-RAL, *adj.* **collateral security.** Assets deposited to guarantee the fulfillment of some contract to pay, and to be returned upon the fulfillment of the contract.

collateral trust bonds. See BOND.

COL-LECT'ING, *p.* **collecting terms.** Grouping terms in a parenthesis or adding like terms. *E.g.*, to collect terms in $2+ax+bx$, we write it in the form $2+x(a+b)$; to collect terms in $2x+3y-x+y$, we write it in the form $x+4y$.

COL-LIN'E-AR, *adj.* **collinear planes.** Planes having a common line. *Syn.* Coaxial planes. Three planes are either *collinear* or *parallel* if the equation of any one of them is a linear combination of the equations of the others. See CONSISTENCY—consistency of linear equations.

collinear points. Points lying on the same line. Two points in the plane are **collinear with the origin** if and only if their corresponding rectangular Cartesian coordinates are proportional, or the determinant, whose first row is composed of the Cartesian coordinates of one of the points and the second of those of the other, is

zero; *i.e.*, $x_1y_2-x_2y_1=0$, where the points are (x_1, y_1) and (x_2, y_2). Two points in space are *collinear with the origin* if, and only if, their corresponding Cartesian coordinates are proportional, *i.e.*, the *matrix*

$$\begin{Vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{Vmatrix}$$

whose columns are composed of the coordinates of the points, is of rank one. Three points in a plane are collinear if, and only if, the third-order determinant whose rows are $x_1, y_1, 1$; $x_2, y_2, 1$; and $x_3, y_3, 1$, where the x's and y's are the coordinates of the three points, is zero. Three points in space are collinear if, and only if, lines through different pairs of the points have their direction numbers proportional, or if, and only if, the coordinates of any one of them can be expressed as a linear combination of the other two, in which the constants of the linear combination have their sum equal to unity.

COL-LIN-E-A′TION, *n.* A transformation of the plane or space which carries points into points, lines into lines, and planes into planes. See TRANSFORMATION—collineatory transformation.

COL-LIN′E-A-TO-RY, *adj.* **collineatory transformation.** See TRANSFORMATION—collineatory transformation.

CO-LOG′A-RITHM, *n.* **cologarithm of a number.** The logarithm of the reciprocal of the number; the negative of the logarithm of the number, written with the decimal part positive. Used in computations to avoid subtracting mantissas and the confusion of dealing with the negatives of mantissas. *E.g.*, to evaluate $\frac{641}{1246}$ by use of logarithms, write $\log \frac{641}{1246}=\log 641+$ colog 1246, where colog $1246=10-\log 1246-10=10-(3.0955)-10=6.9045-10$.

"COLONEL BLOTTO" GAME. See GAME.

COL′UMN, *n.* A vertical array of terms, used in addition and subtraction and in determinants and matrices.

column in a determinant. See DETERMINANT.

COMBESCURE, *adj.* **Combescure transformation of a curve.** A one-to-one continuous mapping of the points of one space curve on another in such a way that the tangents at corresponding points are parallel. It follows that the principal normals and the binormals, respectively, at corresponding points must also be parallel.

Combescure transformation of a triply orthogonal system of surfaces. A one-to-one continuous mapping of the points of three-dimensional Euclidean space on itself, such that the normals to the members of one triply orthogonal system of surfaces are parallel to the normals to the members of another system at points corresponding under the transformation.

COM′BI-NA′TION, *n.* A *combination* of a set of objects is any selection of one or more of the objects *without regard to order.* The **number of combinations of n things r at a time** is the number of sets that can be made up from the n things, each set containing r different things and no two sets containing exactly the same r things. This is equal to the number of permutations of the n things, taken r at a time, divided by the number of permutations of r things taken r at a time; that is,

$${}_nP_r/r!=n!/[(n-r)!r!],$$

which is denoted by ${}_nC_r$, C_r^n, $C(n, r)$, or $\binom{n}{r}$. *E.g.*, the combinations of a, b, c, two at a time, are ab, ac, bc. ${}_nC_r$ is also the coefficient in the $(r+1)$th term of the binomial expansion, $(x-a)^n=(x-a_1)(x-a_2)\cdots(x-a_n)$, with the a's all equal. That is, the coefficient of x^{n-r} is a^r times the number of combinations of the n different a's r at a time. See BINOMIAL—binomial coefficients. The **number of combination of n things, r at a time, when repetitions are allowed** is the number of sets which can be made up of r things chosen from the given n, each being used as many times as desired. The number of such combinations is the same as the number of combinations of $n+r-1$ different things taken r at a time, repetitions not allowed; *i.e.*,

$$\frac{(n+r-1)!}{(n-1)!r!}.$$

The combinations of a, b, c two at a time, repetitions allowed, are: aa, bb, cc, ab, ac, bc. The **total number of combinations of n**

different things (repetitions not allowed) is the sum of the number of combinations taken $1, 2, \cdots, n$ at a time, *i. e.*, $\sum_{r=1}^{n} {}_nC_r$, which is the sum of the binomial coefficients in $(x+y)^n$ less 1, or (2^n-1).

linear combination. See LINEAR.

COM'BI-NA-TO'RI-AL, *adj.* **combinatorial topology.** See TOPOLOGY.

COM-MAND', *n.* An instruction, in machine language, for a computing machine to perform a certain operation.

COM-MEN'SU-RA-BLE, *adj.* **commensurable quantities.** Two quantities which have a common measure; *i.e.*, there is a measure which is contained an integral number of times in each of them. A rule, a yard long, is commensurable with a rod, for they each contain, for instance, 6 inches an integral number of times.

COM-MER'CIAL, *adj.* **commercial bank.** A bank that carries checking accounts.

commercial draft. See DRAFT.

commercial paper. Negotiable paper used in transacting business, such as drafts, negotiable notes, endorsed checks, etc.

commercial year. See YEAR.

COM-MIS'SION, *n.* A fee charged for transacting business for another person.

commission man or merchant. See BROKER.

COM'MON, *adj.* **common difference** in an arithmetic progression. The difference between any term and the preceding term, usually denoted by *d*.

common divisor. See DIVISOR.

common fraction. See FRACTION.

common logarithms. Logarithms having 10 for their base. See LOGARITHM.

common multiple. See MULTIPLE—common multiple.

common stock. Stock upon which the dividends paid are determined by the net profits of the corporation after all other costs, including dividends on preferred stock, have been paid.

least (lowest) common denominator of a set of fractions. The least common multiple of their denominators. Denoted by L.C.D. The L.C.D. of $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{5}$ is 30.

COM-MU-TA'TION, *adj.* **commutation symbols in life insurance.** Symbols denoting the nature of the numbers in the columns of a commutation table. For instance, D_x and N_x. See below, commutation tables.

commutation tables (columns). Tables from which the values of certain types of insurance can be quickly computed. *E.g.*, suppose that one has a commutation table with the values of D_x and N_x for all ages appearing in the mortality tables, where D_x is the product of the number of persons who attain the age x in any year and the present value of a sum of money x years hence, at some given rate, and N_x is the sum of the series $(D_x+D_{x+1}+\cdots$ to end of table). The value of an immediate annuity of \$1.00 at age x is the quotient N_{x+1}/D_x, and that of an annuity due is N_x/D_x. Sometimes (following Davies) N_x is defined as the sum of the series $(D_{x+1}+D_{x+2}+\cdots)$. Using this definition, the annuity values must be N_x/D_x and N_{x-1}/D_x, respectively. Commutation tables based on the latter definition of N_x are called the **terminal form** while those based on the former definition of N_x are called the **initial form.**

COM-MU'TA-TIVE, *adj.* A method of combining objects two at a time is *commutative* if the result of the combination of two objects does not depend on the order in which the objects are given. For example, the **commutative law of addition** states that the order of addition does not affect the sum: $a+b=b+a$ for any numbers a and b (*e.g.*, $2+3=3+2$). The **commutative law of multiplication** states that the order of multiplication does not affect the product: $a\cdot b=b\cdot a$ for any numbers a and b (*e.g.*, $3\cdot 5=5\cdot 3$). See GROUP.

commutative group. See GROUP.

COM'MU-TA-TOR, *n.* **commutator of elements of a group.** The commutator of two elements a and b of a group is the element $a^{-1}b^{-1}ab$, or the element c such that $bac=ab$. The group of all elements of the form $c_1c_2\cdots c_n$, where each c_i is the commutator of some pair of elements, is called the **commutator subgroup.** The commutator subgroup of an Abelian group contains only the identity element. A

group is said to be **perfect** if it is identical with its commutator subgroup. A commutator subgroup is an *invariant subgroup* and the *factor group* formed with it is Abelian.

COM-MUT'ING, *p.* **commuting obligations.** Exchanging one set of obligations to pay a certain sum (or sums) at various times for another to pay according to some other plan. The common date of comparison at which the two sets are equivalent (equal in value at that time) is called the **focal date.**

COM'PACT, *adj.* **compact set.** (1) A set E which either contains only a finite number of points or is such that every infinite set of points of E has at least one accumulation point in E; a set E such that every sequence of points of E contains a subsequence which converges to a point of E. A set which is compact in this sense is also said to be **sequentially compact** or **countably compact.** (2) A set E which has the property that, for any union of open sets which contains E, there is a finite number of these open sets whose union also contains E. A set which is compact in this sense is also said to be **bicompact.** A set which is bicompact is also sequentially compact. A compact subset of a Hausdorff topological space is closed, but closed sets need not be compact in either sense. A space is **locally compact** if each of its points has a neighborhood whose closure is compact. The set $(0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$ is compact. The set of all real numbers is locally compact; it is not compact, since the sequence $1, 2, 3, \cdots$ does not contain a convergent subsequence. See BOLZANO—Bolzano-Weierstrass theorem, and HEINE-BOREL THEOREM.

COM-PACT'I-FI-CA'TION, *n.* A compactification of a topological space T is a *compact* topological space W which contains T (or is such that T is homeomorphic with a subset of W). The *complex plane* (or *sphere*) is the compactification of the *Euclidean plane* obtained by adjoining a single point (usually designated by the symbol ∞) and defining the neighborhoods of ∞ to be sets containing ∞ and the complement of a bounded, closed (*i.e.*, compact) subset of the plane. Likewise, a *locally compact Hausdorff space* H has a **one-point compactification** (also a Hausdorff space) obtained by adjoining a single point, which can be designated by the symbol ∞, whose neighborhoods are sets containing ∞ and the complement of a compact subset of H. The **Stone-Čech compactification** of a *Tychonoff space* T is the closure of the image of T in the space I^ϕ, where I^ϕ is the Cartesian product of the closed unit interval I (taken ϕ times) and ϕ is the cardinal number of the family F of all continuous functions from T to I (the image of a point x of T in I^ϕ is the member of I^ϕ whose f "component" is $f(x)$ for each member f of F). The Stone-Čech compactification is (in a certain real sense) a maximal compactification. The entire space I^ϕ is compact, a consequence of the Tychonoff theorem (see PRODUCT—Cartesian product).

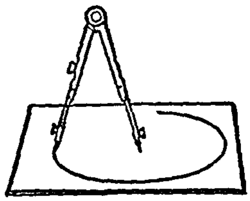
COM-PAC'TUM, *n.* A topological space which is *compact* and *metrizable.* Examples of compacta are closed intervals, closed spheres (with or without their interiors), and closed polyhedra. A Hausdorff topological space which is *bicompact* is sometimes called a **bicompactum** (see COMPACT—compact set).

COM'PA-RA-BLE, *adj.* **comparable functions.** Functions $f(x)$ and $g(x)$ which have real-number values, which have a common domain of definition D, and which are such that either $f(x) \leqq g(x)$ for all x in D or $f(x) \geqq g(x)$ for all x in D.

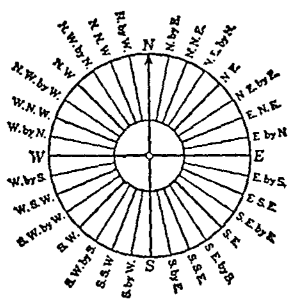
COM-PAR'I-SON, *adj.*, *n.* **comparison date.** See EQUATION—equation of payments.

comparison test for convergence of an infinite series. If, after some chosen term of a series, the *absolute value* of each term is equal to or less than the value of the corresponding term of some convergent series of positive terms, the series converges (and converges absolutely); if each term is equal to or greater than the corresponding term of some divergent series of positive terms, the series diverges.

COM'PASS, *n.* An instrument for describing circles or for measuring distances between two points. Usually used in the plural, as **compasses.**



mariner's compass. A magnetic needle that rotates about an axis perpendicular to a card (see figure) on which the directions are indicated. The needle always indicates the direction of the magnetic meridian.



COM-PAT'I-BIL'I-TY, *adj.*, *n.* Same as CONSISTENCY.

compatibility equations. (*Elasticity.*) The differential equations connecting the components of the strain tensor which guarantee that the state of strain be possible in a continuous body.

COM'PLE-MENT, *n.* **complement of a set of points.** The set of all elements (or points) that do not belong to the given set U, but belong to a given whole space (or set) that contains U. The complement of U is denoted by $C(U)$. The complement of the set of positive numbers with respect to the space of all real numbers is the set containing all negative numbers and zero.

COM'PLE-MEN'TA-RY, *adj.* **complementary acceleration.** See ACCELERATION—acceleration of Coriolis.

complementary angles. Two angles whose sum is a right angle. The two acute angles in a right triangle are always complementary.

complementary function. See DIFFERENTIAL—linear differential equations.

complementary minor. See MINOR—minor of an element in a determinant.

complementary trigonometric functions. Same as TRIGONOMETRIC COFUNCTIONS. See TRIGONOMETRIC.

surface complementary to a given surface. Given a surface S, there is an infinitude of parallel surfaces such that S is a *surface of center* relative to each of them. See SURFACE—surfaces of center relative to a given surface, and PARALLEL—parallel surfaces. The other common surface of center of the family of parallel surfaces is said to be **complementary to** S.

COM-PLETE', *adj.* **complete annuity.** See ANNUITY.

complete field. See FIELD—ordered field.

complete induction. See INDUCTION—mathematical induction.

complete scale. See SCALE—number scale.

complete space A (metric) space such that every *Cauchy sequence* converges to a point of the space. See SEQUENCE—Cauchy sequence. The space of all real numbers (or all complex numbers) is complete. The space of all continuous functions defined on the interval $[0, 1]$ is not complete if the distance between f and g is defined as $\int_0^1 |f-g|\,dx$, since the sequence $f_1, f_2, \cdots$ does not then converge to a continuous function if $f_n(x)=0$ for $0 \leqq x \leqq \frac{1}{2}$ and $f_n(x)=(x-\frac{1}{2})^{1/n}$ for $\frac{1}{2} \leqq x \leqq 1$. A topological space is **topologically complete** if it is homeomorphic to some complete metric space. A subset of a complete metric space is topologically complete if and only if it is a G_δ subset (see BOREL—Borel set).

complete system of functions. See ORTHOGONAL—orthogonal functions.

complete system of representations for a group. See REPRESENTATION—representation of a group.

weakly complete space. See WEAK—weak completeness.

COM-PLET'ING, *n.* **completing the square.** A process used in solving quadratic equations. It consists of transposing all terms to the left side of the equation, dividing through by the coefficient of the square term, then adding to the constant (and to the right side) a number sufficient to make the left member a perfect trinomial square. This method is sometimes modi-

fied by first multiplying through by a number sufficient to make the coefficient of the square of the variable a perfect square, then adding a constant to both sides of the equation, as before, to make the left side a perfect trinomial square. *E.g.*, to complete the square in $2x^2+8x+2=0$, divide both members of the equation by 2, obtaining $x^2+4x+1=0$. Now add 3 to both sides, obtaining

$$x^2+4x+4=(x+2)^2=3.$$

Frequently, *completing* the square refers to writing any polynomial of the form $a_1x^2+b_1x+c_1$ in the form $a_1(x+b_2)^2+c_2$, a procedure used a great deal in reducing the equations of conics to their standard form.

COM'PLEX, *adj., n.* As a *noun,* a complex may mean simply a SET (also see below, simplicial complex).

absolute value of a complex number. See MODULUS—modulus of a complex number.

amplitude, or argument, of a complex number. See POLAR—polar form of a complex number.

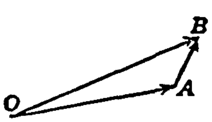
complex conjugate of a matrix. See MATRIX—complex conjugate of a matrix.

complex domain (field). The set of all complex numbers. See FIELD.

complex fraction. See FRACTION.

complex integration. See CONTOUR—contour integral.

complex number. Any number, real or imaginary, of the form $a+bi$, where a and b are real numbers and $i^2=-1$. Called **imaginary numbers** when $b\neq 0$, and **pure imaginary** when $a=0$ and $b\neq 0$ (although complex numbers are not imaginary in the usual sense). Two complex numbers are defined to be **equal** if and only if they are identical. *I.e.*, $a+bi=c+di$ means $a=c$ and $b=d$. A complex number $x+yi$ can be represented in the plane by the vector with components x and y, or by the point (x, y) (see the figure below, and ARGAND DIAGRAM). Thus two complex numbers are equal if and only if they are represented by the same vectors, or by the same points.

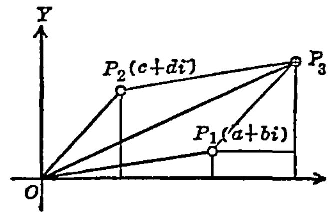


In the above figure, $x=r\cos\theta$ and $y=r\sin\theta$. Therefore

$$x+yi=r(\cos\theta+i\sin\theta),$$

which is called the **polar form** of $x+yi$ (see POLAR). The **sum of complex numbers** is obtained by adding the real parts and the coefficients of i separately; *e.g.*, $(2-3i)+(1+5i)=3+2i$. Geometrically, this is the same as the addition of the corresponding vectors in the plane. In the figure below,

$$OP_1+OP_2=OP_3\ (OP_2=P_1P_3).$$



The **product of complex numbers** is computed by treating the numbers as polynomials in i with the special property $i^2=-1$. Thus:

$$\begin{aligned}(a+bi)(c+di)&=ac+(ad+bc)i+bdi^2\\&=ac-bd+(ad+bc)i.\end{aligned}$$

If the complex numbers are in the form $r_1(\cos A+i\sin A)$ and $r_2(\cos B+i\sin B)$, their product is $r_1r_2[\cos(A+B)+i\sin(A+B)]$, *i.e.*, to multiply two complex numbers, multiply their moduli and add their amplitudes (see DE MOIVRE—De Moivre's theorem). Similarly, the **quotient of two complex numbers** is the complex number whose modulus is the quotient of the modulus of the dividend by that of the divisor and whose amplitude is the amplitude of the dividend minus that of the divisor; that is,

$$\begin{aligned}&r_1(\cos\theta_1+i\sin\theta_1)\div r_2(\cos\theta_2+i\sin\theta_2)\\&\qquad=\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)].\end{aligned}$$

When the numbers are not in polar form, the quotient can be computed by multiplying dividend and divisor by the conjugate of the divisor, as illustrated in the following example:

$$\frac{2+i}{1+i}=\frac{(2+i)(1-i)}{(1+i)(1-i)}=\frac{3-i}{2}.$$

Tech. The system of complex numbers is the set of ordered pairs (a, b) of real numbers in which two pairs are considered equal if, and only if, they are identical [$(a, b) = (c, d)$ if, and only if, $a = c$ and $b = d$], and in which addition and multiplication are defined by

$$(a, b) + (c, d) = (a + c, b + d),$$
$$(a, b)(c, d) = (ac - bd, ad + bc).$$

The system satisfies most of the fundamental algebraic laws, such as the associative and commutative laws for addition and multiplication. It is a *field*, but not an *ordered field*. A remarkable consequence of these definitions is:

$$(0, 1)(0, 1) = (-1, 0);$$
$$(0, -1)(0, -1) = (-1, 0).$$

That is, the number $(-1, 0,)$ or -1, has the two square roots $(0, 1)$ and $(0, -1)$. See FUNDAMENTAL—fundamental theorem of algebra.

complex plane. The plane (of complex numbers) with a single point at infinity whose neighborhoods are exteriors of circles with center at 0. The complex plane is topologically (and conformally) equivalent to a sphere. See PROJECTION—stereographic projection.

complex roots of a quadratic equation $(ax^2 + bx + c = 0)$. Used in contrasting roots of the form $a + bi$ with real roots, although the latter are special cases of the former for which $b = 0$. See DISCRIMINANT—discriminant of a quadratic equation in one variable.

complex sphere. A unit sphere on which the complex plane is represented by a stereographic projection. The complex plane is usually the equatorial plane relative to the pole of projection, or the tangent plane at the point diametrically opposite the pole of projection.

complex unit. A complex number whose modulus is unity; a complex number of the form $\cos \theta + i \sin \theta$; a complex number which, geometrically, is a radius of the unit circle about the pole. The products, or quotients, of unit complex numbers are unit complex numbers.

conjugate complex numbers. Frequently called **conjugate imaginaries**: complex numbers which are identical, except that the pure imaginary terms are opposite in sign or both zero. Numbers of the form $a + bi$ and $a - bi$ are conjugate complex numbers. When $b \neq 0$, $a + bi$ and $a - bi$ are also called *conjugate imaginary numbers*.

modulus of a complex number. See MODULUS.

real and imaginary parts of a complex number. See REAL, and IMAGINARY.

root of a complex number. See ROOT.

simplicial complex. A set which consists of a finite number of simplexes (not necessarily all of the same dimension) with the property that the intersection of any two of the simplexes is either empty or is a face of each of them. This definition is sometimes modified in various ways, *e.g.*, by requiring that each simplex be oriented. A simplicial complex is sometimes called a complex, but a **complex** is sometimes defined with fewer restrictions (*e.g.*, it may be a countable set of simplexes such that the intersection of any two of the simplexes is either empty, or is a face of each of them, and no vertex of a simplex belongs to more than a finite number of the simplexes). The **dimension of a simplicial complex** is the largest of the dimensions of the simplexes making up the simplicial complex. The class of all simplexes which belong to a simplicial complex K and have dimension less than that of K is called the **skeleton** of K. A finite set K of elements $c_0, c_1, \cdots, c_n$ is called an **abstract simplicial complex** (or an **abstract complex** or a **skeleton complex**), and the elements $c_0, \cdots, c_n$ are called **vertices**, if certain nonempty subsets of K, which are called **abstract simplexes** (or **skeletons**) are such that each subset (called a face) of an abstract simplex is an abstract simplex and each of the vertices is an abstract simplex. The **dimension of an abstract simplex** of $r+1$ points is r, and the **dimension of an abstract complex** is the largest of the dimensions of its abstract simplexes. An abstract complex of dimension n can always be represented by a simplicial complex imbedded in the Euclidean space of dimension $2n+1$. A simplicial complex is sometimes called a **geometric complex** (or simply a **complex**), or a **triangulation.** The set of all those points which belong to simplexes of a simplicial complex is called a **polyhedron.** A topological space is said to be **triangulable,** and

is sometimes called a **polyhedron** or a **topological simplicial complex,** if it is homeomorphic to the set of points belonging to simplexes of a simplicial complex K; the homeomorphism together with the complex K is called a **triangulation** of the polyhedron. A simplicial complex is said to be **oriented** if each of its simplexes is oriented. See CHAIN—chain of a complex, MANIFOLD, SIMPLEX, SURFACE, and TRIANGULATION.

COM-PO'NENT, *n.* **component of an acceleration, force, or velocity.** See below, component of a vector.

component of a computing machine. Any physical mechanism or abstract concept having a distinct role in automatic computation. See headings below and under ARITHMETIC, CONTROL, INPUT, OUTPUT, and STORAGE.

component of a set of points. A subset which is *connected* and is not contained in any other connected subset of the given set of points. A component is necessarily a *closed* subset relative to the set.

component of the stress tensor. In linear theory of elasticity, a set of six functions determining the state of stress at any point of the substance.

component of a vector. One of a set of two or more vectors which are equivalent to the given vector (see RESULTANT). The component of a vector in a given direction is the *projection* of the vector on a line in the given direction. The vector may represent a force, velocity, acceleration, etc. See VECTOR.

direction components. See DIRECTION—direction numbers.

elementary potential digital computing component. In a computing machine, any component that can assume any one of a fixed discrete set of stable states, and that can influence and/or be influenced by other components of the machine. See CIRCUIT—flip-flop circuit.

COM-POS'ITE, *adj.* **composite function.** (1) **A composite function of one variable** is a function of a single variable which is itself a function of a second variable—such as $y=f(t)$, where $t=g(x)$. The derivative of such a function, with respect to x, can be obtained by use of the formula

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}.$$

A composite function of two variables is a function in which the independent variables are functions of other independent variables; for instance, $z=f(x, y)$, where $x=u(t, s)$ and $y=v(t, s)$, is a composite function of s and t. (2) A function which is factorable (can be written as the product of two or more functions), as x^2-y^2 or x^2-1 (usually refers only to polynomial functions which are factorable relative to some specified field).

composite life of a plant. The time required for the total annual depreciation charge to accumulate, at a given rate of interest, to the original wearing value.

composite number. A number that can be factored, as 4, 6, or 10, in distinction to prime numbers like 3, 5, or 7. Refers only to integers, not to rational or irrational numbers.

composite quantity. A quantity that is factorable.

derivative and differential of a composite function. See CHAIN—chain rule, and DIFFERENTIAL.

COM'PO-SI'TION, *n.* **composition in a proportion.** Passing from the statement of the proportion to the statement that the sum of the first antecedent and its consequent is to its consequent as the sum of the second antecedent and its consequent is to its consequent; *i.e.*, passing from

$$a/b=c/d \quad \text{to} \quad (a+b)/b=(c+d)/d.$$

composition and division in a proportion. Passing from the statement of the proportion to the statement that the sum of the first antecedent and its consequent is to the difference between the first antecedent and its consequent as the sum of the second antecedent and its consequent is to the difference between the second antecedent and its consequent; *i.e.*, passing from

$$a/b=c/d$$

to

$$(a+b)/(a-b)=(c+d)/(c-d).$$

See above, composition in a proportion, and DIVISION—division in a proportion.

composition of tensors. See INNER—inner product of tensors.

composition of vectors. The same process as *addition of vectors*, but the term composition of vectors is used more when speaking of adding vectors which denote forces, velocities or accelerations; finding the vector which represents the resultant of forces, velocities, accelerations, etc., represented by the given vectors. See SUM—sum of vectors.

graphing by composition of ordinates. See GRAPHING—graphing by composition.

COM'POUND, *adj.* **compound amount.** See INTEREST.

compound event. (1) An event the probability of whose occurrence depends upon the probability of occurrence of two or more independent events. *E.g.*, the probability of getting two heads on each of two tosses of a coin is the product of the separate probabilities, which is $\frac{1}{2}\cdot\frac{1}{2}$. (2) An event consisting of two or more non-mutually exclusive events.

compound interest. See INTEREST.

compound number. The sum of two or more denominations of a certain kind of denominate number; *e.g.*, 5 feet, 7 inches, or 6 pounds, 3 ounces.

compound survivorship life insurance. See INSURANCE—life insurance.

COM-PRES'SION, *n.* See TENSION.

modulus of compression. See MODULUS—bulk modulus.

simple or one-dimensional compressions. Same as ONE-DIMENSIONAL STRAINS. See STRAIN.

COM'PU-TA'TION, *n.* The act of carrying out mathematical processes; used mostly with reference to arithmetic rather than algebraic work. One might say, "Find the formula for the number of gallons in a sphere of radius r and *compute* the result for $r=5$"; or "*Compute* the square root of 3." Frequently used to designate long arithmetic or analytic processes that give numerical results, as *computing* the orbit of a planet.

numerical computation. A computation involving numbers only, not letters representing numbers.

COM-PUTE', *v.* To make a computation. *Syn.* Calculate.

COM-PUT'ER, *n.* Any instrument which performs numerical mathematical operations. A mechanical machine which primarily performs combinations of addition, subtraction, multiplication and division is sometimes called a *calculating machine* in distinction from such more versatile instruments as electronic computers. *Syn.* Computing machine.

analogue computer. A computing machine in which numbers are converted into measurable quantities, such as lengths or voltages, that can be combined in accordance with the desired arithmetic operations; *e.g.*, a slide rule. Generally, if two physical systems have corresponding behavior, and one is chosen for study in place of the other (because of greater familiarity, economy, feasibility, or other factors), then the first is called an analogue device, analogue machine, or analogue computer.

digital computer. A computing machine that performs mathematical operations on numbers expressed by means of digits. *Syn.* Digital device.

CON'CAVE, *adj.* **concave toward a point (or line).** Said of a curve that bulges away from (is hollow toward) the point (or line). *Tech.* An arc of a curve is concave toward a point (or line) if every segment of the arc cut off by a secant lies on the opposite side of the secant from the point (or line). If there exists a horizontal line such that the curve lies above it and is concave toward it (lies below it and is concave toward it) the curve is said to be *concave down* (*concave up*). A circle with center on the x-axis is concave toward that axis, the upper half being concave down and the lower half concave up.

concave function. The negative of a convex function. See CONVEX—convex function.

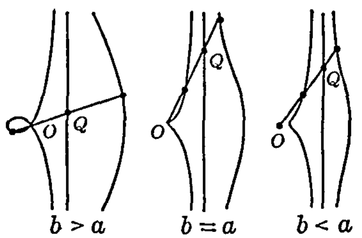
concave polygon and polyhedron. See POLYGON and POLYHEDRON.

CON-CAV'I-TY, *n.* The state or property of being concave.

CON'CEN-TRA'TION, *n.* **concentration method for the potential of a complex.** See POTENTIAL.

CON-CEN'TRIC, *adj.* **concentric circles.** Circles lying in the same plane and having a common center. *Concentric* is applied to any two figures which have centers (that is, are symmetric about some point) when their centers are coincident. *Concentric* is opposed to *eccentric*, meaning not concentric.

CON'CHOID, *n.* The locus of one end point of a segment of constant length, on a line which rotates about a fixed point (O in figure), the other end point of the



segment being at the intersection Q of this line with a fixed line not containing the fixed point. If the polar axis is taken through the fixed point and perpendicular to the fixed line, the length of the segment is taken as b, and the distance from the fixed point to the fixed line as a, the polar equation of the conchoid is $r=b+a \sec \theta$. Its Cartesian equation is

$$(x-a)^2(x^2+y^2)=b^2x^2.$$

The curve is asymptotic to the fixed line in both directions, and on both sides of it. If the line segment is greater than the perpendicular distance from the pole to the fixed line, the curve forms a loop with a node at the pole. If these two distances are equal, it forms a cusp at the pole. The conchoid is also called the **conchoid of Nicomedes.**

CON-CLU'SION, *n.* **conclusion of a theorem.** The statement which follows (or is to be proved to follow) as a consequence of the hypothesis of the theorem. See IMPLICATION.

CON-CORD'ANT-LY, *adv.* **concordantly oriented.** See MANIFOLD and SIMPLEX.

CON'CRETE, *adj.* **concrete number.** A number referring to specific objects or units, as 3 people, or 3 houses. The number and its reference are denoted by *concrete number.*

CON-CUR'RENT, *adj.* Passing through a point.

concurrent lines. Two or more lines which have a point in common. *In a plane*, three lines (of which not more than two are coincident) are concurrent if, and only if, the determinant of the coefficients and constants of the equations of the three lines is zero and one second-order determinant taken from the coefficients (the first two columns of the determinant) is not zero; *i.e.*, the three equations defining the lines have a unique solution. *In space*, three lines are concurrent if, and only if, the six equations defining the three lines have a common solution. See MATRIX—rank of a matrix.

concurrent planes. Three or more planes having a point in common. The condition for three planes to be concurrent is given under CONSISTENCY—consistency of linear equations.

CON'DEN-SA'TION, *adj.* **condensation point.** A point P is a *condensation point* of a set S if each neighborhood of P contains uncountably many points of S. See ACCUMULATION—accumulation point, and COUNTABLE—countable set.

CON-DI'TION, *n.* A mathematical assumption or truth that suffices to assure the truth of a certain statement, or which must be true if this statement is true. A condition from which a given statement logically follows is said to be a **sufficient condition**; a condition which is a logical consequence of a given statement is said to be a **necessary condition.** A **necessary and sufficient condition** is a condition that is both necessary and sufficient. A condition may be necessary but not sufficient, or sufficient but not necessary. It is necessary that a substance be sweet in order that it be called sugar, but it may be sweet and be arsenic; it is sufficient that it be granulated and have the chemical properties of sugar, but it can be sugar without being granulated. In order for a quadrilateral to be a parallelogram, it is necessary, but not sufficient, that two opposite sides be equal, and sufficient, although not necessary, that all

of its sides be equal; but it is *necessary and sufficient* that two opposite sides be equal and parallel. See IMPLICATION.

condition per cent of equipment. The ratio of its present wearing value to its wearing value when it was new. Present wearing value, as used here, means difference between the sale price at the present moment and the scrap value.

CON-DI'TION-AL, *adj.* **conditional convergence of series.** See CONVERGENCE—conditional convergence.

conditional equation and inequality. See EQUATION, and INEQUALITY.

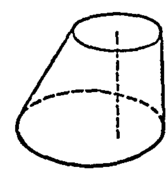
conditional statement. Same as IMPLICATION.

CON-DUC'TOR, *adj., n.* **conductor potential.** For a region R with boundary S, the conductor potential is the function harmonic in the interior of R, continuous on $R \cup S$, and taking on the constant value 1 on S. It describes the potential of an electric charge in equilibrium on the surface of a conductor.

CONE, *n.* (1) A conical surface (see CONICAL—conical surface). (2) A solid bounded by a region (the **base**) in a plane and the surface formed by the straight line segments (the **elements**) which join points of the boundary of the base to a fixed point (the **vertex**) not in the plane of the base (the surface bounding this solid is also called a *cone*). The perpendicular distance from the vertex to the plane of the base is the **altitude** of the cone. If the base has a center, the line passing through the center of the base and the vertex is the **axis** of the cone. The cone is **circular** or **elliptic** in the cases its base is a circle or ellipse (sometimes a circular cone is defined to be a cone whose intersections with planes perpendicular to the axis, but not intersecting the base, are circles). An **oblique circular cone** is a circular cone whose axis is not perpendicular to its base. A **right circular cone** (or **cone of revolution**) is a circular cone whose base is perpendicular to its axis (sometimes called simply a *circular cone*). A right circular cone can be generated by revolving a right triangle about one of its legs, or an isosceles triangle about its altitude. The **slant height** of a right circular cone is the length of an element of the cone. The **lateral area** of a cone is the area of the surface formed by the elements (for a right circular cone this is equal to πrh, where r is the radius of the base and h is the slant height). The **volume** of a cone is equal to one-third the product of the area of the base and the altitude. If the cone has a circular base, the volume is $\frac{1}{3}\pi r^2 s$, where r is the radius of the base and s is the altitude.

frustum of a cone. The part of the cone bounded by the base and a plane parallel to the base (see figure). The volume of a frustum of a cone equals one-third the **altitude** (the distance between the planes) times the sum of the areas of the bases and the square root of the product of the areas of the bases; *i.e.*,

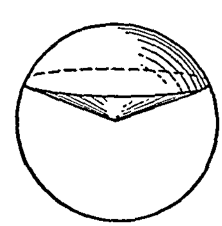
$$\tfrac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2}).$$



The **lateral area** of a frustum of a right circular cone (the area of the curved surface) is equal to $\pi l(r+r')$, where l is the slant height and r and r' are the radii of the bases.

ruling of a cone. See RULING.

spherical cone. A surface composed of the spherical surface of a spherical segment and the conical surface defined by the bounding circle of the segment and the center of the sphere (see CONICAL—conical surface); a spherical sector whose curved base is a zone of one base. The volume of a spherical cone is $\frac{2}{3}\pi r^2 h$, where r is the radius of the sphere and h is the altitude of the zone base.



tangent cone of a quadric surface. A cone whose elements are each tangent to

the quadric. In particular, a tangent cone of a sphere is any circular cone all of whose elements are tangent to the sphere. If a ball is dropped into a cone, the cone is tangent to the ball.

truncated cone. The portion of a cone included between two nonparallel planes whose line of intersection does not pierce the cone. The two plane sections of the cone are called the **bases** of the *truncated cone.*

CON'FI-DENCE, *n.* **asymptotically shortest confidence interval.** If a confidence interval is not shortest for finite sample sizes, but, as $n \to \infty$, its probability of including wrong values of the parameter approaches that of the shortest confidence interval in the limit, then the interval is *asymptotically shortest.* Maximum-likelihood estimates give rise under certain regularity conditions to asymptotically shortest unbiased confidence intervals.

confidence interval. A range of values which is believed, with a preassigned degree of confidence, to include the particular value of some parameter or characteristic being estimated. The degree of confidence is related to the probability of obtaining by random samples ranges which are correct. *Tech.* From a random sample of a specified population, one may set up an interval intended to delimit a characteristic (parameter) of the population. The probability of random samples yielding intervals that do include the correct value is determinable and controllable. If the probability of getting correct intervals is set at .95, *e.g.*, the interval is called a *confidence interval* with the confidence coefficient .95. Given a particular random sample and the derived confidence interval, it is not correct to say that if the probability is .95, the parameter estimated lies in a particular interval, since no distribution of parameter values is implied or used in the confidence-interval method. Regardless of what the parameter value is, the probability of obtaining correct intervals is preassignable, *e.g.*, .95. Roughly, of all the intervals that could be obtained by drawing random samples, a certain fraction will be right; the fraction is .95, and .05 of the intervals will be wrong. Symbolically and precisely: if the functions $\underline{T}(S)$ and $\bar{T}(S)$ are such that whatever is the value of the parameter, T, and regardless of other parameters, the probability that $\underline{T}(S) \leqq T \leqq \bar{T}(S)$ is identically equal to α, then $\underline{T}(S)$ and $\bar{T}(S)$ are the lower and upper bounds of the confidence interval of the parameter T with confidence coefficient α, where: (1) S is a random sample from a distribution with T and other parameters; (2) $\underline{T}(S)$ and $\bar{T}(S)$ are two functions of S; (3) $0<\alpha<1$. It must be noted that the sampling variables are $\underline{T}(S)$ and $\bar{T}(S)$ and not T. As an example, the method of obtaining, from a random sample of size n, the confidence interval with confidence coefficient .95 for the mean of a normal population with known variance, is

$$z_{.975} \leqq \frac{\sqrt{n}(\bar{x}-u)}{\sigma} \leqq z_{.025},$$

where $\bar{x}$ is the sample mean, u is the parameter to be estimated, and $z_{.975}$ and $z_{.025}$ are the normal deviates which are exceeded by .975 and .025 of the normal distribution respectively. If $n=100$, $\bar{x}=10.50$, and $\sigma=25$, then

$$-1.96 \leqq \frac{\sqrt{100}(10.50-u)}{25} \leqq 1.96,$$

$$11.48 \geqq u \geqq 9.52.$$

short unbiased confidence interval. An unbiased confidence interval, whose probability of covering the wrong value of the parameter in the neighborhood of the true value is less than for any other unbiased confidence interval with the same confidence interval, is a *short unbiased confidence interval.* A confidence interval derived from Neyman's type A critical region is a *short unbiased interval.*

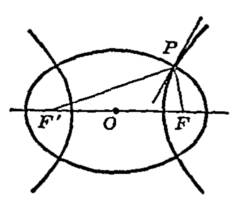
shortest confidence interval. The confidence interval which minimizes some function of $\bar{T}(S)-\underline{T}(S)$. J. Neyman's shortest confidence interval is the interval $\bar{T}(S)$ to $\underline{T}(S)$, where the probability of $\underline{T}(S) < T_1 \leq \bar{T}(S)$ (T_1 being an incorrect value) is less than the probability that $\underline{T}'(S) < T_1 \leq \bar{T}'(S)$ where $\underline{T}'(S)$ and $\bar{T}'(S)$ are any other functions of S and both intervals are of the same confidence coefficient, for a given sample size. Another alternative criterion requires that the ratio $\bar{T}(S)/\underline{T}(S)$ be a minimum for a given confidence coefficient. For simple and composite hypotheses with one constraint, this gives

the same intervals as Neyman's criterion. A shortest confidence interval (in Neyman's sense) yields uniformly most powerful tests of hypotheses. See the example above, under *confidence interval.*

unbiased confidence interval. The confidence interval $\underline{T}(S)$ to $\bar{T}(S)$ with confidence coefficient α is *unbiased* if the probability of containing the correct value is higher than the probability of containing any other value. Otherwise the method leads to biased confidence intervals.

CON-FIG'U-RA'TION, *n.* A general term for any geometrical figure, or any combination of geometrical elements, such as points, lines, curves, and surfaces.

CON-FO'CAL, *adj.* **confocal conics.** Conics having their foci coincident. *E.g.*, the ellipses and hyperbolas represented by the equation $x^2/(a^2-k^2)+y^2/(b^2-k^2)=1$, where $b^2<a^2$, $k^2\neq b^2$, and k takes on all other real values for which $k^2<a^2$, are confocal. These conics intersect at right angles, forming an orthogonal system. (See point P in figure.)



confocal quadrics. Quadrics whose *principal planes* are the same and whose sections by any one of these planes are *confocal conics. E.g.*, if k is a parameter and a, b, and c are fixed, the equation

$$\frac{x^2}{a^2-k}+\frac{y^2}{b^2-k}+\frac{z^2}{c^2-k}=1,$$

$a^2>b^2>c^2$, represents a triply orthogonal system of confocal quadrics: For $c^2>k>-\infty$, the equation represents a family of **confocal ellipsoids**; for $b^2>k>c^2$, it represents a family of **confocal hyperboloids of one sheet**; and for $a^2>k>b^2$, it represents a family of **confocal hyperboloids of two sheets.** Each member of one family is confocal and orthogonal to each member of other families. See ORTHOGONAL—triply orthogonal system of surfaces. For $k=c^2$ we get, by a limiting process, the elliptic portion of the (x, y)-plane (counted twice) bounded by (1) $\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1$; similarly for $k=b^2$ we get the hyperbolic portion of the (x, z)-plane (counted twice) bounded by (2) $\frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1$. Equations (1) and (2) define the **focal ellipse** and the **focal hyperbola,** respectively, of the system. Through each point (x, y, z) of space there pass three quadrics of the system. The corresponding values k_1, k_2, k_3 of k are called the **ellipsoidal coordinates** of (x, y, z). See COORDINATE—ellipsoidal coordinates.

CON-FORM'A-BLE, *adj.* **conformable matrices.** Two matrices A and B such that the number of columns in A is equal to the number of rows in B. It is possible to form the product AB if, and only if, A and B are conformable. Being conformable is not a symmetric relation. See PRODUCT—product of matrices.

CON-FORM'AL, *adj.* **conformal-conjugate representation of one surface on another.** A representation which both is conformal and is such that each conjugate system on one surface corresponds to a conjugate system on the other. *Syn.* Isothermal-conjugate representation of one surface on another.

conformal map or conformal transformation. A map which preserves angles; *i.e.*, a map such that if two curves intersect at an angle θ, then the images of the two curves in the map also intersect at the same angle θ. The functions $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$ map the (u, v)-domain of definition conformally on a surface S if and only if the fundamental quantities of the first order satisfy $E=G=\lambda(u, v)\neq 0$, $F=0$. See ISOTHERMIC—isothermic map. The coordinates u, v are called **conformal parameters.** The correspondence between surfaces S and $\bar{S}$ determined by $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$ and $x=\bar{x}(u, v)$, $y=\bar{y}(u, v)$, $z=\bar{z}(u, v)$ is conformal at regular points if, and only if, the fundamental quantities of the first order satisfy $E:F:G=\bar{E}:\bar{F}:\bar{G}$. The only conformal correspondences between open sets in three-dimensional Euclidean space are obtained

by inversions in spheres, reflections in planes, translations and magnifications. See CAUCHY—Cauchy-Riemann partial differential equations.

conformal parameters. See above, conformal map.

CON-FOUND′, *v.* (*Statistics.*) In the analysis of variance, in which the factors affecting the variance between groups are under investigation, one or more of the factors affecting the variance may be associated with the other factors in such a manner that separation of the items into groups on the basis of one factor results, at the same time, in the separate groups being different with respect to some other factors. These factors, or sources of variation, are thus **confounded,** since it is not possible to tell which of the several factors are responsible for the group differences. Thus, if a set of data is classified into male and female groups, and at the same time age is associated with sex so that the males are older than the females, the difference between the two groups may be a result of age and/or sex. Confounding is often intentional when it leads to reduced sampling error.

CON′GRU-ENCE, *n.* A statement that two quantities are congruent. The congruence between a and b, with modulus c, is written: $a \equiv b$ (mod. c), and is read "a is *congruent* to b, modulus c." Modulo is quite commonly used instead of modulus. *E.g.*, $5 \equiv 3$ (mod 2). (The parenthesis is not always used.) See CONGRUENT—congruent numbers.

linear congruence. See LINEAR.

quadratic congruence. A congruence of the second degree. Thus its general form is $ax^2+bx+c \equiv 0$ (mod n), where $a \neq 0$. See CONGRUENCE.

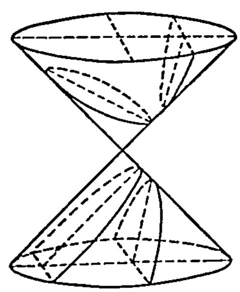
CON′GRU-ENT, *adj.* **congruent figures in geometry.** Figures which can be superposed (placed one upon the other) so that they coincide. This is the definition as given by Euclid. It is embarrassing in that it depends on experience, and such undefined concepts as motion, keeping an object rigid during motion, and ability to move an object to a desired position (if an object is at the center of the sun, a claim that it can be superposed with an object on the earth conflicts with experience). The usual procedure is to take congruence as an undefined concept, restricted by suitable axioms.

congruent matrices. See TRANSFORMATION—congruent transformation.

congruent numbers, or quantities. Two quantities which, when each is divided by a given quantity (called the modulus), give the same remainders; two quantities whose difference is divisible by the modulus. See CONGRUENCE.

congruent transformation. See TRANSFORMATION—congruent transformation.

CON′IC, *n.* Any curve which is the locus of a point which moves so that the ratio of its distance from a fixed point to its distance from a fixed line is constant. The ratio is called the **eccentricity of the curve,** the fixed point the **focus,** and the fixed line the **directrix.** The *eccentricity* is always denoted by e. When e is equal to unity, the conic is a parabola; when less than unity, an ellipse; and when greater than unity, a hyperbola. These are called conics,



or **conic sections,** since they can always be gotten by taking plane sections of a conical surface. The **equation of a conic** can be given in various forms. *E.g.:* (1) If the eccentricity is e, the focus is at the pole, and the directrix perpendicular to the polar axis and at a distance q from the pole, the equation in polar coordinates is

$$\rho = (eq)/(1 + e \cos \theta).$$

This is equivalent to the following equation in Cartesian coordinates (the focus being at the origin and the directrix perpendi-

cular to the x-axis at a distance q from the focus:

$$(1-e^2)x^2+2e^2qx+y^2=e^2q.$$

(2) The general algebraic equation of the second degree in two variables always represents a conic (including here degenerate conics); *i.e.*, an ellipse, hyperbola, parabola, a straight line, a pair of straight lines, or a point, provided it is satisfied by any real points. See DISCRIMINANT—discriminant of a quadratic equation in two variables. (3) See ELLIPSE, HYPERBOLA, and PARABOLA.

acoustical, optical, or focal property of conics. See ELLIPSE—focal property of ellipse, HYPERBOLA—focal property of hyperbola, and PARABOLA—focal property of parabola.

central conics. Conics which have centers—ellipses and hyperbolas. See CENTER—center of a curve.

confocal conics. See CONFOCAL.

degenerate conic. A point, a straight line, or a pair of straight lines, which is a limiting form of a conic. *E.g.*, the parabola approaches a straight line, counted twice, as the plane, whose intersection with a conical surface defines the parabola, moves into a position in which it contains a single element of the conical surface, and the parabola approaches a pair of parallel lines as the vertex of the cone recedes infinitely far; the ellipse becomes a point when the cutting plane passes through the vertex of the cone but does not contain an element; the hyperbola becomes a pair of intersecting lines when the cutting plane contains the vertex of the conical surface. All these limiting cases can be obtained algebraically by variation of the parameters in their several equations. See DISCRIMINANT—discriminant of the general quadratic.

diameter of a conic. See DIAMETER.

focal chords of conics. See FOCAL.

similarly placed conics. Conics of the same type (both ellipses, both hyperbolas, or both parabolas) which have their corresponding axes parallel.

tangent to a general conic. (1) If the equation of the conic in *Cartesian coordinates* is $ax^2+2bxy+cy^2+2\ dx+2ey+f=0$, then the equation of the tangent at the point (x_1, y_1) is

$$ax_1x+b(xy_1+x_1y)+cy_1y \\ +d(x+x_1)+e(y+y_1)+f=0.$$

(2) If the equation of the conic in *homogeneous* Cartesian coordinates is written

$$\sum_{i,j=1}^{3} a_{ij}x_ix_j=0, \quad \text{where } a_{ij}=a_{ji},$$

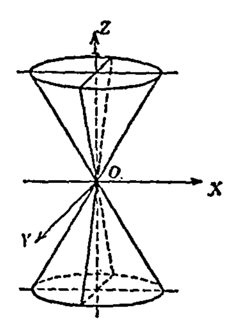
then the equation of the tangent at the point (b_1, b_2, b_3) is:

$$\sum_{i,j=1}^{3} a_{ij}b_ix_j=0.$$

See COORDINATE—homogeneous coordinates.

CON'I-CAL, *adj.* **conical surface.** A surface which can be generated by a line which always passes through a fixed point and intersects a fixed curve. The fixed point is called the **vertex,** or **apex,** of the conical surface, the fixed curve the **directrix,** and the moving line the **generator** or **generatrix.** Any homogeneous equation in rectangular Cartesian coordinates is the equation of a conical surface with vertex at the origin.

circular conical surface. A conical surface whose directrix is a circle and whose vertex is on the line perpendicular to the plane of the circle and passing through the center of the circle. If the vertex is at the origin and the directrix in a plane perpendicular to the z-axis, its equation in rectangular Cartesian coordinates is $x^2+y^2=k^2z^2$.



quadric conical surfaces. Conical surfaces whose directrices are conics.

CON'I-COID, *n.* An ellipsoid, hyperboloid, or paraboloid; usually does not refer to limiting (degenerate) cases.

CON'JU-GATE, *adj.* **complex conjugate of a matrix.** See MATRIX—complex conjugate of a matrix.

conjugate algebraic numbers. Any set of numbers that are roots of the same irreducible algebraic equation with rational coefficients, an equation of the form:

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0.$$

E.g., the roots of $x^2+x+1=0$, which are $\frac{1}{2}(-1+i\sqrt{3})$ and $\frac{1}{2}(-1-i\sqrt{3})$, are conjugate algebraic numbers (in this case conjugate imaginary numbers).

conjugate angles. See ANGLE—conjugate angles.

conjugate arcs. Two arcs whose sum is a complete circle.

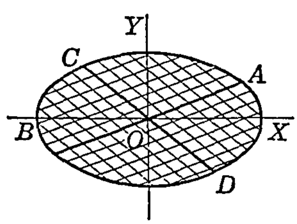
conjugate axis of a hyperbola. See HYPERBOLA.

conjugate complex numbers. See COMPLEX—conjugate complex numbers.

conjugate convex functions. See CONVEX—conjugate convex functions.

conjugate curves. Two curves each of which is a *Bertrand curve* with respect to the other. The only curves having more than one conjugate are plane curves and circular helices.

conjugate diameters. A diameter and the diameter which occurs among the parallel chords that define the given diameter. The conjugate diameters in a circle are perpendicular. The axes of an ellipse are conjugate diameters. But, in general, conjugate diameters are not perpendicular. See DIAMETER—diameter of a conic.



conjugate diameter of a diametral plane of a central quadric. The diameter which contains the centers of all sections of a central quadric by planes parallel to a given diameter. The diametral plane is likewise said to be conjugate to the diameter.

conjugate directions on a surface at a point. The directions of a pair of conjugate diameters of the *Dupin indicatrix* at an elliptic or hyperbolic point P of a surface S. There is a unique direction conjugate to any given direction on S through P, so that there are infinitely many pairs of conjugate directions on S at P. Two conjugate directions which are mutually perpendicular are necessarily principal directions. Conjugate directions are not defined at parabolic or planar points. The characteristic of the tangent plane to S, as the point of contact moves along a curve C on S, is the tangent to the surface in the direction conjugate to the direction of C. See below, conjugate system of curves on a surface.

conjugate dyads and dyadics. See DYAD.

conjugate elements and conjugate subgroups of a group. See TRANSFORM—transform of an element of a group.

conjugate elements of a determinant. *Elements which are interchanged if the* rows and columns of the determinant are interchanged; *e.g.*, the element in the second row and third column is the conjugate of the element in the third row and second column. In general, the elements a_{ij} and a_{ji} are conjugate elements, a_{ij} being the element in the ith row and jth column and a_{ji} the element in the jth row and ith column. See DETERMINANT.

conjugate harmonic functions. See HARMONIC—harmonic functions.

conjugate hyperboloids. Hyperboloids which with suitable choice of coordinate axes have equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

and

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Any plane containing the common axis $z=0$ cuts the two hyperboloids in conjugate hyperbolas. See HYPERBOLOID—hyperboloid of one sheet, hyperboloid of two sheets.

conjugate imaginaries. Imaginary numbers $a+bi$ and $a-bi$, $b\neq 0$. See COMPLEX—conjugate complex numbers.

conjugate points relative to a conic. Two points such that one of them lies on the line joining the points of contact of the two tangents drawn to the conic from the other; two points that are harmonic conjugates of the two points of intersection of the conic and the line drawn through the points; a point and any point on the polar

of the point. *Tech.* If the conic is written in the form

$$\sum_{i,j=1}^{3} a_{ij}x_ix_j, \quad \text{where} \quad x_1, x_2, \quad \text{and} \quad x_3$$

are *homogeneous rectangular Cartesian coordinates* and $a_{ij}=a_{ji}$, then two points, (x_1, x_2, x_3) and (y_1, y_2, y_3), are conjugate points if, and only if,

$$\sum_{i,j=1}^{3} a_{ij}x_iy_j=0.$$

See below, harmonic conjugates with respect to two points.

conjugate radicals. (1) Conjugate binomial surds (see SURD). (2) Radicals that are conjugate algebraic numbers.

conjugate roots. (1) Roots of an equation that are conjugate imaginary numbers. (2) See above, conjugate algebraic numbers.

conjugate ruled surfaces. See RULED.

conjugate space. If f is a *continuous linear functional* defined on a normed linear space N (real or complex), then there is a least number, called the **norm** of f and written $\|f\|$, such that $|f(x)|\leqq\|f\|\cdot\|x\|$ for each x of N. The set of all such functionals is a complete normed linear space, or *Banach space*, which is called the **first conjugate space** of N. The first conjugate space of this space is the **second conjugate space** of N, etc. If N is finite-dimensional, then N and its second conjugate space are identical (*i.e.*, isometric). For any normed linear space N, N is isometric with a subspace of its second conjugate space (see REFLEXIVE—reflexive Banach space). If N is a Hilbert space with a complete orthonormal sequence $u_1, u_2, \cdots$, then the sequence of functions $f_n(x)=(x, u_n)$, $n=1, 2, \cdots$, is a complete orthonormal sequence in the first conjugate space and the correspondence $\sum_1^\infty a_iu_i \leftrightarrow \sum_1^\infty \bar{a}_if_n$ is an isometric correspondence between the two spaces. *Syn.* Adjoint space; dual space.

conjugate subgroups. See ISOMORPHISM.

conjugate system of curves on a surface. Two one-parameter families of curves on a surface S such that through each point P of S there passes a unique curve of each family, and such that the directions of the tangents to these two curves at P are conjugate directions on S at P. See CONJUGATE—conjugate directions on a surface at a point. The parametric curves form a conjugate system if, and only if, $D'\equiv 0$ on S. See SURFACE—fundamental coefficients of a surface. The lines of curvature form a conjugate system, and the only orthogonal conjugate system.

harmonic conjugates with respect to two points. Any two points that divide the line through the two points internally and externally in the same numerical ratio; two points (the 3rd and 4th) which with the given two (the 1st and 2nd) have a *cross ratio* equal to -1 (see RATIO). If two points are harmonic conjugates with respect to two others, the latter two are harmonic conjugates with respect to the first two.

isogonal conjugate lines. See ISOGONAL—isogonal lines.

mean-conjugate curve on a surface. A curve C on a surface S such that C is tangent to a *mean-conjugate direction* on S at each point of C. See below, mean-conjugate directions on a surface at a point.

mean-conjugate directions on a surface. Conjugate directions on the surface S at the point P of S such that the directions make equal angles with the lines of curvature of S at P. The mean-conjugate directions are real if the Gaussian curvature of S is positive at P, and the radius of normal curvature R of S in each of these two directions is the mean of the principal radii there: $R=(\rho_1+\rho_2)/2$. See above, mean-conjugate curves on a surface.

method of conjugate directions. A generalization of the *method of conjugate gradients* for solving a system of n linear equations in n unknowns. In the method of conjugate directions, special restrictions on the conjugate directions to be used do not need to be specified.

method of conjugate gradients. An iterative method, terminating in n steps if there is no round-off error, for solving a system of n equations in n unknowns, $x=(x_1, x_2, \cdots, x_n)$. Starting with an initial estimate x_0 of the solution vector x, the correction steps are in directions that are conjugate to each other relative to the matrix of coefficients, and (to within this constraint) they are successively chosen to be in gradient directions relative to an associated quadratic function that assumes its minimum value 0 at the solution x of the

original problem. The sets of residuals are mutually orthogonal. See CONJUGATE—conjugate points relative to a conic.

method of successive conjugates. In complex variable theory, an iterative method for the approximate evaluation of an analytic function that maps a given nearly circular domain conformally onto the interior of a circle. This mapping might be considered as the second step in a two-step process of mapping a given simply connected domain conformally on the interior of a circle, the mapping of the given domain onto a nearly circular domain having previously been attained through known functions or through a catalogue of conformal maps.

CON-JUNC′TION, *n.* **conjunction of propositions.** The proposition formed from two given propositions by connecting them with the word *and.* *E.g.*, the conjunction of "Today is Wednesday" and "My name is Harry" is the proposition "Today is Wednesday *and* my name is Harry." The conjunction of propositions p and q is usually written as $p \wedge q$, or $p \cdot q$, and read "*p and q.*" The conjunction of p and q is true if and only if both p and q are true. See DISJUNCTION.

CON-JUNC′TIVE, *adj.* **conjunctive transformation.** See TRANSFORMATION—conjunctive transformation.

CON-NECT′ED, *adj.* **arcwise connected set.** A set such that each pair of its points can be joined by a *simple arc* all of whose points are in the set (see SIMPLE—simple arc).

connected set of points. A set that cannot be divided into two sets U and V which have no points in common and which are such that no accumulation point of U belongs to V and no accumulation point of V belongs to U (see DISCONNECTED—disconnected set). The set of all rational numbers is not connected, since the set of rational numbers less than $\sqrt{5}$ and the set greater than $\sqrt{5}$ are both closed in the set of all rational numbers. An *arc-wise connected* set is connected, but a connected set need not be either *arc-wise connected* or *simply connected.*

locally connected set. A set S such that, for any point x of S and neighborhood U of x, there is a neighborhood V of x such that the intersection of S and V is connected and contained in U.

simply connected region. A region such that any closed curve within it can be deformed continuously to a point of the region without leaving the region; a region (area) such that no closed curve lying entirely within the region can enclose a boundary point of the region. A region which is not simply connected is said to be **multiply connected.** See CONNECTIVITY—connectivity number.

CON′NEC-TIV′I-TY, *adj.* **connectivity number.** The connectivity number of a connected curve is 1 *plus* the maximum number of points that can be deleted without separating the curve into more than one piece (this is $2-\chi$, where χ is the Euler characteristic). The connectivity number of a (connected) surface is 1 *plus* the largest number of closed cuts (or cuts joining points of previous cuts, or joining points of the boundary or a point of the boundary to a point of a previous cut, if the surface is not closed) which can be made without separating the surface. This is equal to $3-\chi$ for a closed surface and to $2-\chi$ for a surface with boundary curves. A *simply connected* curve or surface then has connectivity number 1; a connected curve or surface is said to be **doubly connected, triply connected,** etc., according as its connectivity number is 2, 3, etc. The region between two concentric circles in a plane is *doubly connected*; the surface of a doughnut (a torus) is *triply connected.* In the above sense, the connectivity number of a connected *simplicial complex* (which may be a curve or a surface) is 1 *plus* the *1-dimensional Betti number* (modulo 2). However, the connectivity number is sometimes defined to be equal to this Betti number.

CO′NOID, *n.* (1) Any surface generated by a straight line moving parallel to a given plane and always intersecting a given line and given curve. (2) A paraboloid of revolution, a hyperboloid of revolution, or an ellipsoid of revolution. (3) The general paraboloid and hyperboloid, but not the general ellipsoid.

right conoid. A conoid for which the given plane and the given line are mutually orthogonal.

CON′SE-QUENT, *n.* (1) The second term of a ratio; the quantity to which the first term is compared, *i.e.*, the divisor. *E.g.*, in the ratio $\frac{2}{3}$, 3 is the *consequent*, and 2 the *antecedent*. (2) See IMPLICATION.

CON′SER-VA′TION, *n.* **conservation of energy.** See ENERGY.

CON-SER′VA-TIVE, *adj.* **conservative field of force.** A force field such that the work done in displacing a particle from one position to another is independent of the path along which the particle is displaced. In a *conservative field* the work done in moving a particle around any closed path is zero. If the work done on the particle is represented by a line integral

$$\int_C F_x\,dx + F_y\,dy + F_z\,dz,$$

where F_x, F_y, and F_z are the Cartesian components of force in a *conservative field*, then the integrand is an exact differential. The gravitational and electrostatic fields of force are examples of conservative fields, whereas the magnetic field due to current flowing in a wire and fields of force involving frictional effects are non-conservative.

conservative force. The force forming a conservative field.

CON-SIGN′, *v.* **to consign goods, or any property.** To send it to someone to sell, usually at a fixed fee, in contrast to selling on commission.

CON′SIGN-EE′, *n.* A person to whom goods are consigned.

CON-SIGN′OR, *n.* A person who sends goods to another for him to sell; a person who consigns goods.

CON-SIST′EN-CY, *n.* **consistency of equations.** The property possessed by equations when they are all satisfied by at least one set of values of the variables, *i.e.*, their loci all have one or more common points. If they are not satisfied by any one set of values of the variables, they are said to be **inconsistent.** *E.g.*, the equations $x+y=4$ and $x+y=5$ are inconsistent; the equations $x+y=4$ and $2x+2y=8$ are consistent, but are not independent (see INDEPENDENT); and the equations $x+y=4$ and $x-y=2$ are *consistent and independent.* The first pair of equations represents two parallel lines, the second represents two coincident lines, and the third represents two distinct lines intersecting in a point, the point whose coordinates are (3, 1).

consistency of linear equations. A **linear equation in two variables** is the equation of a line in the plane. Therefore a single equation has an unlimited number of solutions. Two equations have a unique simultaneous solution if the lines they represent intersect and are not coincident; there is no solution if the lines are parallel and not coincident; there is an unlimited number of solutions if the lines are coincident. These correspond to the three cases of the following discussion. Consider the equations: $a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$, where at least one of a_1, b_1 and at least one of a_2, b_2 is not zero. Multiply the first equation by b_2 and the second by b_1, then subtract. This gives $(a_1b_2-a_2b_1)x=b_2c_1-b_1c_2$. Similarly, $(a_1b_2-a_2b_1)y=a_1c_2-a_2c_1$,

or $$x\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix},$$

and $$y\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

Three cases follow: I. If the determinant of the coefficients

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is not zero, one can divide by it and secure unique values for x and y. The equations are then consistent and independent. The equations $2x-y=1$ and $x+y=3$ reduce in the above way to

$$3x=4 \quad \text{and} \quad 3y=5$$

and have the unique simultaneous solution $x=\frac{4}{3}$, $y=\frac{5}{3}$. II. If the determinant of the coefficients is zero and one of the determinants formed by replacing the coefficients of x (or of y) by the constant terms is not zero, there is no solution; *i.e.*, the equations are *inconsistent.* The equations $2x-y=1$ and $4x-2y=3$ reduce to

$$0\cdot x=1 \quad \text{and} \quad 0\cdot y=2,$$

which have no solution. III. If all three determinants entering are zero, there results $0 \cdot x = 0$ and $0 \cdot y = 0$. The equations are then consistent but not independent. This is the situation for the equations $x - y = 1$ and $2x - 2y = 2$. An infinite number of pairs of values of x and y can be found that satisfy both of these equations. A **linear equation in three variables** is the equation of a plane in space. Therefore a single equation has an unlimited number of solutions. Two equations either represent parallel planes and have no common solution or else represent planes which intersect in a line or coincide and the equations have an unlimited number of solutions. Eliminating the variables, two at a time, from the equations

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3,$$

gives $Dx = K_1$, $Dy = K_2$, and $Dz = K_3$, where K_1, K_2, and K_3 are the determinants resulting from substituting the d's in the determinant of the coefficients, D, in place of the a's, b's, and c's, respectively. Three cases arise: I. If $D \neq 0$, it can be divided out and a unique set of values for x, y, and z obtained; *i.e.*, the three planes representing the three equations then intersect in a point and the equations are *consistent* (and also independent). II. If $D = 0$ and at least one of K_1, K_2, and K_3 is not zero, there is no solution; the three planes do not have any point in common and the three equations are *inconsistent*. III. If $D = 0$ and $K_1 = K_2 = K_3 = 0$, three cases arise: *a*). Some second-order determinant in D is not zero, in which case the equations have infinitely many points in common; the planes (the loci of the equations) intersect in a line and the equations are *consistent*. *b*). Every second-order minor in D is zero and a second-order minor in K_1, K_2, or K_3 is not zero. The planes are then parallel, but at least one pair do not coincide; the equations are *inconsistent*. *c*). All the second-order minors in D, K_1, K_2 and K_3 are zero. The three planes then coincide and the equations are *consistent* (but not independent). The general situation of m **linear equations in n unknowns** is best handled by consideration of matrix rank (see MATRIX—rank of a matrix): the equations are consistent if and only if the rank of the matrix of the coefficients is equal to the rank of the augmented matrix. If the constant terms in a system of linear equations are all zero (the equations are homogeneous), then the equations have a trivial solution (each unknown equal to zero). For **n homogeneous linear equations in m unknowns**: (1) If $n < m$, the equations have a nontrivial solution (not all unknowns zero). (2) If $n = m$, the equations have a nontrivial solution if, and only if, the determinant of the coefficients is equal to zero. (3) If $n > m$, the equations have a nontrivial solution if, and only if, the *rank* of the *matrix* of the coefficients is less than m. These are simply the special case of the results for n linear equations in m unknowns when the constant terms are all zero.

CON-SIST'ENT, *adj.* **consistent assumptions, hypotheses, postulates.** Assumptions, hypotheses, postulates that do not contradict each other.

consistent equations. See CONSISTENCY.

consistent estimate. (*Statistics.*) A consistent estimate is one that tends to be closer to the true value as the size of the sample increases. Precisely, an estimate of a parameter which *converges in probability* to the true parameter value, as the sample size increases ($n \to \infty$), is a consistent estimate of the parameter. See CONVERGENCE—convergence probability, and PROBABILITY.

CON-SOL'I-DAT'ED, *adj.* **consolidated annuities.** British government bonds irredeemable except at the pleasure of the government.

CON'SOLS, *n.* Same as CONSOLIDATED ANNUITIES. See CONSOLIDATED.

CON'STANT, *n.* A quantity whose value does not change, or is regarded as fixed, during a certain discussion or sequence of mathematical operations.

absolute constant. A constant that never changes in value, such as numbers in arithmetic.

arbitrary constant. A constant that may be assigned different values. See below, constant of integration.

constant of integration. An arbitrary constant that must be added to any function arising from integration to obtain all the primitives. The integral, $\int 3x^2\,dx$, can have any of the values x^3+c, where c is a constant, because the derivative of a constant is zero; further, it follows from the mean value theorem that there are no other values for the integral. See MEAN—mean value theorems for derivatives.

constant of proportionality. See FACTOR—factor of proportionality.

constant speed and velocity. An object is said to have **constant speed** if it passes over equal distances in equal intervals of time (although the object need not move in a straight line). It has **constant velocity** if it passes over equal distances *in the same direction* in equal intervals of time (this means that the *instantaneous velocity* is the same *vector* at each point of the path; see VELOCITY). Constant velocity is also sometimes called **uniform (rectilinear) velocity** and **uniform motion** (although uniform motion is sometimes used in such a sense as uniform circular motion, meaning motion around a circle with constant speed).

constant term in an equation or function. A term which does not contain a variable. *Syn.* Absolute term.

essential constant. A set of essential constants in an equation is a set of *arbitrary constants* which: (1) cannot be replaced by a smaller number (changing the form of the equation if desired) so as to have a new equation which represents essentially the same family of curves, or: (2) are equal in number to the number of points needed to determine a unique member of the family of curves represented by the equation, or: (3) are arbitrary constants in an equation $y=f(x)$ for which the number of arbitrary constants is equal to the minimum order of a differential equation which has $y=f(x)$ as a solution. The linear equation $y=Ax+B$ defines a family of straight lines; it has 2 essential constants, since 2 points (not in a vertical line) determine a unique line of the family, and $y=Ax+B$ is a solution of the differential equation $y''=0$, which is of order 2. The equation $ax+by+c=0$ does not have 3 essential constants, since 2 points determine a line of the family of lines it represents; also, it represents the same family of curves as $y=Ax+B$, except for the lines $x=$constant. The equation $y^2+bxy-abx-(a-c)y-ac=0$ has 4 arbitrary constants; these are not essential, since the equation can be factored as $(y-a)(y+bx+c)=0$ and has the same family of curves as the equation $y=mx+b$. The **number of essential constants** in an equation is the number of essential constants to which the arbitrary constants can be reduced. *E.g.*, the number of essential constants in $y^2+bxy-abx-(a-c)y-ac=0$ is 2. The constants $A_1, \cdots, A_n$ in the equation

$$y=A_1u_1(x)+A_2u_2(x)+\cdots+A_nu_n(x)$$

are essential if and only if the functions $u_1, \cdots, u_n$ are *linearly independent.*

gravitational constant. See GRAVITATION—law of universal gravitation.

Lamé's constants. See LAMÉ'S CONSTANTS.

CON-STRAIN'ING, *p. adj.* **constraining forces (constraints).** (1) Those forces that tend to prevent a particle's remaining at rest or moving at a uniform velocity in a straight line (according to Newton's first law of motion). (2) Those forces that are exerted perpendicularly to the direction of motion of a particle.

CON-STRUCT', *v.* To draw a figure so that it meets certain requirements; usually consists of drawing the figure and proving that it meets the requirements. *E.g.*, to construct a line perpendicular to another line, or to construct a triangle having three given sides.

CON-STRUC'TION, *n.* (1) The process of drawing a figure that will satisfy certain given conditions. See CONSTRUCT. (2) Construction in proving a theorem; drawing the figure indicated by the theorem and adding to the figure any additional parts that are needed in the proof. Such "additional" lines, points, etc., are usually called **construction lines, points, etc.**

CON'TACT, *n.* **chord of contact.** See CHORD.

order of contact. See ORDER—order of contact of two curves.

point of contact. See TANGENT—tangent to a curve.

CON'TENT, *n.* **content of a set of points.** The **exterior content** (or **outer content**) of a set of points E is the greatest lower bound of the sums of the lengths of a finite number of intervals (open or closed) such that each point of E is in one of the intervals, for all such sets of intervals. The **interior content** (or **inner content**) is the least upper bound of the sums of the lengths of a finite number of nonoverlapping intervals such that each interval is completely contained in E, for all such sets of intervals; or (equivalently) the difference between the length of an interval I containing E and the exterior content of the complement of E in I. Also called the **exterior Jordan content** and **interior Jordan content.** If the exterior content is equal to the interior content, their common value is the **(Jordan) content.** If the exterior content is zero, then the interior content is also, and the set is said to have **(Jordan) content zero.** The set of rational numbers in (0, 1) has exterior content of 1 and interior content of zero; the set $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$ has content zero. This definition is for sets of points on a line. A similar definition holds for sets in the plane, or in n-dimensional Euclidean space.

CON-TIN'GENCE, *n.* **angle of contingence.** The angle between the positive directions of the tangents to a given plane curve at two given points of the curve.

angle of geodesic contingence. For two points P_1 and P_2 of a curve C on a surface, the angle of geodesic contingence is the angle of intersection of the geodesics tangent to C at P_1 and P_2. See above, angle of contingence.

CON-TIN'GEN-CY, *n.* **contingency table.** (*Statistics.*) If a set of items can be classified jointly on the basis of two factors, of which one has q subclasses and the other p subclasses, the resulting table of classification is a $q \times p$ or *q by p* table. A bivariate correlation scattergram is a special type of contingency table, in which the two variables are classified by the values, or intervals of values, which they may assume.

two-by-two contingency table. If a group of items can be jointly classified on the basis of two joint factors, and if each factor is a dichotomy, then a two-by-two table results. *E.g.*, individuals may be classified by sex and political party, thus:

Party	Sex		
	Male	Female	
Repub.	10	9	19
Demo.	8	9	17
	18	18	36

Also known as a **fourfold table.**

CON-TIN'GENT, *adj.* **contingent annuity and life insurance.** See ANNUITY, and INSURANCE—life insurance.

CON-TIN'U-A'TION, *adj., n.* **analytic continuation of an analytic function of a complex variable.** see ANALYTIC.

continuation notation. Three center dots or dashes following a few indicated terms. In case there is an infinite number of terms, the most common usage is to indicate a few terms at the beginning of the set, follow these with three center dots, write the general term, and add three center dots as follows:

$$1+x+x^2+\cdots+x^n+\cdots.$$

continuation of sign in a polynomial. Repetition of the same algebraic sign before successive terms.

CON-TIN'UED, *adj.* **continued equality.** See EQUALITY.

continued fraction. See FRACTION—continued fraction.

continued product. A product of an infinite number of factors, or a product such as $(2\times 3)\times 4$ of more than two factors; denoted by Π, that is, capital pi, with appropriate indices.

E.g., $(\frac{1}{2})(\frac{2}{3})(\frac{3}{4})\cdots$

$$[n/(n+1)]\cdots=\prod_{n=1}^{\infty}[n/(n+1)]$$

is a continued product.

CON'TI-NU'I-TY, *n.* The property of being continuous.

axiom of continuity. See AXIOM.

equation of continuity. A fundamental equation of fluid mechanics, namely, $d\rho/dt + \rho\nabla\cdot\eta = 0$, where ρ is the density of the fluid and η is the velocity vector. A more general equation takes account of sources and sinks at which fluid is created and destroyed.

principle of continuity. See AXIOM—axiom of continuity.

CON-TIN'U-OUS, *adj.* **absolutely continuous function.** A function $f(x)$ is absolutely continuous on a closed interval $[a, b]$ if for any positive number ϵ another positive number η can be determined so that if $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ is any finite set of nonoverlapping intervals such that the sum of the lengths of the intervals is less than η, then $\sum_{i=1}^{n} |f(a_i) - f(b_i)| < \epsilon$. The definition remains equivalent to this if it is changed to allow a countable number of intervals. An absolutely continuous function is *continuous* and of *bounded variation.*

continuous annuity. An annuity payable continuously. Such an annuity cannot occur, but has theoretical value. Formulas for this sort of annuity are limiting forms of the formulas for noncontinuous annuities, when the number of payments per year increases without limit while the nominal rate and annual rental remain fixed. The results differ very little from annuities having a very large number of payments per year. Approximate present values for a single life continuous annuity of one dollar is that of a single life annuity payable annually at the end of the year plus $\frac{1}{2}$ of a dollar; or that of a single life annuity payable annually at the beginning of the year minus $\frac{1}{2}$ of a dollar.

continuous conversion of compound interest. See CONVERSION.

continuous correspondence of points. A correspondence (function, mapping, or transformation) which associates with each point of a space D a unique point of a space R is continuous if, whenever x corresponds to x^* and W is a neighborhood of x^*, there is a neighborhood U of x such that W contains all points of R which are associated with points of U. A correspondence which maps D onto R is continuous if and only if the inverse of each open set of R is open in D (or if and only if the inverse of each closed set of R is closed in D), where the inverse of a set W in R is the set of all points of D which are associated with points of W. See OPEN—open mapping.

continuous function. **A function of one variable is continuous at a point** if its value can be made as nearly equal to its value at the point as one pleases by restricting the value of the independent variable to values sufficiently near the given value. *Tech.* $f(x)$ is continuous for $x = a$ if $f(x)$ is defined for all x in some neighborhood of a and $\lim_{x\to a} f(x) = f(a)$, or, equivalently, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta$, then $f(x)$ is defined and

$$|f(x) - f(a)| < \epsilon.$$

A function of a *real variable* is **continuous in an interval** if it is continuous at each point of the interval. A function of a *complex variable* is **continuous in a domain** if it is continuous at each point of the domain. **A function $f(x, y)$ of two variables** x and y **is continuous at a point** (a, b) (*i.e.*, for $x = a$ and $y = b$) if it is defined in the neighborhood of (a, b) and if $f(x, y)$ approaches $f(a, b)$ when x and y approach a and b, respectively, in any way whatever; or, equivalently, if for an $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta$ and $|y - b| < \delta$, then $f(x, y)$ is defined and

$$|f(x, y) - f(a, b)| < \epsilon.$$

A function of two variables is **continuous in a region** if it is continuous at every point of the region. **A function $f(x_1, x_2, \cdots, x_n)$ of the n variables $x_1, x_2, \cdots, x_n$ is continuous at the point $(a_1, \cdots, a_n)$** if it is defined in the neighborhood of the point and the limit of the function as the variables approach their values at the point (in any way whatever) is equal to $f(a_1, \cdots, a_n)$, or equivalently, if for any $\epsilon > 0$ there exists a $\delta > 0$ such that, if the distance between the points $(a_1, \cdots, a_n)$ and $(x_1, \cdots, x_n)$ (*i.e.*, $\sqrt{\sum_{i=1}^{n} |x_i - a_i|^2}$) is less than δ, then $f(x_1, \cdots, x_n)$ is defined and

$$|f(x_1, \cdots, x_n) - f(a_1, \cdots, a_n)| < \epsilon.$$

The function is said to be **continuous in a region** if it is continuous for all points in that region. See DISCONTINUITY.

continuous game. See GAME.

continuous on the left or right. A real-valued function $f(x)$ is **continuous on the right** at a point x_0 if for any $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ if $x_0 < x < x_0 + \delta$, and to be **continuous on the left** if for any $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ if $x_0 - \delta < x < x_0$. A function is continuous on the right (or left) on an interval (a, b) if it is continuous on the right (or left) at each point of (a, b). See LIMIT—limit on the right.

continuous in the neighborhood of a point. A function is *continuous in the neighborhood of a point* if there exists a neighborhood of the point such that the function is continuous at each point of the neighborhood. Thus $f(x_1, x_2, \cdots, x_n)$ is continuous in the neighborhood of $(a_1, \cdots, a_n)$ if there exists a positive number ϵ such that f is continuous at $(x_1, \cdots, x_n)$ if $|x_i - a_i| < \epsilon$ for each i, or if

$$\left[\sum_{i=1}^{n} |x_i - a_i|^2\right]^{1/2} < \epsilon.$$

continuous surface in a given region. The graph of a continuous function of two variables; the locus of the points whose rectangular coordinates satisfy an equation of the form $z = f(x, y)$, where $f(x, y)$ is a continuous function of x and y in the region of the x, y plane which is the projection of the surface on that plane. *E.g.*, a sphere about the origin is a continuous surface, for $z = \sqrt{r^2 - (x^2 + y^2)}$ is a continuous function on, and within, the circle $x^2 + y^2 = r^2$. To determine the entire sphere, both signs of the radical must be considered. Thought of in this way, the sphere is a multiple (two) valued surface.

continuous transformation. See above, continuous correspondence.

piecewise continuous function. See PIECEWISE.

semicontinuous function. If for any arbitrary positive number ϵ a real-valued function $f(x)$ satisfies the relation $f(x) < f(x_0) + \epsilon$ for all x in some neighborhood of x_0, the function is said to be **upper semicontinuous** at x_0; if $f(x) > f(x_0) - \epsilon$ for all x in some neighborhood of x_0, then $f(x)$ is **lower semicontinuous** at x_0. Equivalent conditions are, respectively, that the *limit superior* of $f(x)$ as $x \to x_0$ be $\leqq f(x_0)$ and that the limit inferior be $\geqq f(x_0)$. A function is upper semicontinuous (or lower semicontinuous) on an interval or region R if, and only if, it is so at each point of R. The function defined by $f(x) = \sin x$ if $x \neq 0$ and $f(0) = 1$ is upper semicontinuous, but not lower semicontinuous, at $x = 0$.

CON-TIN'U-UM, *n.* (*pl. continua*). A *compact connected* set. It is usually required that the set contain at least two points, which implies that it contains an infinite number of points. The set of all real numbers (rational and irrational) is called the **continuum of real numbers.** Any closed interval of real numbers is a continuum. A continuum is topologically equivalent to a closed interval of real numbers if and only if it does not contain more than two noncut points (see CUT).

real continuum of numbers. The totality of rational and irrational real numbers.

CON'TOUR, *adj.* **contour integral.** For a complex-valued function $f(z)$ and a curve C joining points p and q in the complex plane (or on a Riemann surface), let $z_0 = p$, $z_1, \cdots, z_n = q$ be $n+1$ arbitrary points on C which divide C into n consecutive segments, $\bar{z}_i$ be a point on the closed segment of C which joins z_{i-1} to z_i, and δ be the largest of the numbers $|z_i - z_{i-1}|$. Then the contour integral

$$\int_p^q f(z)\,dz$$

is the limit of $\sum_{i=1}^{n} f(\bar{z}_i)(z_i - z_{i-1})$ as δ approaches zero, if this limit exists. If $f(z)$ is continuous on C and C is *rectifiable*, this contour integral exists; if it is also true that F is a function such that $dF(z)/dz = f(z)$ at each point of C, then $\int_p^q f(z)\,dz = F(q) - F(p)$. With suitable restrictions on the nature of C, this contour integral can be evaluated as either of the line integrals

$$\int f(z) z'(t)\,dt$$

or

$$\int (u\,dx - v\,dy) + i \int (v\,dx + u\,dy),$$

where $z=z(t)$ is an equation for C and

$$f(z)=u(x, y)+iv(x, y)$$

with $z=x+iy$ and $u(x, y)$ and $v(x, y)$ real. See CAUCHY—Cauchy's integral formula, Cauchy's integral theorem.

contour lines. (1) Projections on a plane of all the sections of a surface by planes parallel to this given plane and equidistant apart; (2) lines on a map which pass through points of equal elevation. Useful in showing the rapidity of ascent of the surface, since the contour lines are thicker where the surface rises faster. *Syn.* Level lines.

CON-TRAC'TION, *n.* **contraction of a tensor.** The operation of putting one contravariant index equal to a covariant index and then summing with respect to that index. The resultant tensor is called the **contracted tensor.**

CON'TRA-POS'I-TIVE, *n.* **contrapositive of an implication.** The implication which results from replacing the antecedent by the negation of the consequent and the consequent by the negation of the antecedent. *E.g.*, the contrapositive of "If x is divisible by 4, then x is divisible by 2" is "If x is not divisible by 2, then x is not divisible by 4." An implication and its contrapositive are *equivalent*—they are either both true or both false. The contrapositive of an implication is the *converse* of the *inverse* (or the *inverse* of the *converse*) of the implication.

CON'TRA-VA'RI-ANT, *adj.* **contravariant derivative of a tensor.** The *contravariant derivative* of a tensor $t^{a_1\cdots a_p}_{b_1\cdots b_q}$ is the tensor

$$t^{a_1\cdots a_p,i}_{b_1\cdots b_q}=g^{i\sigma}t^{a_1\cdots a_p}_{b_1\cdots b_q,\sigma},$$

where the *summation convention* applies, g^{ij} is $1/g$ times the cofactor of g_{ij} in the determinant $g=\{g_{ij}\}$, and $t^{a_1\cdots a_p}_{b_1\cdots b_q,\sigma}$ is the *covariant derivative.* See COVARIANT—covariant derivative of a tensor, and CHRISTOFFEL—Christoffel symbols.

contravariant indices. See TENSOR.

contravariant tensor. See TENSOR.

contravariant vector field. See VECTOR.

CON-TROL', *adj., n.* **control chart.** (*Statistics.*) A graph of the results of sampling the product of a process; usually consists of a horizontal line indicating expected mean value of some characteristic of quality and two lines on either side indicating the allowable extent of sampling and/or random production deviations. Usually used for control of quality of production.

control component. In a computing machine, any component that is used in manual operation, for starting, testing, etc.

control group. (*Statistics.*) In estimating the effect of a given factor it may be necessary to compare the result with another situation in which the tested factor is absent (or held constant). The *control group* is that sample in which the factor is absent.

quality control. A statistical method of directing and testing the output of a production process in order to detect major, nonchance causes of variation in quality of output.

statistical control. A state of *statistical control* exists if a process of obtaining items under essentially the same conditions is such that the variations in the values of the items are random, cannot be attributed to any assignable causes, and the mean values of the subgroups show no trend. Values that conform to what would be expected under a random sampling scheme from a hypothetical normal population also is frequently regarded as characterizing a state of statistical control.

CON-VERGE', *v.* To draw near to. (1) A series is said to converge when the sum of the first n of its terms approaches a limit as n increases without bound (see LIMIT). (2) A curve is said to converge to its asymptote, or to a point, when the distance from the curve to the asymptote, or point, approaches zero; *e.g.*, the polar spiral, $r=1/\theta$, converges to the origin; the curve $xy=1$ converges to the x-axis as x increases and to the y-axis as y increases. (3) A variable is sometimes said to converge to its limit. See various headings under CONVERGENCE.

CON-VER'GENCE, *n.* See CONVERGE.

absolute convergence of an infinite product. See PRODUCT—infinite product.

absolute convergence of an infinite series. The property that the sum of the absolute values of the terms of the series form a

convergent series. Such a series is said to **converge absolutely** and to be **absolutely convergent**; $1-\frac{1}{2}+\frac{1}{2}^2-\frac{1}{2}^3+\cdots+(-1)^{n-1}\frac{1}{2}^{n-1}+\cdots$ is absolutely convergent. See SUM—sum of an infinite series; and below, conditional convergence.

circle of convergence. For a power series,

$$c_0+c_1(z-a)+c_2(z-a)^2+\cdots+c_n(z-a)^n+\cdots,$$

there is an R such that the series converges (absolutely) if $|z-a|<R$ and diverges if $|z-a|>R$. The circle of radius R with center at a in the complex plane is the **circle of convergence** (its equation is $|z-a|=R$); R is the **radius of convergence** (R may be zero or infinite). The series converges uniformly in any circle with center at a and radius less than R. The series may either converge or diverge on the circumference of the circle. *E.g.*, $\sum_{n=1}^{\infty}(3z)^n/n$ converges absolutely within the circle whose radius is $\frac{1}{3}$ and whose center is the origin, and diverges outside this circle. It converges for $z=-\frac{1}{3}$, but diverges for $z=+\frac{1}{3}$. See below, interval of convergence.

conditional convergence. An infinite series is *conditionally convergent* if it is convergent and there is another series which is divergent and which is such that each term of each series is also a term of the other series (the second series is said to be derived from the first by a *rearrangement* of terms); *i.e.*, an infinite series is *conditionally convergent* if its convergence depends on the order in which the terms are written. A convergent series is conditionally convergent if and only if it is not absolutely convergent. *E.g.*, the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is conditionally convergent because it converges and the series $1+\frac{1}{2}+\frac{1}{3}+\cdots$ diverges.

convergence of an infinite product. See PRODUCT—infinite product.

convergence of an infinite sequence. See SEQUENCE—limit of a sequence.

convergence of an infinite series. See SUM—sum of an infinite series.

convergence of an integral. The property that an integral possesses when it approaches a limit as the variable (or variables) which enters into its limits runs through some sequence of values; *e.g.*, the integral

$$\int_2^y (1/x^2)\,dx = -1/y+\tfrac{1}{2}$$

approaches $\frac{1}{2}$ as y increases without bound.

convergence in the mean. A sequence of functions $f_n(x)$ is said to *converge in the mean of order p* to $F(x)$ on the interval or region Ω if

$$\lim_{n\to\infty}\int_\Omega |F(x)-f_n(x)|^p\,dx=0.$$

When the term **convergence in the mean** is used without qualification, it is sometimes understood to mean "convergence in the mean of order two" and sometimes "convergence in the mean of order one."

convergence in measure. A sequence $\{f_n\}$ of measurable functions is said to converge in measure to F on the set S if, for any pair (ϵ,η) of positive numbers, there is a number N such that the measure of E_n is less than η when $n>N$, where E_n is the set of all x for which

$$|F(x)-f_n(x)|<\epsilon.$$

If S is of finite measure, then a sequence $\{f_n\}$ of measurable functions converges in measure to F if

$$\lim_{n\to\infty} f_n(x)=F(x)$$

for all x except a set of measure zero.

convergence probability. See PROBABILITY.

interval of convergence. A power series,

$$c_0+c_1(x-a)+c_2(x-a)^2+\cdots+c_n(x-a)^n+\cdots,$$

either converges for all values of x, or there is a number R such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$. The interval $(a-R, a+R)$ is the *interval of convergence* (R may be zero). The series converges *absolutely* if $|x-a|<R$ and converges *uniformly* in any interval (A, B) with $a-R<A\leqq B<a+R$. See ABEL—Abel's theorem on power series; and above, circle of convergence.

tests for convergence of an infinite series. See ABEL, ALTERNATING, COMPARISON, DIRICHLET, RATIO, and NECESSARY—necessary condition for convergence.

uniform convergence of a series. An infinite series whose terms are functions of a variable is **uniformly convergent** if the

numerical value of the remainder after the first n terms is as small as desired *throughout the given interval* for n greater than a sufficiently large chosen number. *Tech.* If the sum of the first n terms of a series is $s_n(x)$ the series converges uniformly to $f(x)$ in (a, b) if for arbitrary positive ϵ there exists an N (dependent upon ϵ) such that

$$|f(x)-s_n(x)|<\epsilon$$

for all n greater than N and all x in the interval (a, b). Equivalently, $s_n(x)$ converges uniformly on (a, b) if, for arbitrary positive ϵ, there exists an N (dependent upon ϵ) such that $|s_{n+p}(x)-s_n(x)|<\epsilon$ for all $n>N$, for all positive p, and for all x in the interval (a, b). *E.g.*, the series

$$1+x/2+(x/2)^2+\cdots+(x/2)^{n-1}+\cdots$$

converges uniformly for x in any closed interval contained in the interval $(-2, 2)$; but does not converge uniformly for $-2<x<2$, since the absolute value of the difference of

$$f(x)=1/(1-x/2) \quad \text{and}$$
$$s_n(x)=[1-(x/2)^n]/(1-x/2)$$

is $|(x/2)^n/(1-x/2)|$, which (for any fixed n) becomes infinite as x approaches 2. See ABEL, DIRICHLET, and WEIERSTRASS for tests of uniform convergence.

uniform convergence of a set of functions. See UNIFORM.

CON-VER'GENT, *adj., n.* Possessing the property of convergence. See various headings under CONVERGENCE.

convergent of a continued fraction. The fraction terminated at one of the quotients. See FRACTION—continued fraction.

permanently convergent series. Series which are convergent for all values of the variable, or variables, involved in its terms; *e.g.*, the exponential series, $1+x+x^2/2!+x^3/3!+\cdots$, is equal to e^x for all values of x; hence the series is *permanently* convergent.

CON'VERSE, *n.* **converse of a theorem (or implication).** The theorem (or implication) resulting from interchanging the hypothesis and conclusion. If only a part of the conclusion makes up the new hypothesis, or only a part of the old hypothesis makes up the new conclusion, the new statement is sometimes spoken of as *a* converse of the old. *E.g.*, the converse of "If x is divisible by 4, then x is divisible by 2" is the false statement "If x is divisible by 2, then x is divisible by 4." If an implication is true, its converse may be either true or false. If an implication $p\to q$ and its converse $q\to p$ are both true, then the *equivalence* $p\leftrightarrow q$ is true. See INVERSE—inverse of an implication.

CON-VER'SION, *adj., n.* **continuous conversion of compound interest.** Finding the limit of the amount, at the given rate of interest, as the length of the period approaches zero. *I.e.*,

$$\lim_{m\to\infty}(1+j/m)^m$$

where j is the fixed nominal rate and m the number of interest periods per year. This limit is e^j. See *e*.

conversion from centigrade to Fahrenheit (or Fahrenheit to centigrade). Expressing a given temperature as recorded by one scale in terms of the other scale. The formulas for doing this are:

$$T_f=\tfrac{9}{5}T_c+32$$

and

$$T_c=\tfrac{5}{9}(T_f-32).$$

conversion interval, or period. See INTEREST.

conversion tables. Tables such as those giving the insurance premiums (annual or single), at various rates of interest, which are equivalent to a given annuity.

frequency of conversion of compound interest. The number of times a year that interest is compounded.

CON'VEX, *adj.* **convex curve in a plane.** A curve such that any straight line cutting the curve cuts it in just two points.

conjugate convex functions. If $f(x)$, with $f(0)=0$, is *strictly increasing* for $x\geqq 0$, and $g(y)$ is its *inverse*, then the convex functions $F(x)=\int_0^x f(t)\,dt$ and $G(y)=\int_0^y g(t)\,dt$ are said to be conjugate. More generally, for a convex function $F(x_1, x_2, \cdots, x_n)=F(x)$ defined in a domain D, the conjugate convex function is defined by

$$G(y_1, y_2, \cdots, y_n)=\text{l.u.b.}\left[\sum_{i=1}^n x_iy_i-F(x)\right]$$

for x in D. See YOUNG'S INEQUALITY.

convex function. A real function $y=f(x)$, defined in the interval I, $a<x<b$, is said to be convex in I provided that, for each x_1, x_2, x, with $a<x_1<x<x_2<b$, we have $f(x)\leqq l(x)$, where $l(x)$ is the linear function coinciding with $f(x)$ at x_1 and x_2. *I.e.*, $f(x)$ is *convex* provided in each subinterval of I the graph of the function $y=f(x)$ lies nowhere above its secant line. A necessary and sufficient condition that $f(x)$ be convex in I is that for x_1, x_2, λ, with $a<x_1<x_2<b$, $0<\lambda<1$, we have $f[\lambda x_1+(1-\lambda)x_2]\leqq\lambda f(x_1)+(1-\lambda)f(x_2)$. A convex function is necessarily continuous; but see below, convex in the sense of Jensen. If $f(x)$ has a continuous second derivative, it is convex if, and only if, $f''(x)\geqq 0$ at each point of I.

convex hull of a set. The smallest convex set which contains all the points of the set; the intersection of all the convex sets which contain the given set. The **closed convex hull** of a set is the smallest closed convex set which contains the given set, and is also the closure of the convex hull.

convex in the sense of Jensen. A real function $f(x)$ defined in the interval I, $a<x<b$, is said to be convex in the sense of Jensen in I provided that, for each x_1, x_2 with $a<x_1<x_2<b$, we have

$$f\left(\frac{x_1+x_2}{2}\right)\leqq\frac{1}{2}\,[f(x_1)+f(x_2)].$$

A function convex in the sense of Jensen is not necessarily continuous, but, if such a function is bounded in any subinterval of I, it is necessarily continuous in I. See above, convex function.

convex linear combination. See LINEAR—linear combination.

convex polygon and polyhedron. See POLYGON, and POLYHEDRON.

convex sequence. A sequence of numbers $a_1, a_2, a_3, \cdots$ is *convex* if $a_{i+1}\leqq \frac{1}{2}(a_i+a_{i+2})$ for all i (or for $1\leqq i\leqq n-2$ if the sequence is the finite sequence $a_1, a_2, \cdots, a_n$). If the inequality is reversed, the sequence is said to be *concave*.

convex set. A set that contains the line segment joining any two of its points; in a *vector space*, a set such that $rx+(1-r)y$ is in the set for $0<r<1$ if x and y are in the set. A set is **locally convex** if for any point x of the set and any neighborhood U of x there is a neighborhood V of x which is convex and contained in U. A convex set is called a **convex body** if it has an interior point (it is sometimes also required that a convex body be *closed* or *compact*).

convex surface. A surface such that any plane section of it is a convex curve.

convex toward a point (or a given line). Said of a curve which bulges toward the point (or line). *Tech.* An arc of a curve is convex toward a point (or line) when every segment of it, cut off by a chord, lies on the same side of the chord as does the point (or line). If there exists a horizontal line such that a curve lies above (below) it and is convex toward it, the curve is said to be *convex down* (*convex up*). A sufficient condition that a curve, whose equation is $y=f(x)$, be convex upward (or downward) in a given interval is that the second derivative of the function, d^2y/dx^2, be negative (or positive) at all but a finite number of points of the interval. A surface is said to be convex toward (or away from) a plane when every plane perpendicular to this plane cuts it in a curve which is convex toward (or away from) the line of intersection of the two planes.

generalized convex function. Let $\{F\}$ be a family of functions which are continuous on an interval (a, b) and such that, for any two points (x_1, y_1) and (x_2, y_2) with x_1 and x_2 two different numbers of the interval (a, b), there is a unique member F of $\{F\}$ satisfying

$$F(x_1)=y_1;\quad F(x_2)=y_2.$$

A function f is a *generalized convex function* relative to $\{F\}$, or a *sub-F* function in (a, b), provided that for any numbers x_1, ξ, x_2 with $a<x_1<\xi<x_2<b$ we have $f(\xi)\leqq F(\xi)$, where F is the member of $\{F\}$ for which $F(x_1)=f(x_1)$ and $F(x_2)=f(x_2)$.

logarithmically convex function. A function whose logarithm is convex. The *gamma function* is the only logarithmically convex function which is defined and positive for $x>0$, satisfies the functional equation $\Gamma(x+1)=x\Gamma(x)$, and for which $\Gamma(1)=1$.

strictly convex space. A normed linear space which has the property that if $\|x+y\|=\|x\|+\|y\|$ and $y\neq 0$, then there is a number t such that $x=ty$. A finite-dimensional space is *strictly convex* if and only if it is *uniformly convex*, but an infinite-

dimensional space can be strictly convex without being uniformly convex.

uniformly convex space. A normed linear space is *uniformly convex* if for any $\epsilon>0$ there is a $\delta>0$ such that $\|x-y\|<\epsilon$ if $\|x\|<1+\delta$, $\|y\|<1+\delta$, and $\|x+y\|>2$. A finite-dimensional space is uniformly convex if and only if elements x and y are proportional whenever $\|x+y\|=\|x\|+\|y\|$. Hilbert space is uniformly convex. Any uniformly convex Banach space is reflexive, but there are reflexive Banach spaces which are not isomorphic with any uniformly convex space.

CON-VO-LU'TION, *n.* **convolution of two functions.** The function

$$h(x)=\int_0^x f(t)g(x-t)\,dt=\int_0^x g(t)f(x-t)\,dt$$

is the convolution of $f(x)$ and $g(x)$. The function $H(x)=\int_{-\infty}^{\infty} f(t)g(x-t)\,dt$ is sometimes also called a convolution of $f(x)$ and $g(x)$, but is also called a **bilateral convolution.** *Syn.* Faltung (German), resultant.

convolution of two power series. The convolution of two series of the form

$$\sum_{n=-\infty}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=-\infty}^{\infty} b_n z^n$$

is the series

$$\sum_{n=-\infty}^{\infty} c_n z^n, \quad \text{where} \quad c_n=\sum_{p=-\infty}^{\infty} a_p b_{n-p}.$$

This is the formal term-by-term product of the series.

CO-OP'ER-A-TIVE, *adj.* **cooperative game.** See GAME.

CO-OR'DI-NATE, *n.* One of a set of numbers which locate a point in space. If the point is known to be on a given line, only one coordinate is needed; if in a plane, two are required; if in space, three. See CARTESIAN and POLAR.

barycentric coordinates. See BARYCENTRIC.

Cartesian coordinates. See CARTESIAN.

complex coordinates. (1) Coordinates which are complex numbers. (2) Coordinates used in representing complex numbers in the plane (see COMPLEX—complex numbers).

coordinate paper. Paper ruled with graduated rulings to aid in plotting points and drawing the loci of equations. See CROSS—cross-section paper, and LOGARITHMIC—logarithmic coordinate paper.

coordinate planes. See CARTESIAN—cartesian coordinates.

coordinate system. Any set of numbers which locate a point, line, or any geometric element, in space. See COORDINATE, CARTESIAN, and POLAR.

coordinate trihedral. See TRIHEDRAL.

curvilinear coordinates. See CURVILINEAR.

cylindrical coordinates. See CYLINDRICAL—cylindrical coordinates.

ellipsoidal coordinates. Through each point in space there passes just one of each of the families of confocal quadrics whose equations are the following, if $a^2>b^2>c^2$:

$$\frac{x^2}{a^2-k}+\frac{y^2}{b^2-k}+\frac{z^2}{c^2-k}=1, \quad k<c^2,$$

$$\frac{x^2}{a^2-l}+\frac{y^2}{b^2-l}-\frac{z^2}{l-c^2}=1, \quad c^2<l<b^2,$$

$$\frac{x^2}{a^2-m}-\frac{y^2}{m-b^2}-\frac{z^2}{m-c^2}=1, \quad b^2<m<a^2.$$

The values of k, l, m, which determine these three quadrics, are called the *ellipsoidal coordinates* of the given point. However, three such quadrics intersect in eight points, so further restrictions are necessary to determine a point from a given set of quadrics, such perhaps as the quadrant in which the point shall lie. See CONFOCAL—confocal quadrics.

geodesic coordinates. See GEODESIC.

geographical coordinates. See SPHERICAL—spherical coordinates.

homogeneous coordinates. In a plane, the homogeneous coordinates of a point, whose Cartesian coordinates are x and y, are any three numbers (x_1, x_2, x_3) for which $x_1/x_3=x$ and $x_2/x_3=y$. See LINE—line at infinity. The coordinates are called homogeneous since any equation in Cartesian coordinates becomes homogeneous when the transformation to homogeneous coordinates is made; *e.g.*, $x^3+xy^2+9=0$ becomes $x_1{}^3+x_1x_2{}^2+9x_3{}^3=0$. Homogeneous coordinates are defined analogously for spaces of three or more dimensions.

inertial coordinate system. In mechanics, any system of coordinate axes moving with constant velocity with respect to a system of axes fixed in space relative to the positions of "fixed" stars. The latter system is called the **primary inertial system.**

left-handed coordinate system. A coordinate system in which the positive directions of the axes form a left-handed trihedral. See TRIHEDRAL—directed trihedral.

logarithmic coordinates. Coordinates using the logarithmic scale; used in plotting points on logarithmic paper. See LOGARITHMIC—logarithmic coordinate paper.

normal coordinates. Coordinates y^i such that the parametric equations of any geodesic going through the origin $y^i=0$ have the linear form $y^i=\xi^i s$ in terms of the arc-length parameters. Normal coordinates are special kinds of *geodesic coordinates.* Also see ORTHOGONAL—orthogonal transformation.

oblique coordinates. See CARTESIAN—Cartesian coordinates.

polar coordinate paper. Paper ruled with concentric circles about the point which is to serve as the pole and with radial lines through this point at graduated angular distances from the initial line. Used to graph functions in polar coordinates. See POLAR—polar coordinates in the plane.

polar coordinates. See POLAR, and SPHERICAL—spherical coordinates.

rectangular Cartesian coordinates. See CARTESIAN, and above, coordinate planes.

right-handed coordinate system. A coordinate system in which the positive directions of the axes form a right-handed trihedral. See TRIHEDRAL—directed trihedral.

spherical coordinates. See SPHERICAL—spherical coordinates.

symmetric coordinates. Coordinates u, v of a surface S: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, such that the element of length is given by $ds^2=F\,du\,dv$; that is, such that $E=G=O$. See LINEAR—linear element of a surface. We have symmetric coordinates if, and only if, the parametric curves are minimal. See PARAMETRIC—parametric curves of a surface, and MINIMAL—minimal curve.

tangential coordinates of a surface. Let X, Y, Z denote the direction cosines of the normal to a surface S: $x=x(u, v)$ $y=y(u, v)$, $z=z(u, v)$, and let W denote the algebraic distance from the origin to the plane tangent to S at the point P: (x, y, z) of S, $W=xX+yY+zZ$. The surface S is uniquely determined by the functions X, Y, Z, W, which are called the **tangential coordinates** of S.

transformation of coordinates. See TRANSLATION, and TRANSFORMATION—transformation of coordinates.

CO-PLA'NAR, *adj.* Lying in the same plane. *E.g.*, **coplanar lines** are lines which lie in the same plane; **coplanar points** are points which lie in the same plane. Three points are necessarily coplanar; four points described by their rectangular Cartesian coordinates are coplanar if and only if the following determinant is zero (otherwise, the absolute value of the determinant is the volume of a parallelepiped with the four points as four of the eight vertices and three of the points adjacent to the fourth).

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

CO-PUNC'TAL, *adj.* **copunctal planes.** Three or more planes having a point in common.

CORD, *n.* A stack of wood (with the sticks parallel, each to each) 8 feet long, 4 feet high, and 4 feet wide. See DENOMINATE NUMBERS in the appendix.

CORIOLIS. Coriolis acceleration. See ACCELERATION—acceleration of Coriolis, and below, Coriolis force.

Coriolis force. (*Astronomy.*) A force on terrestrial particles arising from the rotation of the earth about its axis. Its magnitude is $2m\omega v$, where ω is the angular velocity of rotation of the earth and v is the speed of the particle of mass m relative to the earth. Because of the small angular velocity of the earth (2π radians per day, or 7.27×10^{-5} radians per second) the effects of the *Coriolis force* are negligible in most technical applications, but are important

in meteorological and geographical considerations since they account for the trade winds.

COR'OL-LA-RY, *n.* A theorem that follows so obviously from the proof of some other theorem that no, or almost no, proof is necessary; a by-product of another theorem.

COR-RECT', *adj.* Without error in principle or computation. One speaks of a correct proof, correct solution, correct answer or correct computation. See ACCURATE.

COR-REC'TION, *n.* **correction in interpolation.** See INTERPOLATION.

Sheppard's correction, Yates' correction. (*Statistics.*) See the respective names.

COR'RE-LA'TION, *n.* (1) In pure mathematics: A linear transformation which, in the plane, carries points into lines and lines into points and, in space, carries points into planes and planes into points. (2) (*Statistics.*) The interdependence between two sets of numbers; a relation between two quantities, such that when one changes the other does (simultaneous increasing or decreasing is called **positive correlation** and one increasing, the other decreasing, **negative correlation**); a relation similar to that denoted by the functional concept, but usually not as explicitly defined. *Tech.* Let $f(x, y)$ be a joint frequency function of the two variables. The study of correlation is a study of the properties of that joint frequency function. Let $f(x \mid y)$ be a joint conditional frequency function. If $f(x \mid y)$ is invariant for all possible values of y, then x and y are *stochastically independent* and $f(y \mid x)$ is invariant for all x. If $f(x \mid y)$ varies with y, x is correlated with y. If $f(x \mid y)$ varies with y, $f(y \mid x)$ varies with x Hence correlation under this general definition is mutual. Variables that are independent (or correlated) in this general sense are also independent (or correlated) in all more special or restricted definitions, but not conversely. K. Pearson defined x as being *correlated* with y if the conditional mathematical expectation of x takes different values as y changes; x is *uncorrelated* with y if the conditional mean expectation of x is constant. Correlation, in Pearson's sense, of x with y does not imply the correlation of y with x. Noncorrelation in Pearson's sense does not imply noncorrelation under the preceding more general definition. Under both of the above definitions of correlation, x is correlated with y if the expected conditional variance of x for given possible values of y is less than the unconditional variance of x.

canonical correlation. See CANONICAL.

correlation coefficient. See COEFFICIENT—correlation coefficient.

correlation ellipse. The contour for which $f(x_1, x_2)$ is a constant, where $f(x_1, x_2)$ is a normal bivariate frequency function, forms an ellipse, called the correlation ellipse.

correlation ratio. $1-\sigma_{x\cdot y}^2/\sigma_x^2$ is the *correlation ratio* and is equal to the square of the *correlation coefficient* if the regression of x on y is linear. Otherwise the correlation ratio is greater than r^2. Here σ_x denotes standard deviation, r the correlation coefficient, and $\sigma_{x\cdot y}$ the standard deviation of x categorized by intervals of the y variable.

curvilinear correlation. If the regression function (the function relating the *expected value* of x to the given value of y) is not a linear function of y, the variables are curvilinearly correlated.

interclass correlation. The correlation between two or more variables, each variable considered as a separate class. *E.g.*, the correlation between height and weight, in which heights are regarded as one class and weights as the other; also the correlation between ages of elder and younger brothers, in which the older brothers are in one class (and their ages one variable), and the younger in the other (and their ages the other variable).

intraclass correlation. If there are several classes of items, with more than one item in each class (each item being measured in terms of the same variable), the **intraclass correlation** r_c (not squared) is equal to $\sigma_w^2/(\sigma_w^2+\sigma_c^2)$, where σ_w^2 is the variance within the classes and σ_c^2 is the variance between means of the classes (expressed in terms of individual observation variance). It is especially useful when there is no basis for distinguishing between the items within the classes. The range of

r_c is $1/(k-1)$ to 1, if each class has k members. This is a special case of the *analysis of variance.*

linear correlation. See below, normal correlation.

multiple correlation. The concept of correlation as applied to bivariate correlation may be generalized to k variables and as such is called multiple correlation. In the set of n variables $x_1, \cdots, x_n$, of which at least k_1 are stochastic variables, let

$$x_1 - b_{12}x_2 - b_{13}x_3 - \cdots - b_{1n}x_n - k = x_1 - x',$$

where x' is the linear regression function of $x_2, \cdots, x_n$ which minimizes $E(x_1 - x')^2$. The b_{1i} are **partial regression coefficients.** If the x_i are measured in standard deviation units, the regression coefficients are **beta coefficients** or **beta weights,** which are in general equal to $\beta_{1i} = b_{1i}\dfrac{\sigma_i}{\sigma_1}$. If the conditional expectation of x, for x', where $f(x, x')$ is a joint frequency function, is a linear function of x' over all x', the variable x is linearly and multiply correlated with the set $x_2, \cdots, x_n$. The extent of multiple correlation is measured by the relative reduction in the expected conditional variance of x for given x'. Specifically, the **multiple correlation coefficient** is $R_{x \cdot x'} = \sqrt{1 - \sigma_{x \cdot x'}^2/\sigma_x^2}$, or

$$R_{x \cdot x'} = \frac{E[(x - E(x))(x' - E(x'))]}{\sqrt{[E(x - E(x))^2][E(x' - E(x'))^2]}}.$$

Also,

$$R_{1 \cdot 234}^2 = \beta_{12 \cdot 34} r_{12} + \beta_{13 \cdot 24} r_{13} + \beta_{14 \cdot 23} r_{14},$$

which can be generalized for n variables.

nonsense correlation. Correlation between two variables may be due to the fact that each variable is correlated with a third variable. *E.g.*, the population of South Africa and the number of kilowatts of electrical energy consumed in California may be correlated, since both are positively correlated with time (*i.e.*, they both have a positive trend). This may be a nonsense-making situation in so far as any causal relationship is concerned. Sometimes the correlation observed in a random sample, if assignable to random sampling fluctuations, is called **nonsense correlation.**

normal correlation. Two variables, each of which is normally distributed, are *normally correlated* if the joint frequency function is

$$f(x, y) = \frac{1}{\sqrt{1 - r^2}\, 2\pi} \exp\left[\frac{1}{2(1 - r^2)}\left(\frac{x^2}{\sigma^2} - 2r\frac{xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right],$$

where x and y are each normally distributed with mean of zero and variances σ_x^2 and σ_x^2, and $r = \dfrac{E(xy)}{\sigma_x\sigma_y}$. Also known as the **product moment correlation coefficient.** In *normal correlation*, r contains all the required information about the degree of stochastic dependence between the two variables. The correlation is mutual and symmetrical, thus $r_{xy} = r_{yx}$. The **coefficient of alienation** is $\sqrt{1 - r^2}$. The conditional frequency functions are all normal with variance $\sigma_{xy}^2 = \sigma_x^2(1 - r^2)$, the square root of which is known as the **standard error of estimate** of x on y, for linear correlation. The conditional expectation of x for given values of y is a function of y and is called the **mean regression function** of x on y. If the function is linear $[E(x \mid y) = ax + by]$, the correlation of x on y is called **linear correlation**; b is the regression coefficient of x on y. Where x and y are expressed in standard-deviation units, the regression coefficient of x on y is the **beta weight** of x on y, which is otherwise equal to $b\sigma_y/\sigma_x$.

partial correlation. Let the two stochastic variables x_1 and x_2 each be multiply correlated with the set of variables $x_3, \cdots, x_k$. Then instead of the variables x_1 and x_2, consider the residual variables y_1 and y_2 obtained by subtracting from x_1, x_2 the respective linear functions of the set of variables $x_3, \cdots, x_k$ which maximize the multiple correlations with the two variables x_1 and x_2 individually. The residuals y_1 and y_2, when correlated, form the **partial correlation** of x_1 and x_2 apart from $x_3, \cdots x_k$. It is sometimes also called the **net correlation.** The partial-correlation coefficient of x_1 and x_2 apart from $x_3, \cdots, x_k$ is determinable from several alternative formulas. One useful formula for computing it from lower order correlation coefficients is

$$r_{12:34\cdots k} = \frac{r_{12:34\cdots k-1} - r_{1k:34\cdots k-1}}{\sqrt{(1 - r^2_{1k:34\cdots k-1})(1 - r^2_{2k:34\cdots k-1})}}.$$

The **order** of a correlation coefficient is equal to the number of terms in the secondary subscript of the correlation coefficient, and similarly for the regression coefficient. Thus r_{12} is a *zero order* coefficient; $r_{12:34}$ is a *second order* partial correlation coefficient.

product moment correlation coefficient. See above, normal correlation.

rank correlation. If the n joint values of two variables are ranked, each in terms of its own set of observations, then the Spearman **rank correlation** between the ranks of the variables is defined as

$$r = 1 - \frac{6\sum^{n} d^2}{n^3 - n},$$

where d is difference between the ranks of the paired items. Other less well-known correlation coefficients based on mere order rather than rank and order are available.

spurious correlation. Correlation between two variables which are related in that both variables have a common multiplicative or additive variable. It is termed spurious correlation, although there is nothing spurious in the sense of false about the correlation. It may, however, be misleading. *E.g.*, the correlation between x/y and z/y is due to the common element y, where x and z are uncorrelated.

tetrachoric correlation. A correlation adapted to a bivariate sample in which, although both variables are continuous and normally distributed, they are each measured in dichotomous form. The formula for estimating the correlation coefficient from the four categories resulting from the double dichotomizing is very complex. Tables are available for estimating the correlation coefficient from the double dichotomy table. Essentially, it consists of determining the value of the correlation coefficient of the bivariate normal distribution which, if dichotomized in both variables, would yield the same proportions as in the observed double dichotomy "fourfold table."

COR'RE-SPOND'ENCE, *n.* **one-to-one correspondence.** A correspondence (relation) between two sets of things such that pairs can be removed, one member from each group, until both groups have been simultaneously exhausted; *e.g.*, (a, b, c, d) and $(1, 2, 3, 4)$ can be put into one-to-one correspondence in many ways. *Tech.* A one-to-one correspondence is said to have been set up between two classes of elements, C and D, when a pairing has been set up between them such that each element of D has been made to correspond to one and only one element of C, and each element of C has been made to correspond to one and only one element of D.

COR'RE-SPOND'ING, *adj.* **corresponding angles, lines, points, etc.** Points, angles, lines, etc., in different figures, similarly related to the rest of the figures. *E.g.*, in two right triangles the hypotenuses are *corresponding* sides.

corresponding angles of two lines cut by a transversal. See ANGLE—angles made by a transversal.

corresponding rates. Rates producing the same amount on the same principal in the same time with different conversion periods. The nominal rate of 6%, money being converted semiannually, and the effective rate of 6.09%, are corresponding rates. *Syn.* Equivalent rates.

CO-SE'CANT, *adj.*, *n.* **cosecant of an angle.** See TRIGONOMETRIC—trigonometric functions.

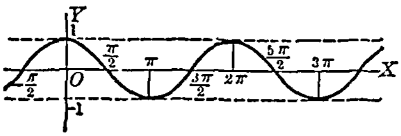
cosecant curve. The graph of $y = \operatorname{cosec} x$; the same as the curve obtained by moving the secant curve $\pi/2$ radians to the right, since $\operatorname{cosec} x = \sec(x - \pi/2)$. See SECANT—secant curve.

CO-SET, *n.* **coset of a subgroup of a group.** A set consisting of all products hx, or of all products xh, of elements h of the subgroup by a fixed element x of the group. If the multiplication by x is on the right, the set is a **right coset.** If the multiplication by x is on the left, the set is called a **left coset.** Two cosets are either identical or have no elements in common. Each element of the group belongs to one of the cosets. See GROUP.

CO'SINE, *n.* **cosine of an angle.** See TRIGONOMETRIC—trigonometric functions.

cosine curve. The graph of $y = \cos x$ (see figure). The curve has a y-intercept

1, is concave toward the x-axis, and cuts this axis at odd multiples of $\frac{1}{2}\pi$ (radians).



direction cosine (in space). See DIRECTION—direction cosines.

law of cosines. For a **plane triangle,** if a, b, c are the sides and C the angle opposite c, the law of cosines is

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

This formula is useful for solving a triangle when two sides and an angle, or three sides, are given. For a **spherical triangle,** the cosine laws are:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

where a, b, c are the sides of the spherical triangle, and $A, B,$ C are the corresponding opposite angles.

COST, *n.* **capitalized cost.** First cost plus the *present value* of the perpetual replacements to be made at the end of regular periods.

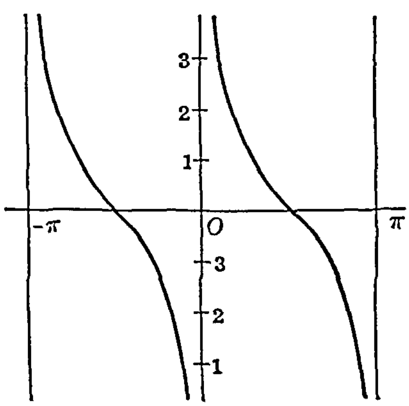
first cost. The amount paid for an article, not including the expense of holding or handling.

per cent profit on cost. See PER CENT.

replacement cost. See REPLACEMENT.

CO-TAN′GENT, *n.* **cotangent of an angle.** See TRIGONOMETRIC—trigonometric functions.

cotangent curve. The graph of $y = \cot x$; the same as the curve obtained by *reciprocating* the tangent curve. It is



asymptotic to the lines $x=0$ and $x=n\pi$ and cuts the x-axis at odd multiples of $\pi/2$ (radians).

CO-TER′MI-NAL, *adj.* **coterminal angles.** Angles having the same terminal line and the same initial line; two angles generated by the revolution of two lines about the same point in the initial line in such a way that the final positions of the revolving lines are identical; *e.g.*, 30°, 390°, and −330° are coterminal angles.

COTES. Newton-Cotes integration formulas. See NEWTON.

COU-LOMB′, *n.* A unit of electrical charge, abbreviation coul or Cb. The **absolute coulomb** is defined as the amount of electrical charge which crosses a surface in one second if a steady current of one **absolute ampere** is flowing across the surface. The absolute coulomb has been the legal standard of quantity of electricity since 1950. The **International coulomb,** the legal standard before 1950, is the quantity of electricity which, when passed through a solution of silver nitrate in water, in accordance with certain definite specifications, deposits 0.00111800 gm of silver.

1 Int. coul = 0.999835 abs. coul.

Coulomb's law for point-charges. A point-charge of magnitude E_1 located at a point P_1 exerts a force on a point-charge of magnitude e_2 located at a point P_2, which is given by the expression $ke_1e_2r^{-2}\rho_1$, where k is a positive constant depending on the units used, r is the distance between the charges, and ρ_1 is a unit vector having the direction of the displacement P_1 to P_2. Thus the force is a repulsion or attraction according as the charges are of the same or opposite sign. If we replace the positive constant k with the negative constant $-G$ and replace e_1 and e_2 with m_1 and m_2, the magnitudes of two point-masses, the foregoing formula becomes *Newton's law of gravitation for particles.*

COUNT, *v.* To name a set of consecutive integers in order of their size, usually beginning with 1.

count by twos (threes, fours, fives, etc.). To name, in order, a set of integers that

have the difference 2 (3, 4, 5, etc.); *e.g.*, when counting by two's, one says "2, 4, 6, 8, · · ·"; when counting by three's, "3, 6, 9, 12, · · ·."

COUNT-A-BIL'I-TY, *n.* **first and second axioms of countability.** See BASE—base for a topological space.

COUNT'A-BLE, *adj.* **countable set.** (1) A set of objects whose members can be put into one-to-one correspondence with the positive integers; a set whose members can be arranged in an infinite sequence $p_1, p_2, p_3, \cdots$ in such a way that every member occurs in only one position. *Syn.* Countably infinite. (2) A set of objects which either has a finite number of members or which can be put into one-to-one correspondence with the set of positive integers. The sets of all integers and of all rational numbers are countable, but the set of all real numbers is not. See DENUMERABLE, ENUMERABLE.

COUN'TER-CLOCK'WISE, *adj.* In the direction of rotation opposite to that in which the hands move around the dial of a clock.

COUN'TER, *adj., n.* In a computing machine, an adder that receives only unit addends, or addends of amount one. Counters are usually built up by means of simple modulo 2 counters, a **modulo 2 counter** being a simple arithmetic component that is in one of its two steady states according as the number of impulses it has received is even or odd. See ADDER.

counter life. See INSURANCE—life insurance.

COU'PON, *n.* **bond coupons.** See BOND.

COURSE, *n.* **course of a ship.** See SAILING—plane sailing.

CO-VAR'I-ANCE, *n.* **analysis of covariance.** Statistical analysis of variance of a variable (as in the analysis of variance) which is affected by linearly related variables. As in the *analysis of variance*, the effects of linearly related variables may be considered. *E.g.*, the effect of variation in carbon content on the tensile strength of steel may be removed at the same time that other nonlinearly related factors, such as different manufacturers or blast furnaces, are also controlled. In this example, the intent is (1) to determine if there is any relationship between tensile strength and carbon content, (2) to determine if that relationship varies from manufacturer to manufacturer, and (3) to remove the effect of carbon content in determining the relationship between tensile strength and manufacturer.

CO-VAR'I-ANT, *adj.* **covariant derivative of a tensor.** The *covariant derivative* of a tensor $t^{a_1 \cdots b_p}_{b_1 \cdots a_q}$ is the tensor

$$t^{a_1 \cdots a_p}_{b_1 \cdots b_q, j} = \frac{\partial t^{a_1 \cdots a_p}_{b_1 \cdots b_q}}{\partial x_j} - \sum_{r=1}^{q} t^{a_1 \cdots a_p}_{b_1 \cdots b_{r-1} i b_{r+1} \cdots b_q} \begin{Bmatrix} i \\ b_r\ j \end{Bmatrix} + \sum_{r=1}^{p} t^{a_1 \cdots a_{r-1} i a_{r+1} \cdots a_p}_{b_1 \cdots b_q} \begin{Bmatrix} a_r \\ i\ \ j \end{Bmatrix},$$

where the summation convention applies and $\begin{Bmatrix} i \\ j\,k \end{Bmatrix}$ is the *Christoffel symbol* of the second kind. This tensor is contravariant of rank p and covariant of rank $q+1$. Covariant differentiation is not commutative. *E.g.*, $t_{,j,k}{}^i \neq t_{,k,j}{}^i$ in general, since $t_{,j,k}{}^i - t_{,k,j}{}^i = R_{rjk}{}^i t^r$, where $R_{jkl}{}^i$ is the *Riemann-Christoffel tensor*. If $t_i(x^1, x^2, \cdots, x^n)$ is a covariant tensor of rank one (*i.e.*, a *covariant vector field*), then the covariant derivative of t_i is

$$t_{i,j} = \frac{\partial t_i}{\partial x^j} - \begin{Bmatrix} \sigma \\ i\ \ j \end{Bmatrix} t_\sigma,$$

a covariant tensor of rank two. If $t^i(x^1, \cdots, x^n)$ is a contravariant tensor of rank one (*i.e.*, a *contravariant vector field*), then the covariant derivative of t^i is

$$t_{,j}{}^i = \frac{\partial t^i}{\partial x^j} + \begin{Bmatrix} i \\ \sigma\ j \end{Bmatrix} t^\sigma,$$

which is contravariant of rank one and covariant of rank one. In Cartesian coordinates, or for *scalar fields*, covariant differentiation is ordinary differentiation. See CONTRAVARIANT—contravariant derivative of a tensor.

covariant indices. See TENSOR.

covariant tensor. See TENSOR.

covariant vector field. See VECTOR.

Stokian covariant derivative. If

$$t_{a_1a_2\cdots a_p}(x^1, \cdots, x^n)$$

is an alternating covariant tensor field, then the covariant tensor field $t_{a_1a_2\cdots a_p|\beta}$ of rank $p+1$ defined by

$$t_{a_1a_2\cdots a_p|\beta}=\frac{\partial t_{a_1\cdots a_p}}{\partial x^\beta} - \sum_{r=1}^{n}\frac{\partial t_{a_1\cdots a_{r-1}\beta a_{r+1}\cdots a_p}}{\partial x^{a_r}}$$

is an alternating tensor, called the **Stokian covariant derivative** of $t_{a_1\cdots a_p}$. The terminology is appropriate, since in the generalized Stokes' theorem on multiple integrals we have

$$\int_{B_p}\cdots\int t_{a_1\cdots a_p}\,dx^{a_1}\cdots dx^{a_p} = \int_{V_{p+1}}\cdots\int t_{a_1\cdots a_p|\beta}\,dx^{a_1}\cdots dx^{a_p}\,dx^\beta.$$

It is to be observed that Stokian covariant differentiation does not depend on the machinery put at one's disposal by a metric tensor field g_{ij}.

COV'ER-ING, *n.* A covering of a set is a set of subsets such that each point of the set belongs to at least one of the subsets (sometimes it is required that a covering contain only a finite number of sets). A covering is **closed** (or **open**) according as each of the covering sets is closed (or each is open). An **ϵ-covering** of a metric space M is a covering of M by a finite number of sets each of which is such that the distance between any two of its points is less than ϵ. An ϵ-covering is of **order n** if there is a point which is contained in n of the sets of the covering, but no point is contained in $n+1$ of these sets. Also see VITALI—Vitali covering.

CO'VERSED, *adj.* **coversed sine.** *One minus* the *sine* of an angle; geometrically the difference between the radius and the sine of an angle constructed in a unit circle. See TRIGONOMETRIC—trigonometric functions.

CO'VER-SINE'. Same as COVERSED SINE.

CRAMER'S RULE for the solution of any given number of linear algebraic equations in the same number of unknowns. A simple rule for writing out, in determinant form, the value of each of the unknowns. The rule for n equations is: Each unknown is equal to the fraction in which the denominator is the determinant of the coefficients of the n unknowns and the numerator is the same determinant, except that the coefficients of the unknown which is being found are replaced by the constant terms if these appear as the right-hand members of the system of equations and by their negatives if they appear in the left members. *E.g.*, the values of x and y which satisfy

$$x+2y=5$$
$$2x+3y=0 \quad \text{are}$$

$$x=\begin{vmatrix}5 & 2\\ 0 & 3\end{vmatrix} \div \begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix} = -15,$$

$$y=\begin{vmatrix}1 & 5\\ 2 & 0\end{vmatrix} \div \begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix} = 10.$$

This rule gives the solution of equations for which there is a unique solution, that is, for which the determinate of the coefficients is not zero. See DETERMINANT—determinant of the second order, and CONSISTENCY.

CRED'IT, *adj.* **credit business.** A business in which goods are sold without immediate payment, but with a promise to pay later, generally at some specified time.

CRED'I-TOR, *n.* One who accepts a promise to pay in the future in place of immediate payment; a term most commonly applied to retail merchants who do a *credit* business.

CRI-TE'RI-ON, *n.* [*pl.* cri-te'ri-a.] A law or principle by which a proposition can be tested.

CRIT'I-CAL, *adj.* **biased critical region.** (*Statistics.*) A *critical region* of size α is *biased* if the probability of rejecting the null hypothesis is less than α when the null hypothesis is false. *E.g.*, the use of two equal tails of the chi-square distribution is a biased critical region for the test of the hypothesis that a variance of a normal population is equal to some specified value.

critical ratio. (*Statistics.*) A statistic used to determine the probability of a

sample, or one even less probable, under a given hypothesis about the population from which the sample is taken. Used in tests of hypotheses and tests of significance. *E.g.*, the ratio of the difference between a sample mean and the hypothesized value to the standard deviation of the population is often called the **critical ratio.**

critical region. See HYPOTHESIS—test of hypothesis.

critical value or point. (1) A point at which a curve has a maximum, a minimum, or a point of inflection. (2) A point at which a curve has a maximum or a minimum. (3) A point at which dy/dx is either zero or infinite. See STATIONARY—stationary point.

CROSS, *adj.* **cross product.** See MULTIPLICATION—multiplication of vectors.

cross ratio. See RATIO—cross ratio.

cross section of an area or solid. A plane section perpendicular to the axis of symmetry or to the longest axis, if there be more than one; rarely used except in cases where all cross sections are equal, as in the case of a circular cylinder or rectangular parallelepiped.

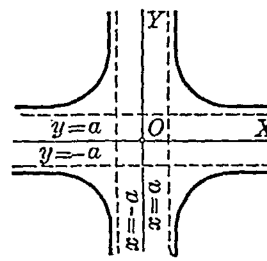
cross-cap. A Möbius strip has a boundary which is a simple closed curve. It can be deformed into a circle, although in the process the strip must be allowed to intersect itself (the curve of intersection is regarded as two different curves, each belonging to exactly one of the two parts of the surface which "cross" along the curve). The surface which results is a nonorientable surface called a *cross-cap.* It can be described as a hemisphere which has been pinched together along a short vertical line starting at the pole; the surface then appears to intersect along this line and the line is to be regarded as two lines, each belonging to one of the two parts of the surface which cross along the line. See GENUS—genus of a surface.

cross-section paper. Paper ruled with vertical and horizontal lines equally spaced; used in graphing equations in rectangular coordinates. *Syn.* Ruled paper, squared paper.

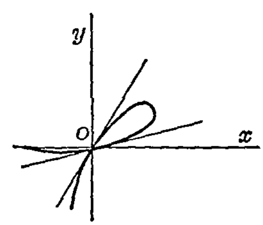
CRU'CI-FORM, *adj.* **cruciform curve.** The locus of the equation

$$x^2y^2-a^2x^2-a^2y^2=0.$$

The curve is symmetric about the origin and the coordinate axes, has four branches, one in each quadrant, and is asymptotic to each of the four lines $x=\pm a, y=\pm a$. It is called the *cruciform curve* because of its resemblance to a cross.



CRU'NODE, *n.* A point on a curve through which there are two branches of the curve with distinct tangents.



CUBE, *n.* A solid bounded by six planes, with its twelve edges all equal and its face angles all right angles. A cube in n-dimensional Euclidean space is a set consisting of all those points $x=(x_1, x_2, \cdots, x_n)$ for which $a_i \leqq x_i \leqq b_i$ for each i, where the numbers $\{a_i\}$ and $\{b_i\}$ are such that b_i-a_i has the same value k for each i. The number k is the length of an edge of the cube and the *volume* (or *measure*) of the cube is equal to k^n. Such a cube is the *Cartesian product* of n closed intervals, each of length k.

cube of a number. The third power of the number. *E.g.*, the cube of 2 is $2\times2\times2$, written 2^3.

cube of a quantity. The third power of the quantity; *e.g.*, the cube of $(x+y)$ is $(x+y)(x+y)(x+y)$, written $(x+y)^3$ or $x^3+3x^2y+3xy^2+y^3$.

cube root of a given quantity. A quantity whose cube is the given quantity. See ROOT—root of a number.

duplication of the cube. See DUPLICATION.

CU'BIC, *adj., n.* **Cardan's solution of the cubic.** See CARDAN.

bipartite cubic. See BIPARTITE.

cubic curve. See CURVE—algebraic plane curve.

cubic equation. A polynomial equation of the third degree, such as

$$2x^3+3x^2+x+5=0.$$

reduced cubic. See REDUCED.

resolvent cubic. See FERRARI's solution of the general quartic.

twisted cubic. A curve which cuts each plane in three points, real or imaginary, distinct or not. *E.g.*, the equations $x=at$, $y=bt^2$, $z=ct^3$, $abc\neq 0$, represent such a curve.

CU'BI-CAL, *adj.* **coefficient of volume (cubical) expansion.** See COEFFICIENT.

cubical expansion. Same as VOLUME EXPANSION.

cubical and semicubical parabola. See PARABOLA—cubical parabola.

CU'MU-LANTS, *n.* A set of parameters k_i of a distribution which measures its properties and frequently specifies it. In terms of moments u_i, $k_1=u_1$, $k_2=u_2-u_1^2$, $k_3=u_3-3u_2u_1+2u_1^3$. Precisely, k_i is the coefficient of $(it)^r/r!$ in $\log\phi(t)$, where $\phi(t)$ is the *characteristic function* derived from the frequency function of the distribution (if $\phi(t)$ can be expanded in a power series).

CU'MU-LA'TIVE, *adj.* **cumulative frequency.** The sum of all preceding frequencies, a certain order having been established. *E.g.*, if the number of students making the grades of 60% to 70%, 70% to 80%, 80% to 90%, and 90% to 100% are, respectively, 2, 4, 7, and 3 (which are called the frequencies), then *cumulative frequencies* are 2, 6, 13, and 16. The sum of the absolute (or relative) frequencies of values of x equal to or less than x_i is the upward cumulative absolute (or relative) frequency of x. It may also be cumulated in a downward direction. See FREQUENCY—absolute frequency, and relative frequency.

cumulative frequency curve. The curve whose ordinates are the cumulative frequencies and whose abscissas are the class intervals. *Syn.* Ogive curve.

CURL, *n.* **curl of a vector function,** $F(x, y, z)$. It is written $\nabla\times F$ and defined as

$$i\times\frac{\partial F}{\partial x}+j\times\frac{\partial F}{\partial y}+k\times\frac{\partial F}{\partial z}$$

where ∇ is the operator,

$$i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}.$$

E.g., if F is the velocity at a point $P(x, y, z)$ in a moving fluid, $\frac{1}{2}\nabla\times F$ is the vector angular velocity of an infinitesimal portion of the fluid about P. See VECTOR—vector components.

CUR'RENT, *adj.* **current rate.** Same as PREVAILING INTEREST RATE. See INTEREST.

current yield rate. The ratio of the dividend (bond interest) to the purchase price.

CUR'TATE, *adj.* **curtate annuity.** See ANNUITY.

curtate expectation of life. See EXPECTATION—expectation of life.

CUR'VA-TURE, *n.* **average curvature of a curve in a plane.** The ratio of the change in inclination of the tangent, over a given arc, to the length of the arc. The limit of the *average curvature* as the length of the arc approaches zero is the *curvature*.

center of curvature. See below, curvature of a plane curve.

curvature of a plane curve. The rate of change of the inclination of the tangent with respect to change of arc length; *i.e.*, the derivative of $\tan^{-1}(dy/dx)$ with respect to the arc (see above, average curvature). In rectangular Cartesian coordinates the curvature, K, is given by

$$K=(d^2y/dx^2)/[1+(dy/dx)^2]^{3/2};$$

in parametric coordinates it is

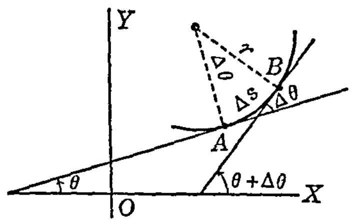
$$\frac{(dx/dt)(d^2y/dt^2)-(dy/dt)(d^2x/dt^2)}{[(dx/dt)^2+(dy/dt)^2]^{3/2}}$$

where $y=h(t)$, $x=g(t)$; and in polar coordinates it is

$$\frac{r^2+2(dr/d\theta)^2-r(d^2r/d\theta^2)}{[r^2+(dr/d\theta)^2]^{3/2}}$$

The numerical value of the curvature of a circle is constant and equal to the reciprocal of its radius. The circle tangent to a

curve on the concave side and having the same curvature at the point of tangency is called the **circle of curvature** of the curve (at that point). Its radius is the numerical value of the **radius of curvature,** and its center is the **center of curvature.** The curvature at A is the limit of $\Delta\theta/\Delta s$ as Δs approaches zero (where $\Delta s = AB$ and θ is measured in radians). The sign of K



depends upon that of $\Delta\theta$, which is positive or negative according as the curve is convex or concave down (according as d^2y/dx^2 is positive or negative). Some writers define the curvature to be $|K|$.

curvature of a space curve at a point. If P is a fixed point, and P' a variable point, on a (directed) space curve C, s the length of arc on C from P to P', and $\Delta\theta$ the angle between the positive directions of the tangents to C at P and P', then the curvature $K = 1/\rho$ of C at P is defined by $K = \frac{1}{\rho} = \lim_{\Delta s \to 0} \left|\frac{\Delta\theta}{\Delta s}\right|$. Thus the curvature may be taken as a measure of the rate of turning of the tangent to C relative to the arc length s. The number ρ is called the **radius of curvature.** Also called *first curvature of a space curve at a point.* See TORSION.

Gaussian curvature of a surface at a point. See below, total curvature of a surface at a point.

integral curvature of a geodesic triangle on a surface. See below, integral curvature of a region on a surface. For a geodesic triangle, the value of the integral curvature is the sum of the angles of the triangle, diminished by π. With proper account of sign, the integral curvature is equal to the area of the part of the unit sphere covered by the spherical image of the region. See below, total curvature of a geodesic triangle on a surface.

integral curvature of a region on a surface. The integral of the Gaussian curvature over the region: $\iint K\,dA$. *Syn.* Curvatura integra.

lines of curvature of a surface. The curves on a surface S: $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ defined by $(ED' - FD)\,du^2 + (ED'' - GD)\,du\,dv + (FD'' + GD')\,dv^2 = 0$. See SURFACE—fundamental coefficients of a surface. The curves form an orthogonal system on S, and the two curves of the system through a point P of S determine the principal directions on S at P. See DIRECTION—principal directions on a surface at a point.

mean curvature of a surface at a point. The sum of the principal curvatures of the surface at the point:

$$K_m = \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{ED'' + GD - 2FD'}{EG - F^2}.$$

See below, principal curvatures of a surface at a point. *Syn.* Mean normal curvature.

mean normal curvature of a surface. See above, mean curvature of a surface.

normal curvature of a surface at a point in a given direction. The curvature, with proper choice of sign, of the normal section C of the surface S at the point and in the given direction. The sign is positive if the positive direction of the principal normal of C coincides with the positive direction of the normal to S, otherwise negative. The normal curvature $1/R$ is given by

$$\frac{1}{R} = \frac{D\,du^2 + 2D'\,du\,dv + D''\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}.$$

The reciprocal R of the normal curvature is called the **radius of normal curvature** of the surface at the point in the given direction. See SURFACE—fundamental coefficients of a surface.

principal curvatures of a surface at a point. The *normal curvatures* $1/\rho_1$ and $1/\rho_2$ in the *principal directions* at the point. The numbers ρ_1 and ρ_2 are called the **principal radii of normal curvature** of the surface at the point. See DIRECTION—principal directions on a surface at a point.

radius of curvature. See various headings under RADIUS; and below, radius of curvature of a plane curve.

radius of curvature of a plane curve. The reciprocal of the curvature. See above, curvature of a plane curve.

radius of curvature of a space curve. See TORSION; and above, curvature of a space curve.

Riemannian curvature. See RIEMANN.

second curvature of a space curve. Same as TORSION.

surface of negative curvature. A surface on which the *total curvature* is negative at every point. Such a surface lies part on one side and part on the other side of the tangent plane in the neighborhood of a point; *e.g.*, the inner surface of a torus, and the hyperboloid of one sheet.

surface of positive total curvature. A surface on which the *total curvature* is positive at every point, such as the sphere and ellipsoids.

surface of zero total curvature. A surface on which the total curvature is zero at every point; such as cylinders or, in fact, all developable surfaces.

total curvature of a geodesic triangle on a surface. See above, integral curvature of a geodesic triangle on a surface.

total curvature of a surface at a point. The product of the *principal curvatures* of the surface at the point:

$$K=\frac{1}{\rho_1\rho_2}=\frac{DD''-D'^2}{EG-F^2},$$

See above, principal curvatures of a surface at a point. In tensor notation this is the scalar field

$$K=\frac{R_{1221}}{g_{11}g_{22}-g_{12}{}^2}.$$

if $ds^2=g_{\mu\gamma}\,dx^\mu\,dx^\gamma$ is the line element of the surface as a two-dimensional Riemannian space and, within sign, R_{1221} is the only nonzero component of the covariant Riemann-Christoffel tenor $R_{\alpha\beta\gamma\delta}$. Since R_{1221} is the determinant of the coefficients of the second fundamental differential form of the surface, it follows that the total curvature of a surface is the ratio of the determinant of its second fundamental differential form to the determinant of its first fundamental differential form $ds^2=g_{\alpha\beta}\,dx^\alpha\,dx^\beta$. See RADIUS—radius of total curvature of a surface. *Syn.* Total normal curvature; Gaussian curvature.

CURVE, *n.* The locus of a point which has one degree of freedom. *E.g.*, in a plane, a straight line is the locus of points whose coordinates satisfy a linear equation, and a circle with radius 1 is the locus of points which satisfy $x^2+y^2=1$.

algebraic plane curve. A plane curve which has an equation in Cartesian coordinates of the form $f(x, y)=0$, where $f(x, y)$ is a polynomial in x and y. If $f(x, y)$ is a polynomial of the nth degree, the curve is said to be an **algebraic curve of degree *n*.** If n is one, the curve is a **straight line,** if n is two, the curve is called a **quadratic** or **conic;** and if it is of the third, fourth, fifth, sixth degree, etc., it is called a **cubic, quartic, quintic, sextic,** etc. When n is greater than 2, the curve is called a **higher plane curve.**

analytic curve. See ANALYTIC.

angle between two intersecting curves. See ANGLE—angle of intersection.

curvature of a curve. See various headings under CURVATURE.

curve fitting. Determining empirical curves. See EMPIRICAL.

curve in a plane, or plane curve. A curve all points of which lie in a plane; the locus of an equation (in Cartesian coordinates) of the form $y=f(x)$ or $f(x, y)=0$. See above, CURVE.

curve tracing. Plotting or graphing a curve by finding points on the curve and, in a more advanced way, by investigating such matters as symmetry, extent, and asymptotes and using the derivatives to determine critical points, slope, change of slope, and concavity and convexity.

curve of zero length. Same as MINIMAL CURVE. See MINIMAL.

derived curve. See DERIVED.

empirical curves. See EMPIRICAL.

family of curves. See FAMILY.

growth curve. See GROWTH.

integral curves. See INTEGRAL—integral curves.

length of a curve. See LENGTH—length curve.

normal frequency curve. See FREQUENCY.

parabolic curve. An algebraic curve which has an equation in Cartesian coordinates of the following type:

$$y=a_0+a_1x+\cdots+a_nx^n.$$

parallel curves (in a plane). Two curves which have their points paired on the same normals, always cutting off the same length segments on these normals. Their

tangents at points where they cut a common normal are parallel. See INVOLUTE.

path curves. See PATH.

pedal curve. See PEDAL.

periodic curves. See PERIODIC.

primitive curve. See PRIMITIVE.

quadric (or quadratic) curve. A curve whose equation is of the 2nd degree. *Syn.* Conic, or conic section. See above, CURVE.

simple closed curve. See SIMPLE.

skew curve. Same as TWISTED CURVE. See below.

space curve. A curve in space, but not necessarily a twisted curve. See below, twisted curve, and above, plane curve.

spherical curve. A curve that lies wholly on the surface of a sphere.

turning points on a curve. See TURNING.

twisted curve. A space curve that does not lie in a plane. *Syn.* Skew curve. A twisted curve is of the **nth order** if it cuts each plane in n points, real or imaginary, distinct or not. See CUBIC—twisted cubic.

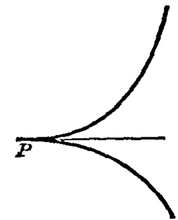
u-curves on a surface. See PARAMETRIC—parametric curves on a surface.

CUR'VI-LIN'E-AR, *adj.* **curvilinear coordinates of a point in space.** The surfaces of a triply orthogonal system of surfaces may be determined by three parameters. The values of these three parameters which determine the three surfaces of the system through a given point P in space are called the **curvilinear coordinates** of P. See ORTHOGONAL — triply orthogonal system of surfaces, and CONFOCAL—confocal conics.

curvilinear coordinates of a point on a surface. Parametric coordinates u, v of a point on a surface S: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$. See PARAMETRIC—parametric equations of a surface.

curvilinear motion. See MOTION.

CUSP, *n.* A *double point* at which the two tangents to the curve are coincident. *Syn.* Spinode. A **cusp of the first kind** (or **simple cusp**) is a cusp in which there is a branch of the curve on each side of the double tangent in the neighborhood of the point of tangency; *e.g.*, the semicubical parabola, $y^2=x^3$, has a cusp of the *first kind* at the origin.



A **cusp of the second kind** is a cusp for which the two branches of the curve lie on the same side of the tangent in the neighborhood of the point of tangency; the curve $y=x^2 \pm \sqrt{x^5}$ has a cusp of the second kind at the origin.

A **double cusp** is the same as a *point of osculation* (see OSCULATION). For a given family of curves, a **cusp locus** is a set of points each of which is a cusp for one of the members of the family. See DISCRIMINANT—discriminant of a differential equation.

hypocycloid of four cusps. See HYPOCYCLOID.

CUT, *n.* A **cut** (or **cutting**) of a set T is a subset C of T such that $T-C$ is not connected. If a cut C is a point or a line, then C is called a **cut point** or a **cut line.** A point which is not a cut point is said to be a **noncut point.**

Dedekind cut. See DEDEKIND CUT.

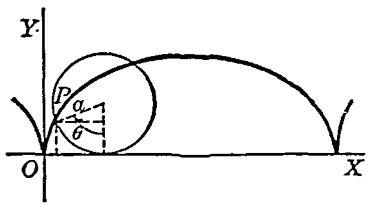
CY'CLE, *n.* See PERMUTATION (2), and CHAIN—chain of simplexes.

CY'CLIC, *adj.* **cyclic change, interchange, or permutation, of objects.** See PERMUTATION (2).

CYC'LI-DES, *n.* **cyclides of Dupin.** The envelope of a family of spheres tangent to three fixed spheres.

CY'CLOID, *n.* The plane locus of a point which is fixed on the circumference of a circle, as the circle rolls upon a straight line. *E.g.*, the path described by a point

on the rim of a wheel. The cycloid is a special case of the **trochoid**, although the two words are sometimes used synonymously. If a is the radius of the rolling circle and θ is the central angle of this circle, which is subtended by the arc OP that has contacted the line upon which the circle rolls, the parametric equations of the cycloid are:



$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$$

The cycloid has a cusp at every point where it touches the *base line*. The distance from cusp to cusp is $2\pi a$, and the distance traveled by the point, as the center of the circle moves between two positions, is independent of a, provided the point starts and finishes at a cusp; *i.e.*, the length of the path traced out by a point on the rim of a large wheel is the same as the length of the path traced out by a point on the rim of a small wheel, provided the hubs travel over the same distance and both paths have cusps at the beginning and finishing point.

base of a cycloid. The line upon which the generating circle rolls.

curtate cycloid. A *trochoid* which has no *loops*. See TROCHOID.

prolate cycloid. See TROCHOID.

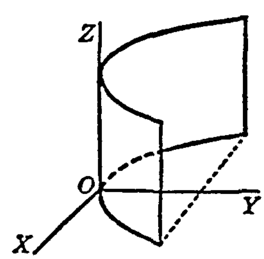
CY′CLO-SYM′ME-TRY, *n.* See SYMMETRIC—symmetric function.

CY′CLO-TOM′IC, *adj.* **cyclotomic equation.** An equation of the form

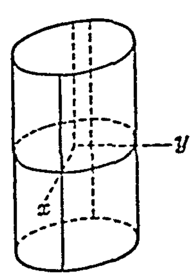
$$x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1 = 0,$$

where n is a prime number. A cyclotomic equation is *irreducible* (in the field of real numbers).

CYL′IN-DER, *n.* (1) A cylindrical surface (see CYLINDRICAL). (2) Either the solid bounded by two parallel planes and a cylindrical surface whose directrix is a closed curve, or the surface consisting of the portion of the cylindrical surface between the planes and the regions of the planes bounded by the cylindrical surface (these are the **bases** of the cylinder). The **altitude** of a cylinder is the perpendicular distance between the bounding planes (or between the bases) and the **elements** (or **rulings**) are the segments of the elements of the cylindrical surface between the planes. The **volume** of a cylinder is equal to the product of the area of a base and the altitude; the **lateral area** (the area of the cylindrical surface between the planes) is equal to the product of the length of an element and the perimeter of the intersection of the cylindrical surface with a plane perpendicular to the elements of the surface. The cylinder is **circular, elliptic, hyperbolic,** or **parabolic,** according as the directrix is a circle, ellipse, hyperbola, or parabola (it is a **quadric cylinder** if the Cartesian equation of the directrix is of the second degree). Sometimes a **circular cylinder** is defined to be a cylinder whose intersections with planes perpendicular to the elements are circles. The figure shows a parabolic cylinder whose equation is $x^2 = 2py$, where $\frac{1}{2}p$ is the distance from the z-axis of the focus of a cross section.



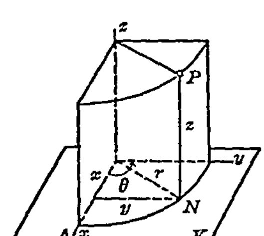
The **axis** of a cylinder is a line of symmetry of the cylindrical surface; *e.g.*, the line joining the centers of the bases is the axis of an elliptic or a circular cylinder. A **right circular cylinder** (or **cylinder of revolution**)



is a circular cylinder whose bases are perpendicular to the axis. This is the surface generated by revolving a rectangle about one of its sides. (The equation of the cylindrical surface in the figure is $x^2+y^2=a^2$, where a is the radius of a cross section.) Its volume is $\pi a^2 h$ and its lateral area is $2\pi ah$, where h is its altitude and a the radius of the base.

similar right circular cylinders. Cylinders for which the ratio between the radius of the base and the length of an element in any one cylinder is the same as the corresponding ratio in any of the other cylinders.

CY-LIN'DRI-CAL, *adj.* **cylindrical coordinates.** Space coordinates making use of polar coordinates in one coordinate plane, usually the x, y plane, the third coordinate being simply the rectangular coordinate measured from this plane. These are called cylindrical coordinates because when r is fixed and z and θ vary, they develop a cylinder; *i.e.*, $r=c$ is the equation of a cylinder. The locus of points for which θ has a fixed value is a plane, PNO, containing the z-axis; the points for which z is constant define a plane parallel to the x, y plane. The three surfaces for r, θ, and z constant, respectively, locate the point $P(r, \theta, z)$ as their intersection. The **transformation of cylindrical into rectangular coordinates** is given by the formulas $x=r\cos\theta$, $y=r\sin\theta$, $z=z$.



cylindrical function. Any solution of Bessel's differential equation. Sometimes taken to be synonymous with Bessel function.

cylindrical map. For a spherical surface S with longitude and latitude denoted by θ and ϕ, respectively, a **cylindrical map** is a continuous one-to-one map of the points of S onto a set of points of a (u, v)-plane given by formulas of the type $u=\theta$ and $v=v(\phi)$, with $v(0)=0$ and $v(\phi)>0$ for $\phi>0$. A cylindrical map that is given by the formulas $u=\theta$, $v=\tan\phi$ is said to be a **central cylindrical projection.** This is a projection of a sphere from its center onto a tangent right circular cylinder that is then slit along one of its elements and spread out on a plane. A cylindrical map that is given by the formulas $u=\theta$, $v=\phi$ is said to be an **even-spaced map.** Lines of longitude and latitude with equal angular increments appear as squares, such as the squares on a checkerboard. See MERCATOR—Mercator's projection.

cylindrical surface. The surface generated by a straight line moving always parallel to a given straight line, and intersecting a given curve (if the curve is a plane curve with its plane parallel to the given line, the cylinder is a plane). The line is called the **generator** or **generatrix.** The curve is called the **directrix.** The generator in any one fixed position is called an **element.** A cylindrical surface is not necessarily closed, since the directrix is not restricted to being a closed curve. See above, for example, parabolic cylinder. A cylindrical surface is named after its right sections; *e.g.*, if the right section is an ellipse, it is called an **elliptical cylindrical surface,** or simply an **elliptic cylinder** (the word cylinder not always being restricted to the solid bounded by a cylindrical surface and two parallel cutting planes). The **equation of a cylindrical surface,** when one of the coordinate planes is perpendicular to the elements, is the equation of the trace of the cylinder in this plane. *E.g.*, the equation $x^2+y^2=1$ is the equation of a right circular cylindrical surface since, for every pair, (x, y), of numbers that satisfies this equation, z may take all values; similarly, $y^2=2x$ is the equation of a parabolic cylindrical surface with its elements parallel to the z-axis, and

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is the equation of an elliptical cylindrical surface with its elements parallel to the z-axis.

CYL'IN-DROID, *n.* (1) A cylindrical surface whose sections perpendicular to the

elements are ellipses. (2) The surface which is the locus of a straight line moving so as to intersect two curves and remain always parallel to a given plane.

D

D'ALEMBERT'S TEST for convergence (or divergence) of an infinite series. Same as the *generalized ratio test.* See RATIO—ratio test.

DAMPED, *p.* **damped harmonic motion.** Harmonic motion having its amplitude continually reduced. See HARMONIC—harmonic motion.

damped oscillations. See OSCILLATION.

DARBOUX. Darboux's monodromy theorem. The theorem states that if the function $f(z)$ of the complex variable z is analytic in the finite domain D bounded by the simple closed curve C, and $f(z)$ is continuous in the closed region $D+C$ and takes on no value more than once for z on C, then $f(z)$ takes on no value more than once for z in D. See MONODROMY—monodromy theorem.

Darboux's theorem. If $f(x)$ is bounded on (a, b), $M_1, M_2, \cdots, M_n$ and $m_1, m_2, \cdots m_n$ are the upper and lower limits of $f(x)$ on the intervals (a, x_1), $(x_1, x_2), \cdots, (x_{n-1}, b)$, and the lengths of these subintervals converge to zero uniformly as n increases, then

$$\lim_{n\to\infty} [M_1(x_1-a)+M_2(x_2-x_1)+ \cdots + M_n(b-x_{n-1})]$$

and

$$\lim_{n\to\infty} [m_1(x_1-a)+m_2(x_2-x_1)+ \cdots + m_n(b-x_{n-1})]$$

both exist. The former is called the *upper* Darboux integral of $f(x)$ and is written

$$\overline{\int_a^b} f(x)\, dx;$$

the latter is called the *lower* Darboux integral of $f(x)$ and is written

$$\underline{\int_a^b} f(x)\, dx.$$

A necessary and sufficient condition that $f(x)$ be Riemann integrable is that these two integrals be equal.

DATE, *n.* **after date draft.** See DRAFT.

average date. Same as EQUATED DATE.

dividend date. See DIVIDEND—dividend date.

due date of a note or other promise to pay. The date when it is to be paid.

equated date. See EQUATED.

focal date. See COMMUTING—commuting obligations.

DEATH, *n.* **death rate.** Same as RATE OF MORTALITY. See MORTALITY.

central death rate, during one year. The ratio of the number of persons dying during that year to the number living at some particular time during the year; denoted by M_x, where x is the year. Usually M_x is defined as $d_x/[\frac{1}{2}(l_x+l_{x+1})]$, where d_x is the number of the group dying during the year x, l_x is the number living at the beginning of the year, and l_{x+1} the number living at the end. Compare MORTALITY—rate of mortality.

DE-BEN'TURE, *adj., n.* A written recognition of a debt or loan; usually carries the seal of a corporation or other firm and represents funds raised in addition to ordinary stocks and bonds. Such a **debenture bond** is usually unsecured and protected only by the credit and earning power of the issuer.

DEBT, *n.* An obligation to pay a certain sum of money.

DEBT'OR, *n.* One who owes a debt.

DEC'ADE, *n.* (1) A division or group of ten. Thus the numbers from 1 to 10 inclusive form one decade, those from 11 to 20 inclusive another, etc. (2) Ten years.

DEC'A-GON, *n.* A polygon having ten sides. It is a **regular decagon** if the polygon is a regular polygon.

DEC'A-ME'TER, *n.* A term used in the metric system; 10 meters or approximately 32.808 feet. See DENOMINATE NUMBERS in the appendix.

DE-CEL'ER-A'TION, *n.* Negative acceleration. See ACCELERATION.

DEC'I-MAL, *adj., n.* A decimal fraction. Sometimes used of any decimal number.

accurate to a certain decimal place. See ACCURATE.

addition and multiplication of decimals. See ADDITION—addition of decimals, PRODUCT—product of real numbers.

decimal equivalent of a common fraction. A decimal fraction equal to the common fraction. *E.g.,* $\frac{1}{8}=.125$; $\frac{1}{3}=.333\cdots$.

decimal fraction (or decimal). A proper fraction whose denominator (not written) is a power of 10, the power being the number of digits on the right-hand side of a dot, called the **decimal point.** If the number of digits in the numerator of a fraction, whose denominator is a power of 10, is not as large as the power, the required number is obtained by inserting zeros immediately to the right of the decimal point; *e.g.,* $\frac{2}{10}$ is written as a decimal in the form .2, $\frac{23}{100}$ as .23, and $\frac{23}{1000}$ as .023.

decimal measure. Any system of measuring in which each unit is derived from some standard unit by multiplying or dividing the latter by some power of 10. See METRIC—metric system.

decimal number. A decimal fraction or any number containing a decimal point, as .23 or 5.23. *Tech.* Any number written using base 10.

decimal place. The position of a digit to the right of the decimal point; *e.g.,* in 2.357, 3 is in the *first decimal place,* 5 in the *second,* and 7 in the *third.*

decimal point. See above, decimal fraction.

decimal system. (1) The number system which uses ten as its base, the ordinary number system. (2) A number system in which all fractions are expressed as **decimal numbers.** (3) Any system of decimal measurement; the metric system, for instance.

floating decimal point. See FLOATING.

mixed decimal. An integer plus a decimal, as 23.35.

recurring decimal. Same as REPEATING DECIMAL.

repeating decimal. A decimal in which all the digits (after a certain one) consist of a set of one or more digits repeated indefinitely. *E.g.,* $.333\cdots$, $.030303\cdots$, and $.235235\cdots$ are repeating decimals, while π and the square root of 2 cannot be so represented. A repeating decimal can be written as a decimal with a finite number of nonzero digits *plus* a geometric series having the ratio 1/10, or 1/100, or 1/1000, etc. *E.g.,* $.333\cdots=.3+.03+.003+\cdots$, $7.030303\cdots=7+.03+.0003+\cdots$, and $.235235\cdots=.235+.000235\cdots$. Using this property any repeating decimal can be shown to be equal to an ordinary fraction (a quotient of integers) and is therefore a rational number. See SERIES—geometric series. *Syn.* Circulating decimal.

similar decimals. Decimals having the same number of decimal places, as 2.361 and .253. Any decimals can be made similar by annexing the proper number of zeros; *e.g.,* .36 can be made similar to .321 by writing it .360. See DIGIT—significant digits.

DEC'I-ME'TER, *n.* A term used in the metric system; one-tenth of a meter; approximately 3.937 inches. See DENOMINATE NUMBERS in the appendix.

DEC'LI-NA'TION, *n.* **declination of a celestial point.** Its angular distance north or south of the celestial equator, measured along the hour circle passing through the point. See HOUR—hour angle and hour circle.

north declination. The celestial declination of a point which is north of the celestial equator. It is always regarded as *positive.*

south declination. The celestial declination of a point which is south of the celestial equator. It is always regarded as *negative.*

DE'COM-PO-SI'TION, *n.* **decomposition of a fraction.** Breaking a fraction up into **partial fractions.**

DE-CREAS'ING, *adj.* **decreasing function of one variable.** A function whose value decreases as the independent variable increases; a function whose graph falls as the abscissa increases. If the function possesses a derivative, then the function is decreasing in an interval if the derivative is nonpositive throughout the interval, provided the derivative is not identically zero in any interval. A decreasing function is often said to be **strictly decreasing,** to distinguish it from a *monotonic decreasing*

function. *Tech.* A function f is strictly decreasing on an interval (a, b) if

$$f(y) < f(x)$$

for any two numbers x and y of this interval, for which $x < y$.

decreasing the roots of an equation. See ROOT—root of an equation.

monotonic decreasing. See MONOTONIC.

DEC′RE-MENT, *n.* The decrease, at a given age, of the number of lives in a given group—such as the number in the service of a given company.

DEDEKIND CUT. A subdivision of the rational numbers into two nonempty disjoint sets A and B such that: (a) if x belongs to A and y to B, then $x < y$; (b) set A has no largest member (this condition can be replaced by the requirement that B have no least member). *E.g.*, A might be the set of all rational numbers less than 3 and B the set of all rational numbers greater than or equal to 3; or A might be the set of negative rational numbers together with all positive rational numbers x for which $x^2 < 2$ and B the set of all positive rational numbers x for which $x^2 > 2$. In the first example, B has a least member; in the second example, it does not. The **real numbers** can be defined as the set of all Dedekind cuts. It is then convenient to use the notation (A, B) for the real number or cut consisting of the sets A and B. Inequality, addition, and multiplication of real numbers can then be defined as follows: **Inequality**: $(A_1, B_1) > (A_2, B_2)$ if there is an x which belongs to A_1 but does not belong to A_2. **Addition**: If (A_1, B_1) and (A_2, B_2) are real numbers, their sum is the real number (A, B) for which A is the set of all $x+y$ where x belongs to A_1 and y belongs to A_2. **Multiplication**: If (A_1, B_1) and (A_2, B_2) are real numbers, their product is the real number (A, B) for which A is the set of all rational numbers x with the property that, for any positive number ϵ, there are numbers a_1, b_1, a_2, b_2, belonging to A_1, B_1, A_2, B_2, respectively, such that $b_1 - a_1 < \epsilon$, $b_2 - a_2 < \epsilon$, and $x < a_1 a_2$ (note that if A_1 and A_2 each contain positive numbers, then A is the set of all negative rational numbers and all products xy of positive numbers x and y which belong to A_1 and A_2, respectively. A real number (A, B) for which there is a rational number a which is the least upper bound of A is usually identified with a and called a rational number, since the correspondence $a \leftrightarrow (A, B)$ preserves order, sums, and products. See IRRATIONAL—irrational number. Dedekind cuts of the real numbers might now be similarly defined as subdivisions of the real numbers into two subsets, but this would yield a set of objects isomorphic with the real numbers themselves and would not lead to a further extension of the number system.

DE-DUC′TIVE, *adj.* **deductive method or proof.** The method which makes inferences (arrives at conclusions) from accepted principles. *Syn.* Synthetic method.

DE-FAULT′ED, *adj.* **defaulted payments.** (1) Payments made on principal after the due date; occurs most frequently in installment plan paying. (2) Payments never made.

DE-FEC′TIVE, *adj.* **defective equation.** See EQUATION—defective equation.

defective (or deficient) number. See NUMBER—perfect number.

DE-FERRED′, *adj.* **deferred annuity and life insurance.** See ANNUITY, and INSURANCE—life insurance.

DE-FI′CIEN-CY, *n.* **premium deficiency reserve.** See RESERVE.

DEF′I-NITE, *adj.* **definite integral.** See INTEGRAL—definite integral.

definite integration. The process of finding definite integrals.

partial definite integral. One of the definite integrals constituting an ITERATED INTEGRAL.

positive definite and semidefinite quadratic form. See QUADRATIC—quadratic form.

DEF′OR-MA′TION, *adj., n.* **continuous deformation.** A transformation which shrinks, twists, etc., in any way without tearing. *Tech.* A *continuous deformation* of an object A into an object B is a continuous mapping $T(p)$ of A onto B for which there is a function $F(p, t)$ which is defined and

continuous (simultaneously in p and t) for real numbers t with $0 \leqq t \leqq 1$ and points p of A and for which $F(p, 0)$ is the identity mapping of A onto A, $F(p, 0) \equiv p$, and $F(p, 1)$ is identical with $T(p)$. With this definition, a circle in the plane can be continuously deformed to a point (although a circle around the outer circumference of a torus can not be deformed continuously into a point, or into one of the small circles around the body of the torus, without leaving the torus—*i.e.*, with all values of $F(p, t)$ being points on the torus). It is frequently required that a continuous deformation not bring points together; *i.e.*, that the above function $F(p, t)$ be a one-to-one correspondence for each value of t. Then a circle in the plane can be continuously deformed into a square, but not into a point or a figure "8"; a sphere with one hole can be continuously deformed into a disc (a circle and its interior), but not into a cylinder or a sphere. It is said that **two mappings** T_1 and T_2 of a topological space A into a topological space B can be continuously deformed into each other if there is a function $F(x, t)$ which has values in B, which is continuous simultaneously in x and t for x in A and $0 \leqq t \leqq 1$, and for which $F(x, 0) \equiv T_1(x)$ and $F(x, 1) \equiv T_2(x)$ for each x of A. Two mappings are said to be **homotopic** if they can be continuously deformed into each other. If A is contained in B and T_1 is the identity mapping of A onto A, then T_2 is a continuous deformation (in the above sense) of A into the range of T_2 if T_1 can be continuously deformed into T_2. See INESSENTIAL—inessential mapping.

deformation (*in elasticity*). The change in the position of the points of a body accompanied by a change in the distance between them. See STRAIN.

deformation ratio. In a conformal map the magnification is the same in all directions at a point, *i.e.*, $ds^2 = [M(x, y)]^2(dx^2 + dy^2)$. The function $M(x, y)$ is called the **linear deformation ratio.** The **area deformation ratio** is $[M(x, y)]^2$. If the map is given by the analytic function $w = f(z)$ of the complex variable z, then $M = |f'(z)|$. *Syn.* Ratio of magnification.

DE-GEN'ER-ATE, *adj.* **degenerate conics.** See CONIC—degenerate conic.

DE-GREE', *n.* (1) A unit of angular measure. See SEXAGESIMAL—sexagesimal measure of an angle. (2) A unit of measure of temperature. (3) Sometimes used in the same sense as *period* in arithmetic. (4) See the various headings below.

degree of a curve. See CURVE—algebraic plane curve.

degree of a differential equation. The degree of the equation with respect to its highest-order derivative; the greatest power to which the highest-order derivative occurs. The degree of

$$\left(\frac{d^4y}{dx^4}\right)^2 + 2\left(\frac{dy}{dx}\right)^3 = 0$$

is two. See DIFFERENTIAL—differential equation (ordinary).

degree of a polynomial, or equation. The degree of its highest-degree term. The degree of a term in one variable is the exponent of that variable; a term in several variables has degree equal to the sum of the exponents of its variables (or one may speak of the degree with respect to a certain variable, meaning the exponent of that variable). *E.g.*, $3x^4$ is of degree four; $7x^2yz^3$ is of degree six, but of degree two in x. The equation $3x^4 + 7x^2yz^3 = 0$ is of degree six, but is of degree four in x, of degree one in y, and of degree three in z.

general equation of the *n*th degree. See EQUATION—polynomial equation.

spherical degree. See SPHERICAL.

degree of freedom. (*Statistics.*) See FREEDOM.

DEL, *n.* The operator $i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}$, denoted by ∇. See GRADIENT—gradient of a function, and DIVERGENCE—divergence of a vector function.

DELAMBRE'S ANALOGIES. Same as GAUSS'S FORMULAS.

DEL'TA, *adj., n.* The fourth letter of the Greek alphabet, written δ; capital, Δ.

delta method. See FOUR—four-step rule.

DE-MAND', *adj.* **demand note.** See NOTE.

DE MOIVRE. De Moivre's hypothesis of equal decrements. (*Life Insurance.*) The hypothesis that, for practical purposes,

monetary computations can be made on the assumption that the number of a given group that die is the same during each year; in other words, if births are ignored, the numbers of members of a group which are living at the beginnings of successive years form a decreasing arithmetical progression. (Not a sufficiently accurate hypothesis to meet present-day demands.) See MAKEHAM'S LAW.

De Moivre's theorem. A rule for raising a complex number to a power when the number is expressed in polar form. The rule is: Raise the modulus to the given power and multiply the amplitude by the given power; *i.e.*,

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

E.g.,

$$\begin{aligned}(\sqrt{2}+i\sqrt{2})^2 &= [2(\cos 45° + i\sin 45°)]^2\\ &= 4(\cos 90° + i\sin 90°)\\ &= 4i.\end{aligned}$$

DE MORGAN. De Morgan formulas. Let A and B be subsets of a set S. Then the complement of the union of A and B is the intersection of the complements of A and of B, and the complement of the intersection of A and B is the union of the complements of A and of B. These formulas are also valid for any collection of subsets $\{A_\alpha\}$ of S, and may be stated symbolically as

$$(\cup A_\alpha)' = \cap A'_\alpha \quad\text{and}\quad (\cap A_\alpha)' = \cup A'_\alpha,$$

where the complement of a set A is indicated by A'.

DE-NOM'I-NATE, *adj.* **denominate number.** A number whose unit represents a unit of measure—such as 3 inches, 2 pounds, or 5 gallons.

addition, subtraction or multiplication of denominate numbers. The process of reducing them to the same denomination and then proceeding as with ordinary (abstract) numbers. *E.g.*, to find the number of square yards in a room 17′ 6″ by 12′ 4″, the length in yards is $5\frac{5}{6}$ and the width is $4\frac{1}{9}$. The required number of square yards is $5\frac{5}{6}\times 4\frac{1}{9}$. See PRODUCT—product of real numbers.

DE-NOM'I-NA'TION, *n.* **denomination of a bond.** Its par value.

denomination of a number. See NUMBER—denominate number.

DE-NOM'I-NA'TOR, *n.* The term below the line in a fraction; the term that divides the numerator. The denominator of $\frac{2}{3}$ is 3.

common denominator. A quantity divisible by all the denominators under consideration.

DENSE, *adj.* **dense set.** A set E in a space M is **dense** (or **dense in** M, or **everywhere dense**) if every point of M is a point of E or a limit point of E, or (equivalently) if the closure of E is M, or if every neighborhood in M contains a point of E. A set E is **dense in itself** if every point of E is a limit point of E, *i.e.*, if each neighborhood of any point of E contains another point of E. A set E is **nondense** (or **nowhere dense**) relative to M if no neighborhood in M is contained in the closure of E, or (equivalently) if the complement of the closure of E is dense in M. The set of rational numbers is dense in itself and dense in the set R of all real numbers. The set $(0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$ is nowhere dense in R.

DEN'SI-TY, *n.* The mass or amount of matter per unit volume. Since the mass of 1 cc. of water at 4°C. is one gram, density in the *metric system* is the same as *specific gravity*. See SPECIFIC.

density of a sequence of integers. Let $0, a_1, a_2, \cdots$ be an increasing sequence A of integers. Let $F(n)$ be the number of integers (other than zero) which are in this sequence and not larger than n. Then $0 \leqq F(n)/n \leqq 1$ and the greatest lower bound of $F(n)/n$ is said to be the *density* $d(A)$ of the sequence A. Then $d(A) = 0$ if $a_1 \neq 1$, or if A contains "very few" of the integers, *e.g.*, if A is a geometric sequence, a sequence of primes, or a sequence of perfect squares. Let the sum of two sequences A and B of the above type be defined as the sequence of all numbers (arranged in order of size) which can be represented as a sum of a term of one sequence and a term of the other sequence. It can be proved that $d(A+B) \geqq d(A)+d(B)$ if $d(A)+d(B) \leqq 1$. A sequence has density 1 if and only if it contains all nonzero integers.

mean density. The mass divided by the volume. *Tech.*

$$\int_v \rho \, dv \div \int_v dv,$$

where ρ is the density and $\int_v$ denotes the integral taken over the total volume.

metric density. See METRIC.

surface density of charge. Charge per unit area. It is sometimes advantageous to think of a body as having a skin of definite thickness at its boundary. If we then think of all of the charge in the skin as being shifted and concentrated on the outer surface of the skin, then so far as total charge is concerned we may replace the original charge per unit volume of the skin with the charge per unit area obtained by the shift. The volume integral of the former density would equal the surface integral of the latter.

surface density (or moment per unit area) of a double layer (or dipole distribution). Polarization per unit area. Instead of charge being concentrated on the surface, we may consider that we have a continuous distribution of dipoles spread over the surface.

volume density of charge. Charge per unit volume. The most fundamental property of density of charge is that its volume integral taken throughout any given volume V gives the total charge in V. If we start with point charge as the fundamental concept, instead of density, it is found that we may approximate the electric field at points external to the complex as closely as we please by introducing a sufficiently complicated density function. See POTENTIAL—concentration method for the potential of a complex. In terms of total charge, density may be defined as the limit of e_i/V_i, where V_i is a sequence of regions having the property that each V_i is within a sphere of radius r_i with center at P and $\lim r_i = 0$, e_i is the total charge in V_i, and it is required that the limit of e_i/V_i be independent of the sequence of regions V_i.

DE-NU'MER-A-BLE, *adj.* **denumerable set.** Same as COUNTABLE SET. *Denumerably infinite* is used in the same sense as *countably infinite*, to denote an infinite set whose elements can be put into one-to-one correspondence with the positive integers.

DE-PAR'TURE, *n.* **departure between two meridians on the earth's surface.** The length of the arc of the parallel of latitude subtended by the two meridians. This grows shorter the nearer the parallel of latitude is to a pole. Used in parallel sailing.

DE-PEND'ENCE, *n.* **domain of dependence.** For an initial-value problem for a partial differential equation, the value of the solution at a point p and time t might be determined by the initial values on only a portion of their entire range, called the *domain of dependence*. *E.g.*, for the wave equation $(1/c^2)u_{tt} = u_{xx}$, with initial conditions $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, the value of the solution at the point x and time t depends only on the initial values in the interval $[x - ct, x + ct]$. See HUYGEN—Huygen's principle.

DE-PEND'ENT, *adj.* **dependent event.** See EVENT.

dependent functions. A set of functions, one of which can be expressed as a function of the others; *e.g.*,

$$u = (x+1)/(y+1)$$

and

$$v = \sin [(x+1)/(y+1)]$$

are dependent functions, for $v = \sin u$. *Syn.* Interdependent functions. Functions that are not *independent* are said to be *dependent*. See INDEPENDENT.

dependent variable. See FUNCTION—function of one variable.

linearly dependent. A set of quantities $z_1, z_2, \cdots, z_n$ (vectors, matrices, polynomials, etc.) is said to be **linearly dependent** (in a given region) if there is a linear combination of them,

$$a_1 z_1 + a_2 z_2 + \cdots + a_n z_n,$$

which is identically zero (zero for all values of the variables in the given region). The coefficients a_i must belong to a specified field (real numbers, complex numbers, etc.), and at least one of them must be nonzero. A set of quantities is said to be **linearly independent** if they are not linearly depen-

dent. The quantities $x+2y$ and $3x+6y$ are linearly dependent, since

$$-3(x+2y)+(3x+6y)\equiv 0.$$

The numbers 3 and π are linearly independent with respect to rational numbers, since $a_1\cdot 3+a_2\cdot\pi$ can not be zero if a_1 and a_2 are rational numbers, not both zero. Since $-1\cdot 3+(3/\pi)\pi=0$, 3 and π are linearly dependent with respect to real numbers. Similarly, $1+i$ and $3-5i$ are *linearly independent* with respect to the field of real numbers and *linearly dependent* with respect to the field of complex numbers. If $v^k=(x_1^k, x_2^k, \cdots, x_n^k)$, $k=1, 2, \cdots, r$, are vectors (or points) of n-dimensional space, then these vectors are linearly dependent if there exist numbers $\lambda_1, \lambda_2, \cdots, \lambda_r$, not all zero, such that $\lambda_1 v^1+\lambda_2 v^2+\cdots+\lambda_r v^r=0$. This means that a similar equation holds for each component: $\lambda_1 x_p^1+\lambda_2 x_p^2+\cdots+\lambda_r x_p^r=0$ for each p. See GRAM DETERMINANT, and WRONSKIAN.

DE-PRE'CI-A'TION, *adj., n.* The loss in value of equipment; the difference between the *cost value* and the *book value*.

depreciation charge. A decrease in the book value, usually annual, such that the total of these decreases, without interest, will equal the original book value (or cost) minus the scrap value at the end of a certain number of years (the **total depreciation**). There are various methods for computing depreciation charges. For the **straight line method,** equal depreciation charges are made each year. For the **declining balance method** (or **constant percentage method**), each depreciation charge is computed as a constant percent of the book value at the time of computation; this percent is equal to $(1-\sqrt[n]{R/C})\times 100$, where C is the cost, R the scrap value, and n the number of years.

DE-PRESSED', *adj.* **depressed equation.** The equation resulting from reducing the number of roots in an equation, *i.e.*, by dividing out the difference of the unknown and a root; *e.g.*, $x^2-2x+2=0$ is the *depressed* equation obtained from $x^3-3x^2+4x-2=0$ by dividing the left member of the latter by $x-1$.

DE-PRES'SION, *n.* **angle of depression.** See ANGLE—angle of depression.

DE-RIV'A-TIVE, *n.* The instantaneous rate of change of a function with respect to the variable. Let $y=f(x)$ be a given **function of one variable** and let Δx denote a number (positive or negative) to be added to the number x. Let Δy denote the *corresponding increment of* y:

$$\Delta y=f(x+\Delta x)-f(x).$$

Form the increment ratio

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

Then let Δx approach zero. If $\Delta y/\Delta x$ approaches a *limit* as Δx approaches zero, this limit is called the derivative of $f(x)$ at the point x. This derivative is denoted by

$$y',\ D_x y,\ \frac{dy}{dx},\ f'(x),\ D_x f(x),$$

$$\frac{d}{dx}f(x) \quad\text{or}\quad \frac{df(x)}{dx}.$$

The derivative, evaluated at a point, $x=a$, is written

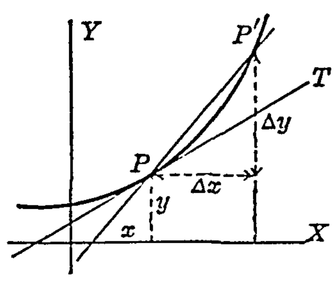
$$f'(a),\quad D_x f(x)_{x=a},\quad \left[\frac{df(x)}{dx}\right]_{x=a},\quad f'(x)|_{x=a},$$

etc.

The definition of the derivative of $f(x)$ at the point a can also be written in the form

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

Two special interpretations of the derivative are: (1) As the **slope** of a curve: In the figure, the ratio $\Delta y/\Delta x$ is the slope of the



line PP'. Therefore the limit of this ratio as Δx approaches zero is the slope of the tangent PT. It follows that a function is increasing at points where the derivative is positive and decreasing at points where the derivative is negative. If the derivative is

zero, the function may be either increasing or decreasing, or it may have a **maximum** or **minimum** at the point (see MAXIMUM). (2) As the **speed** and **acceleration** of a moving particle: If the function $y=f(t)$ is equal to the distance traversed by the particle in time t, then the derivative of $f(t)$ at $t=t_1$ is the speed of the particle at the time t_1; the increment ratio $\Delta y/\Delta t$ is the *average speed* during the time interval Δt. The derivative of the speed (the second derivative of the distance) at $t=t_1$ is the acceleration of the particle at the time t_1. There are many powerful and useful formulas for evaluating derivatives (see DIFFERENTIATION FORMULAS in the appendix). *E.g.*, the **derivative of a sum** is the sum of the derivatives, the derivative of x^n is nx^{n-1}, and the derivative of a function $F(u)$, where u is a function of x, is given by the following formula (**chain rule**):

$$\frac{dF(u)}{dx}=\frac{dF(u)}{du}\frac{du}{dx}$$

From these rules, it follows that $D_x(x^3+x^2)=3x^2+2x$, $D_x[x^{1/2}+(x^2+7)^\pi]=\frac{1}{2}x^{-1/2}+\pi(x^2+7)^{\pi-1}(2x)$, etc. *Syn.* Differential coefficient. See various headings under ACCELERATION, DIFFERENTIAL, DIFFERENTIATION, LEIBNIZ' THEOREM, TANGENT, VELOCITY. Similar definitions of the derivative are used when the function is of a different type (see below, derivative of a function of a complex variable, derivative of a vector).

chain rule for derivatives. See CHAIN.

covariant derivative of a tensor. See COVARIANT.

derivative of a function of a complex variable. The complex function $w=f(z)$, defined at all points in a neighborhood of z_0, has a derivative at z_0 provided

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

exists. The limit is called the derivative of $f(z)$ at z_0 and is denoted by $f'(z)|_{z=z_0}$, $f'(z_0)$, $\left.\frac{df}{dz}\right|_{z=z_0}$, etc. See ANALYTIC—analytic function of a complex variable.

derivative of higher order. Derivatives of other derivatives, the latter being considered as functions of the independent variable just as was the function of which the first derivative was taken. *E.g.*, $y=x^3$ has its first derivative $y'=3x^2$ and its second derivative $y''=6x$, gotten by taking the derivative of $3x^2$; similarly $y'''=6$, and $y^{[4]}=0$.

derivative of an integral. (1) The derivative of $\int_a^x f(t)\,dt$ at the point x_0 exists and is equal to $f(x_0)$, provided $f(x)$ is integrable on the interval (a, b) and is continuous at x_0, where x_0 is on the open interval (a, b). See FUNDAMENTAL—fundamental theorem of the integral calculus. (2) If $f(t, x)$ has a partial derivative $\partial f/\partial t=f_t(t, x)$ which is continuous in both x and t and $\int_a^b f(t, x)\,dx=F(t)$ exists, then dF/dt exists and equals $\int_a^b f_t(t, x)\,dx$. This is sometimes called *Leibniz's rule*, although he did not specify the conditions on $f(t, x)$. (3) Combining (1) and (2) by using the *chain rule* for partial differentiation gives the formula:

$$D_t\int_u^v f(t, x)\,dx=D_tv\cdot f(t, v)-D_tu\cdot f(t, u)+\int_u^v f_t(t, x)\,dx.$$

E.g., the derivative of $\int_1^2 (x^2+y)\,dx$, with respect to y, is $\int_1^2 dx$, and the derivative, with respect to y, of $\int_y^{y^2} (x^2+y)\,dx$ is $\int_y^{y^2} dx+(y^4+y)2y-(y^2+y)$.

derivative from parametric equations. See PARAMETRIC—differentiation of parametric equations.

derivative of a vector. Let t be the arc-length of a curve and suppose that corresponding to each point of the curve there is a vector. Let $v(t)$ denote the vector at the point t of the curve. Then

$$\lim_{\Delta t\to 0}\frac{v(t+\Delta t)-v(t)}{\Delta t}$$

is the derivative of the vector, relative to the arc-length of the curve, at the point t, provided this limit exists. See ACCELERATION and VELOCITY.

directional derivative. See DIRECTIONAL.

normal derivative. See NORMAL—normal derivative.

partial derivative. See PARTIAL.

particle derivative. The rate of change of a quantity with respect to the time, along a particle of a fluid. *E.g.*, if $\rho(x, y, z, t)$ is the density of the fluid and $x=x(t)$, $y=y(t)$, $z=z(t)$ are the equations of motion of a certain particle of this fluid, then the total derivative $d\rho/dt=\partial\rho/\partial x\, dx/dt + \partial\rho/\partial y\, dy/dt+\partial\rho/\partial z\, dz/dt+\partial\rho/\partial t$ is a **particle derivative.** It gives the rate of change of the density along a given particle of the fluid as opposed to the rate of change at a fixed geometrical point.

total derivative. See CHAIN—chain rule (for partial differentiation).

DE-RIVED′, *adj.*, *n.* **derived curve** (first derived curve). The curve whose ordinates, corresponding to given abscissas, are equal in sign and numerical value to the slope of some given curve for the same values of the abscissas. *E.g.*, the curve whose equation is $y=3x^2$ is the derived curve of the curve whose equation is $y=x^3$. The derived curve of the first derived curve is called the **second derived curve,** etc.

derived equation. (1) *In algebra,* an equation obtained from another one by transposing terms, powering both sides, or multiplying or dividing by some quantity. A derived equation is not always equivalent to the original, *i.e.*, does not always have the same set of roots. (2) *In calculus,* the equation resulting from differentiating the given equation. See above, derived curve.

derived set. See CLOSURE.

DESARGUE'S THEOREM **on perspective triangles.** If the lines joining corresponding vertices of two triangles pass through a point, the intersections of the three pairs of corresponding sides lie on a line, and conversely.

DESCARTES. **Descartes' rule of signs.** A rule determining an upper bound to the number of positive roots and to the number of negative roots of an equation. The rule as ordinarily used states that an algebraic equation, $f(x)=0$, cannot have a greater number of positive roots than it has variations in sign, nor a greater number of negative roots than the equation $f(-x)=0$ has variations in sign. For instance, the equation

$$x^4-x^3-x^2+x-1=0$$

has three variations in sign and hence cannot have more than three positive roots. Since $f(-x)=0$ takes the form $x^4+x^3-x^2-x-1=0$, in which there is only one variation of sign, the original equation cannot have more than one negative root. Descartes' Rule of Signs in full shows more than is indicated above, the complete criteria being: The number of positive real roots of an equation with real coefficients is either equal to the number of its variations of sign, or is less than that number by an even integer, a root of multiplicity m being counted as m roots. As before, the equation can be tested for negative roots by applying this rule to the equation $f(-x)=0$.

folium of Descartes. See FOLIUM.

DE-TACHED′, *adj.* **detached coefficient.** See COEFFICIENT—detached coefficients.

DE-TACH′MENT, *n.* **rule of detachment.** If an implication is true and the antecedent is true, then the consequent is true. *E.g.*, if the statements "If my team lost, I will eat my hat" and "My team lost" are both true, then the statement "I will eat my hat" is true.

DE-TER′MI-NANT, *n.* A square array of quantities, called **elements,** symbolizing the sum of certain products of these elements. The number of rows (or columns) is called the **order** of the determinant. The diagonal, from the upper left corner to the lower right corner, is called the **principal** (or **leading**) **diagonal.** The diagonal from the lower left corner to the upper right corner is called the **secondary diagonal.** A determinant of the second order is a square array of type

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

whose value is $a_1b_2-a_2b_1$. A determinant of the third order is a square array of type

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

whose value is $(a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2)$. This expression is equal to the sum of the products of the elements in a given column (or row) by their cofactors (see below, expansion of a determinant by minors). The element in row i and column j of a determinant is usually indicated by some such symbol as a_{ij}, where i is called the row index and j the column index. The value of the determinant is then the algebraic sum of all products obtained by taking one and only one factor from each row and each column and attaching the positive or negative sign to each product according as the column (or row) indices form an even or an odd permutation when the row (or column) indices are in natural order (1, 2, 3, etc.). *E.g.*, the term $a_{13}a_{21}a_{34}a_{42}$ of the expansion of a determinant of order four has the column indices in order (3, 1, 4, 2). This term should have a negative sign attached, since three successive interchanges will change the column indices to (1, 3, 4, 2), (1, 3, 2, 4), and (1, 2, 3, 4), the last being in natural order. In practice, a determinant is usually evaluated by using minors (see below, expansion of a determinant by minors, Laplace's expansion of a determinant) after simplifying the determinant by use of certain properties of determinants. Some of the simple properties of determinants are: (1) If all the elements of a column (or row) are zero, the value of the determinant is zero. (2) Multiplying all the elements of a column (or row) by the same quantity is equivalent to multiplying the value of the determinant by this quantity. (3) If two columns (or rows) have their corresponding elements alike, the determinant is zero. (4) The value of a determinant is unaltered if the same multiple of the elements of any column (row) are added to the corresponding elements of any other column (row). (5) If two columns (or rows) of a determinant are interchanged, the sign of the determinant is changed. (6) The value of a determinant is unaltered when all the corresponding rows and columns are interchanged.

cofactor of an element in a determinant. See MINOR—minor of an element in a determinant.

conjugate elements of a determinant. See CONJUGATE—conjugate elements of a determinant.

determinant of the coefficients of a set of linear equations. For n equations in n unknowns, the determinant whose element in the ith row and jth column is the coefficient of the jth variable in the ith equation (the variables being written in the same order in each equation). This determinant is not defined if the number of equations is not equal to the number of variables (see MATRIX—matrix of the coefficients). The determinant of the coefficients of the unknowns of

$$\begin{matrix} 2x+3y-1=0 \\ 4x-7y+5=0 \end{matrix} \quad \text{is} \quad \begin{vmatrix} 2 & 3 \\ 4 & -7 \end{vmatrix}.$$

(the two equations joined by "and")

determinant of a matrix. See MATRIX.

elementary operations on determinants. See ELEMENTARY.

expansion of a determinant by minors. The expansion of the determinant by writing it in terms of determinants of one lower order, using the elements of a selected row (or column) as coefficients. The determinant is equal to the sum of the products of the elements of that row (or column) by their *signed minors* (or *cofactors*) (see MINOR—minor of an element in a determinant). *E.g.*,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$=a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}+a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

Fredholm's determinant. (*Integral Equations.*) See FREDHOLM.

functional determinant. Same as JACOBIAN.

Gram determinant. Same as GRAMIAN.

Laplace's expansion of a determinant. Let A be a determinant of order n and $A^{r_1r_2,\cdots r_k}_{s_1s_2,\cdots s_k}$ be the determinant formed from A by using the elements in rows $r_1, r_2, \cdots r_k$ and columns $s_1, s_2, \cdots, s_k$. Laplace's expansion is

$$A=\Sigma(-1)^h(A^{r_1\cdots r_k}_{i_1\cdots i_k})(A^{r_{k+1}\cdots r_n}_{i_{k+1}\cdots i_n}),$$

where $(r_1, r_2, \cdots, r_n)$ and $(i_1, i_2, \cdots, i_n)$ are permutations of the integers $(1, 2, \cdots, n)$, h is the number of inversions necessary to bring the order $(i_1, i_2, \cdots, i_n)$ into the order $(r_1, r_2, \cdots, r_n)$, and the summation is to be

taken over the $n!/[k!(n-k)!]$ ways of choosing the combinations $(i_1, i_2, \cdots, i_k)$ from the integers $(1, 2, \cdots, n)$. See above, expansion of a determinant by minors, which is the special case of Laplace's expansion for $k=1$.

minor of an element in a determinant. See MINOR.

multiplication of determinants. See MULTIPLICATION—multiplication of determinants.

numerical determinant. A determinant whose elements are numbers (absolute numbers).

skew-symmetric determinant. See SKEW.

symmetric determinant. See SYMMETRIC—symmetric determinant.

Vandermonde determinant. A determinant having unity in each place of the first row, the second row unspecified, and the elements of the ith row the $(i-1)$th power of the corresponding elements of the second row, or the transpose of such a determinant.

DE-VEL'OP-A-BLE, *adj.*, *n.* **developable surface.** The envelope of a one-parameter family of planes; a surface that can be developed, or rolled out, on a plane without stretching or shrinking; a surface for which the total curvature vanishes identically. See below, rectifying developable of a curve, polar developable of a curve, and TANGENT—tangent surface of a curve.

polar developable of a space curve. The envelope of the normal planes of the space curve; the totality of points on the polar lines of the curve. See above, developable surface, and NORMAL—normal plane to a space curve at a point.

rectifying developable of a space curve. The envelope S of the rectifying planes of the space curve C. This developable surface S is called the **rectifying developable** of C because the process of developing S on a plane results in rolling C out along a straight line. See above, developable surface, and RECTIFYING—rectifying plane of a space curve at a point.

DE'VI-ATE, *n.* **standard deviate.** (*Statistics.*) The standard deviate value of a particular value x_1 of the variable x is $(x_1-\bar{x})/\sigma$, where the *mean* and *standard deviation* of x are $\bar{x}$ and σ, respectively.

DE'VI-A'TION, *n.* (*Statistics.*) (1) The variation from the trend. (2) The difference between the particular number and the average of the set of numbers under consideration. *Syn.* Measure of dispersion.

algebraic deviation. (*Statistics.*) Deviation which is counted positive if the magnitude is greater than the average or trend, and negative if less.

absolute mean deviation. The arithmetic mean of the *numerical* values of the deviations. For continuous variables, the absolute mean deviation is

$$\int_{-\infty}^{\infty} |x-E(x)|f(x)\,dx,$$

or, for discrete variables,

$$\sum_{i=1}^{n} |x_i-E(x_i)|/n,$$

where $f(x)$ is the *frequency function* and $E(x)$ the *expected value* of x.

mean deviation. The quantity

$$\sum_{i=1}^{n} |x_i-\bar{x}|/n,$$

where $\bar{x}$ is the arithmetic mean or the median. The use of the latter eliminates certain computational difficulties. The sum of deviations (with regard to sign) is zero about the mean and, without regard to sign, it is a minimum around the median. Either method yields an inefficient estimate of the standard deviation of a normal distribution.

probable deviation. The deviation that will be exceeded by a random variable with probability $\frac{1}{2}$. In a normal distribution, the probable deviation around the mean is $\pm .675$ standard-deviation units. It is not the deviation that is most probable, or even "probable." No longer generally used. *Syn.* Probable error.

quartile deviation. One-half of the difference between the two quartile magnitudes.

standard deviation. The square root of the arithmetic mean of the squares of the deviations from the mean. *Syn.* Root mean square deviation. In a normal distribution the parameter σ is the standard deviation, where the frequency function is

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/\sigma^2}.$$

Same as the square root of the *second moment* around the mean expectation of the variable; $\sigma^2 = \int_{-\infty}^{\infty} (x - E(x))^2 f(x)\,dx$. To estimate the standard deviation of a normal distribution from a random sample, the usually used formula is $\sigma = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$, where the mean x is estimated from the sample. If the mean of the population is known, $n-1$ is replaced by n. This yields a maximum-likelihood estimate of the standard deviation and an unbiased minimum-variance estimate of the variance.

DEXTRORSUM [*Latin*] or **DEX'TRORSE,** *adj.* Same as RIGHT-HANDED CURVE. See RIGHT.

DI-AG'O-NAL, *adj., n.* **diagonal of a determinant.** See DETERMINANT.

diagonal of a matrix. See MATRIX.

diagonal of a polygon. A line connecting two nonadjacent vertices. In *elementary geometry* it is thought of as the **line segment** between nonadjacent vertices; in *projective geometry* it is the straight line (of infinite length) passing through two nonadjacent vertices.

diagonal of a polyhedron. A line segment between any two vertices that do not lie in the same face. See PARALLELEPIPED.

diagonal scale, for a rule. A scale in which the rule is divided crosswise and diagonally by systems of parallel lines. *E.g.*, suppose that there are 11 longitudinal (lengthwise of the ruler) lines per inch (counting the lines at the beginning and end of the inch interval) and one diagonal line per inch. Then the intersections of the diagonal lines with the longitudinal lines are $\frac{1}{10}$ inch apart longitudinally, for the 10 segments cut off on any one diagonal by the horizontal lines are equal, and hence the 10 corresponding distances measured along the longitudinal lines must be equal. Thus the inch is divided into 10 equal parts. Similarly one diagonal per $\frac{1}{10}$ inch scales the ruler in $\frac{1}{100}$ inch, etc.

DI'A-GRAM, *n.* A drawing representing certain data and, perhaps, conclusions drawn from the data; a drawing representing pictorially (graphically) a statement or a proof; used to aid readers in understanding algebraic explanations.

Argand diagram. See ARGAND.

indicator diagram. See INDICATOR.

DI'A-LYT'IC, *adj.* **Sylvester's dialytic method.** See SYLVESTER.

DI-AM'E-TER, *n.* **conjugate diameters.** See CONJUGATE—conjugate diameters.

diameter of a central quadric surface. The locus of the centers of parallel sections of the central quadric. This locus is a straight line.

diameter of a circle. See CIRCLE.

diameter of a conic. Any straight line which is the locus of the midpoints of a family of parallel chords; a chord joining the points of tangency of two parallel tangents to the conic. Any conic has infinitely many diameters. In the central conics, ellipses and hyperbolas, they form a pencil of lines through the center of the conic. See CONJUGATE—conjugate diameters.

diameter of a set of points. See BOUNDED—bounded set of points.

DI-AM'E-TRAL, *adj.* **conjugate diametral planes.** Two diametral planes, each of which is parallel to the set of chords defining the other.

diametral line in a conic (ellipse, hyperbola or parabola). Same as DIAMETER.

diametral plane of a quadric surface. A plane containing the middle points of a set of parallel chords.

DIDO'S PROBLEM. The problem of finding the curve, with a given perimeter, which incloses the maximum area. The required curve is a circle. If part of the boundary is freely given as a straight-line segment of arbitrary length, as along a river, then the solution is a semicircle.

DIF'FER-ENCE, *adj., n.* The result of subtracting one quantity from another. *Syn.* Remainder.

difference equation. See below, ordinary difference equation, and partial difference equation.

difference of like powers of two quantities, factorability of. If the power is odd, the

difference of like powers of two quantities is divisible by the difference of the two quantities; whereas if the power is even, the difference is divisible by both the sum and the difference of the two quantities. *E.g.*,

$$x^3-y^3=(x-y)(x^2+xy+y^2), \quad \text{while}$$
$$x^4-y^4=(x-y)(x+y)(x^2+y^2).$$

See SUM—sums of like powers of two quantities.

difference of two sets. The difference $A-B$ of two sets A and B is the set of all points which belong to A and do not belong to B. The **symmetric difference** of two sets A and B is the set which contains all the points that belong to one of the sets but not to the other; *i.e.*, the symmetric difference of A and B is the union of the sets $A-B$ and $B-A$. Some of the notations used for the symmetric difference of A and B are $A \ominus B$, $A \nabla B$, $A+B$. See RING—ring of sets.

difference of two squares. The result of subtracting the square of one number from the square of another. If a and b denote the numbers, the difference of the squares, (a^2-b^2), is equal to $(a+b)(a-b)$.

difference quotient. The increment of the function, corresponding to an increment of the independent variable, divided by the latter; *e.g.*, if the function, f, is defined by $f(x)=x^2$, the *difference quotient* is

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{(x+\Delta x)^2-x^2}{\Delta x}$$
$$=2x+\Delta x.$$

See DERIVATIVE.

differences of the first order or **first-order differences.** The sequence formed by subtracting each term of a sequence from the next succeeding term. The first-order differences of the sequence $(1, 3, 5, 7, \cdots)$ would be $(2, 2, 2, \cdots)$.

differences of the second order or **second-order differences.** The *first-order differences* of the first-order differences; *e.g.*, the first-order differences of the sequence $(1, 2, 4, 7, 11, \cdots)$ are $(1, 2, 3, 4, \cdots)$, while the second-order differences are $(1, 1, 1, \cdots)$. Similarly, the *third-order differences* are the first-order differences of the second-order; and, in general, the rth-*order differences* are the first-order differences of the $(r-1)$th order. If the sequence is $(a_1, a_2, a_3, \cdots, a_n, \cdots)$, the first-order differences are a_2-a_1, a_3-a_2, $a_4-a_3, \cdots$, the second-order are $a_3-2a_2+a_1$, $a_4-2a_3+a_2, \cdots$, and the rth order are:

$$[a_{r+1}-ra_r+\{r(r-1)/2!\}a_{r-1}- \cdots \pm a_1],$$
$$[a_{r+2}-ra_{r+1}+\{r(r-1)/2!\}a_r- \cdots$$
$$\pm a_2], \cdots.$$

finite differences. The differences derived from the sequence of values obtained from a given function by letting the variable change by arithmetic progression. If f is the given function, the arithmetic progression $(a, a+h, a+2h, \cdots)$ gives the sequence of values: $f(a), f(a+h), f(a+2h), \cdots$. The differences may be of any given order. The first-order differences are $f(a+h)-f(a), f(a+2h)-f(a+h), \cdots$. The successive differences of order one, two, three, etc., are written: $\Delta f(x), \Delta^2 f(x), \Delta^3 f(x)$, etc. In the study of difference equations, it is sometimes understood that $\Delta f(x)=f(x+1)-f(x)$, $\Delta^2 f(x)=\Delta\Delta f(x)=f(x+2)-2f(x+1)+f(x)$, etc.

ordinary difference equation. An expressed relation between an independent variable x and one or more dependent variables $f(x), g(x), \cdots$, and any successive differences of f, g, etc., as $\Delta f(x)=f(x+h)-f(x)$, $\Delta^2 f(x)=f(x+2h)-2f(x+h)+f(x)$, etc., or equivalently, the results of any successive applications of the operator E, where $Ef(x)=f(x+h)$. The **order** of a difference equation is the order of the highest difference (or exponent of the highest power of E), and the **degree** is the highest power to which the highest difference is involved. A difference equation is **linear** if it is of the first degree with respect to all of the quantities $f(x)$, $\Delta f(x)$, $\Delta^2 f(x)$, etc.; or $f(x)$, $Ef(x)$, etc. The equation $f(x+1)=xf(x)$ is a linear difference equation. See below, partial difference equation.

partial difference equation. An expressed relation between two or more independent variables $x, y, z, \cdots$, one or more dependent variables $f(x, y, z, \cdots)$, and partial differences of these dependent variables.

partial differences. Partial differences of a function $f(x, y, z, \cdots)$ of two or more variables are any of the expressions arising from successive derivation of ordinary differences, holding all the variables but one fixed at each step.

tabular difference. See TABULAR.

DIF′FER-ENC-ING, *p.* **differencing a function.** Taking the successive differences. See DIFFERENCE—finite differences.

DIF′FER-EN′TI-A-BLE, *adj.* For a function of one variable, possessing a derivative. For a function of several variables, see DIFFERENTIAL.

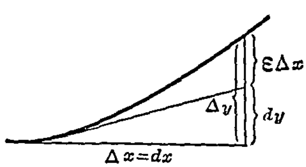
DIF′FER-EN′TIAL, *adj., n.* Let $y=f(x)$ be a **function of one variable** for which the derivative $f'(x)$ exists. Then the differential of y is

$$dy=f'(x)\,dx,$$

where dx is an independent variable. Thus dy is a function of the two variables x and dx. Since the derivative of x is 1, it follows that the differential of x is dx. The differential dy has the property that, if x is changed to Δx, the resulting change Δy in y differs from dy (with dx set equal to Δx) by an infinitesimal of higher order than Δx; for since

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=f'(x),\quad \frac{\Delta y}{\Delta x}=f'(x)+\epsilon,$$

where ϵ is an infinitesimal. Hence



$$\Delta y=f'(x)\,\Delta x+\epsilon\,\Delta x,$$

from which dy is obtained by dropping the infinitesimal, $\epsilon\,\Delta x$, and writing dx in the place of Δx. The **differential** (the **total differential**), **of a function of several variables,** $f(x_1, x_2, \cdots, x_n)$, is the function

$$df=\frac{\partial f}{\partial x_1}\,dx_1+\frac{\partial f}{dx_2}\,dx_2+\cdots+\frac{\partial f}{dx_n}\,dx_n,$$

which is a function of the independent variables $x_1, \cdots, x_n, dx_1, \cdots, dx_n$. Each of the terms $\frac{\partial f}{\partial x_i}\,dx_i$ is called a **partial differential.** If $u=f(x, y, z)$ and z is a function of x and y, then

$$du=\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}\right)dy.$$

Each term on the right is a partial differential, but is sometimes called an **intermediate differential** in cases such as the above where at least one of the variables of f is dependent on the others. The formulas for the differentials of functions of one or more variables hold when the functions are **composite.** One may, in that case, replace the differentials of the variables by their total differentials in terms of the variables of which they are functions. *E.g.*, if $z=f(x, y)$, $x=u(s, t)$ and $y=v(s, t)$, then

$$\begin{aligned} dz&=\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy\\ &=\frac{\partial f}{\partial x}\left[\frac{\partial u}{\partial s}\,ds+\frac{\partial u}{dt}\,dt\right]\\ &\qquad+\frac{\partial f}{\partial y}\left[\frac{\partial v}{\partial s}\,ds+\frac{\partial v}{\partial t}\,dt\right]. \end{aligned}$$

For a function $f(x, y)$ of two variables,

$$df=\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy.$$

The function $f(x, y)$ is said to be **differentiable** at (x, y) if, for any $\epsilon>0$, there is a $\delta>0$ such that if Δx and Δy are numbers with $|\Delta x|$ and $|\Delta y|$ each less than δ, then

$$\left|\Delta f-\left(\frac{\partial f}{\partial x}\cdot\Delta x+\frac{\partial f}{\partial y}\,\Delta y\right)\right|<\epsilon(|\Delta x|+|\Delta y|),$$

where $\Delta f=f(x+\Delta x, y+\Delta y)-f(x, y)$. Thus if the independent variables are changed by small amounts, the change in a *differentiable function* can be approximated by its differential with an error which is small relative to the changes in the variables. A function with continuous partial derivatives is differentiable. This concept of approximation can be used to define differentials for more general situations (see FUNCTIONAL—differential of a functional). *Syn.* Total differential. See INCREMENT—increment of a function, and ELEMENT—element of integration.

adjoint of a differential equation. See ADJOINT.

binomial differential. See BINOMIAL.

differential analyzer. An instrument for solving differential equations (or systems of differential equations) by mechanical means. The **Bush differential analyzer,** designed in the 1920's by Vannevar Bush, was the first differential analyzer ever built. It was based on the two fundamental operations of addition and integration, performed respectively by differential gear boxes and wheel and disc mechanisms.

differential of arc, area, attraction, mass, moment, moment of inertia, pressure, volume, and work. Same as ELEMENT OF ARC, AREA, ATTRACTION, etc. See ELEMENT—element of integration.

differential calculus. See CALCULUS.

differential coefficient. Same as DERIVATIVE.

differential equation (ordinary). An equation containing at most two variables, and derivatives of the first or higher order of one of the variables with respect to the other, such as $y(dy/dx)+2x=0$. The order of a differential equation is the order of the highest derivative which appears. When an equation contains only derivatives of the first order it is frequently written in terms of differentials. This is permissible because the first derivative may be treated as the quotient of the differentials. Thus the equation above may be written $y\,dy+2x\,dx=0$. See the headings below, and PARTIAL—partial differential equation.

differential equations of Bessel, Clairaut, Gauss, Hermite, Laguerre, Laplace, Legendre, Mathieu, Sturm-Liouville, Tchebycheff. See the respective names.

differential equations with variables separable. Ordinary differential equations which can be written in the form $P(x)\,dx+Q(y)\,dy=0$, by means of algebraic operations performed on the given equation. Its general solution is obtainable directly by integration.

differential form. A homogeneous polynomial in differentials. *E.g.*, if $g_{i_1i_2\cdots i_n}$ is a symmetric covariant tensor field and $t_{\beta_1\beta_2\cdots\beta_q}$ is an alternating covariant tensor field, then

$$g_{i_1i_2\cdots i_r}\,dx^{i_1}\,dx^{i_2}\cdots dx^{i_r}$$

and

$$t_{\beta_1\beta_2\cdots\beta_q}\,dx^{\beta_1}\,dx^{\beta_2}\cdots dx^{\beta_q}$$

transform like scalar fields and are a symmetric differential form and alternating differential form, respectively.

differential of a functional. See FUNCTIONAL.

differential geometry. The theory of the properties of configurations in the neighborhood of one of its general elements. See GEOMETRY—metric differential geometry; and below, projective differential geometry.

differential operator. A polynomial in the operator D, where D stands for d/dx and Dy for dy/dx. *E.g.*, $(D^2+xD+5)y=d^2y/dx^2+x(dy/dx)+5y$. Symbols of the form $1/f(D)$, where $f(D)$ is a polynomial in D, are called **inverse differential operators.** *E.g.*, the symbol $1/(D-a)$ arises from the equation $dy/dx-ay=f(x)$. The equation is written in the form $(D-a)y=f(x)$. Then

$$y=\frac{1}{(D-a)}f(x)$$

is a solution, where

$$\frac{1}{(D-a)}f(x)=Ce^{ax}+e^{ax}\int e^{-ax}f(x)\,dx.$$

differential parameter of a surface. For a given function $f(u,v)$ and a given surface S: $x=x(u,v)$, $y=y(u,v)$, $z=z(u,v)$, the function

$$\Delta_1 f\equiv\left(\frac{df}{ds}\right)^2$$
$$=\frac{E\left(\frac{\partial f}{\partial v}\right)^2-2F\frac{\partial f}{\partial u}\frac{\partial f}{\partial v}+G\left(\frac{\partial f}{\partial v}\right)^2}{EG-F^2},$$

where the derivative df/ds is evaluated in the direction perpendicular to the curve $f=$const. on S, is invariant under change of parameters: $u=u(u_1,v_1)$, $v=v(u_1,v_1)$. See VARIATION—variation of a function on a surface. The invariant $\Delta_1 f$ is called the **differential parameter of the first order** for the function f relative to the surface S. See below, mixed differential parameter of the first order.

The **differential parameter of the second order** is the invariant

$$\Delta_2 f\equiv\frac{\frac{\partial}{\partial u}\left(\frac{G\frac{\partial f}{\partial u}-F\frac{\partial f}{\partial v}}{(EG-F^2)^{1/2}}\right)+\frac{\partial}{\partial v}\left(\frac{E\frac{\partial f}{\partial v}-F\frac{\partial f}{\partial u}}{(EG-F^2)^{1/2}}\right)}{(EG-F^2)^{1/2}}.$$

For a conformal map of the (u,v)-domain of definition on the surface S, which is a map with $E=G=\sigma(u,v)\neq 0$, $F=0$, the numerator of $\Delta_2 f$ reduces to the Laplacian of f: $\Delta_2 f=\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right)/\sigma$. There are other differential invariants of the second and higher orders, such as $\Delta_1\Delta_1(f,g)$, $\Delta_1\Delta_2 f$.

etc. See below, mixed differential parameters of the first order.

exact differential equation. A differential equation which is obtained by setting the total differential of some function equal to zero. An exact differential equation in two variables can be put in the form:

$$[\partial f(x\ y)/\partial x]\ dx+[\partial f(x\ y)/\partial y]\ dy=0.$$

A necessary and sufficient condition that an equation of the form $M\,dx+N\,dy=0$, where M and N have continuous first-order partial derivatives, be exact is that the partial derivative of M with respect to y be equal to the partial derivative of N with respect to x; *i.e.*, $D_yM=D_xN$. The equation

$$(2x+3y)\,dx+(3x+5y)\,dy=0$$

is exact. If a differential equation in three variables is of the form

$$P\,dx+Q\,dy+R\,dz=0,$$

where P, Q, and R have continuous first-order partial derivatives, then a necessary and sufficient condition that it be exact is that $D_yP=D_xQ$, $D_zQ=D_yR$ and $D_xR=D_zP$, where D_yP, etc., denote partial derivatives. This can be generalized to any number of variables.

homogeneous differential equation. A name usually given to a differential equation of the first degree and first order which is homogeneous in the variables (the derivatives not being considered) such as

$$y^2+(xy+x^2)\frac{dy}{dx}=0$$

and

$$\frac{x}{y}+\left(\sin\frac{x}{y}\right)\frac{dy}{dx}=0.$$

Solvable by use of the substitution $y=ux$. Equations of the type

$$\frac{dy}{dx}=\frac{ax+by+c}{dx+ey+f}$$

can be reduced to a homogeneous equation by the substitutions: $x=x'+h$, $y=y'+k$, where h and k are to be chosen so as to remove the constant terms in the numerator and denominator of the fraction.

homogeneous linear differential equation. A *linear differential equation* which does not contain a term involving only the independent variable. *E.g.*, $y'+yf(x)=0$.

integrable differential equation. A differential equation that is exact, or that can be made so by multiplying through by an integrating factor.

linear differential equation. A linear differential equation of first order is an equation of the form

$$\frac{dy}{dx}+P(x)y=Q(x).$$

Such an equation has an integrating factor of the form $e^{\int P dx}$ See example above, under *differential operator.* The general linear differential equation is an equation of the first degree in y and its derivatives, the coefficients of y and its derivatives being functions of x alone. *I.e.*, an equation of the form

$$L(y)\equiv p_0\frac{d^ny}{dx^n}+p_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+p_ny=Q(x).$$

The *general solution* can be found by finding n linearly independent particular solutions of the homogeneous equation $L(y)=0$, multiplying each of these functions by an arbitrary parameter, and adding to the sum of these products (called the **complementary function**) some particular solution of the original differential equation. The equation $L(y)=0$ is called an **auxiliary equation** or a **reduced equation.** The original equation $L(y)=Q$ is called the **complete equation.** Another method of finding a general solution, after having the complementary function, consists in assuming that the arbitrary parameters in the complementary function are undetermined functions of x, then substituting the complementary function in the original differential equation and determining these undetermined functions of x so that the result is an identity. This method is called **variation of parameters.**

metric differential geometry. See GEOMETRY.

mixed differential parameter of the first order. The invariant

$$\Delta_1(f,g)\equiv \frac{E\dfrac{\partial f}{\partial v}\dfrac{\partial g}{\partial v}-F\left(\dfrac{\partial f}{\partial u}\dfrac{\partial g}{\partial v}+\dfrac{\partial f}{\partial v}\dfrac{\partial g}{\partial u}\right)+G\dfrac{\partial f}{\partial u}\dfrac{\partial g}{\partial u}}{EG-F^2},$$

for given functions $f(u, v)$ and $g(u, v)$ and a given surface S: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$. See above, differential parameter of the first order. The invariance of

$\Delta_1(f, g)$ under change of parameters u, v follows from its geometrical significance:

$$\cos\theta = \frac{\Delta_1(f, g)}{[\Delta_1 f]^{1/2}[\Delta_1 g]^{1/2}},$$

where θ is the angle between the curves $f=$const. and $g=$const. through a point of S. Another mixed differential parameter of the first order is

$$\Theta(f, g) \equiv \frac{\partial(f, g)}{\partial(u, v)} / (EG-F^2)^{1/2}.$$

We have

$$\Delta_1{}^2(f, g) + \Theta^2(f, g) = [\Delta_1 f][\Delta_1 g].$$

order of a differential equation. The order of its highest-order derivative.

partial differential equation. See PARTIAL—partial differential equations.

Picard and Runge-Kutta methods for solving differential equations. See the respective names.

projective differential geometry. The theory of the differential properties of configurations, which are invariant under projective transformations.

primitive of a differential equation. See PRIMITIVE—primitive of a differential equation.

simultaneous (or systems of) differential equations. Two or more differential equations involving the same number of dependent variables, taken as a system in the sense that solutions are sought which will satisfy them simultaneously.

solution of a differential equation. Any function which reduces the differential equation to an identity when substituted for the dependent variable; $y=x^2+cx$ is a solution of

$$x\frac{dy}{dx} - x^2 - y = 0, \quad \text{for} \quad \frac{dy}{dx} = 2x+c$$

and substituting $2x+c$ for $\frac{dy}{dx}$, and x^2+cx for y, in the differential equation reduces it to the identity $0=0$. *Syn.* **primitive integral.** The constant c in the solution $y=x^2+cx$ is an arbitrary constant in the sense that $y=x^2+cx$ is a solution whatever value is given to c. The **general solution** of a differential equation is a solution in which the number of *essential* arbitrary constants is equal to the order of the differential equation. A **particular solution** is a solution obtained from the general solution by giving particular values to the arbitrary constants. A **singular solution** is a solution not obtainable by assigning particular values to the parameters in the general solution; it is the equation of an envelope of the family of curves represented by the general solution. This envelope satisfies the differential equation because at every one of its points its slope and the coordinates of the point are the same as those of some member of the family of curves representing the general solution. See DISCRIMINANT—discriminant of a differential equation.

total differential. See above, DIFFERENTIAL.

DIF'FER-EN'TI-A'TION, *adj.*, *n.* The process of finding the derivative, or differential coefficient. See DERIVATIVE.

differentiation formulas. Formulas that give the derivatives of functions or enable one to reduce the finding of their derivatives to the problem of finding the derivatives of simpler functions. See DIFFERENTIATION FORMULAS in the appendix, CHAIN—chain rule, and DERIVATIVE.

differentiation of an infinite series. See SERIES—differentiation of an infinite series.

differentiation of parametric equations. See PARAMETRIC—parametric equations.

differentiation of an integral. See DERIVATIVE—derivative of an integral.

implicit differentiation. The process of finding the derivative of one of two variables with respect to the other by differentiating all the terms of a given equation in the two variables, leaving the derivative of the dependent variable (with respect to the independent variable) in indicated form, and solving the resulting identity for this derivative. *E.g.*, if

$$x^3+x+y+y^3=4,$$

then

$$3x^2+1+y'+3y^2y'=0,$$

whence

$$y' = -(3x^2+1)/(3y^2+1).$$

In cases where an equation cannot be solved for one of the variables, this method is indispensable. It generally facilitates the work even when the equation can be so

solved. For the equation $f(x, y)=0$ one may also use the formula:

$$dy/dx = -D_x f(x, y)/D_y f(x, y),$$

if

$$D_y f(x, y) \neq 0;$$

this is easily seen to be equivalent to the above method. See DIFFERENTIAL (when the point (x, y, z) moves along a curve parallel to the xy-plane, *i.e.*, if z is constant, $dz = df(x, y)$ is zero and the above formula results).

indirect differentiation. See INDIRECT.

logarithmic differentiation. Finding derivatives by the use of logarithms. Consists of taking the logarithm of both sides of an equation and then differentiating. It is used for finding the derivatives of variable powers of variable bases, such as x^x, and to simplify certain differentiation processes. *E.g.*, if $y = x^x$, one can write $\log y = x \log x$ and find the derivative of y with respect to x from the latter equation by means of the usual method of *implicit differentiation.*

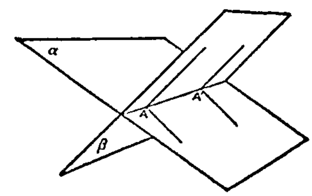
successive differentiation. The process of finding higher-order derivatives by differentiating lower-order derivatives.

DIG'IT, *n.* A term applied to any of the integers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 occurring in a number. The number 23 has the digits 2 and 3.

significant digits. (1) The digits which determine the mantissa of the logarithm of the number; the digits of a number beginning with the first digit on the left that is not zero and ending with the last digit on the right that is not zero. (2) The digits of a number which have a significance; the digits of a number beginning with the first nonzero digit on the left of the decimal point, or with the first digit after the decimal point if there is no nonzero digit to the left of the decimal point, and ending with the last digit to the right. *E.g.*, the significant digits in 230 are 2, 3, and 0. The significant digits in .230 are 2, 3, and 0, the 0 meaning that, to third place accuracy, the number is .230. In 0.23, the 0 is not significant, but in 0.023 the second zero is significant.

DIG'IT-AL, *adj.* **digital device.** See COMPUTER—digital computer.

DI-HE'DRAL, *adj.* **dihedral angle.** The angle between two planes. If the planes are parallel, the angle is said to be zero. A dihedral angle is measured by the plane angle formed by the lines of intersection of the two planes with a plane perpendicular to their line of intersection. The planes α and β form a dihedral angle, which is measured by the angle drawn at A, or at A'.



DIL'A-TA'TION, *n.* The change in volume per unit volume of the element of a deformed substance. If the principal strains are denoted by e_1, e_2, and e_3, the dilatation $\vartheta = (1+e_1)(1+e_2)(1+e_3)-1$, and for small strains $\vartheta = e_1 + e_2 + e_3$, approximately.

DI-MEN'SION, *n.* Refers to those properties called length, area, and volume. A configuration having length only is said to be of one dimension; area and not volume, two dimensions; volume, three dimensions. A geometric configuration is of dimension n if n is the least number of real-valued parameters which can be used to (continuously) determine the points of the configuration; *i.e.*, if there are n degrees of freedom, or the configuration is (locally) topologically equivalent to a subspace of n-dimensional Euclidean space. There are various definitions of dimension of a topological space, the most important of which give the same number for a compact metric space. *E.g.*, "A metric space is of dimension n if (1) for each positive number ϵ there is a closed ϵ-covering of order less than or equal to $n+1$, and (2) there is a positive number ϵ for which each closed ϵ-covering of M is of order greater than n" (see COVERING). This definition of dimension is such that the dimension of a metric space is topologically invariant and a subset of n-dimensional Euclidean space which contains the interior of a sphere is of dimension n. Also see BASIS—basis of a vector space, and SIMPLEX.

DI-MEN′SION-AL′I-TY, *n.* The number of dimensions of a quantity. See DIMENSION.

DIM′I-NU′TION, *n.* **diminution of the roots of an equation.** See ROOT—root of an equation.

DI-O-PHAN′TINE, *adj.* **diophantine analysis.** A method for finding *integral* solutions of certain algebraic equations. Depends mostly upon an ingenious use of arbitrary parameters.

diophantine equations. See EQUATION—indeterminate equation.

DI′POLE, *n.* See POTENTIAL—concentration method for the potential of a complex.

DI-RECT′, *adj.* **direct product (or sum).** See headings under PRODUCT.

direct trigonometric functions. The trigonometric functions *sine*, *cosine*, *tangent*, etc., as distinguished from the *inverse trigonometric functions.* See TRIGONOMETRIC—trigonometric functions.

DI-RECT′ED, *adj.* **directed angle.** An angle that has been indicated as positive or negative. See ANGLE.

directed line, or line segment. A line (or line segment) on which the direction from one end to the other has been indicated as positive, and the reverse direction as negative.

directed numbers. Numbers having signs, positive or negative, indicating that the negative numbers are to be measured, geometrically, in the direction opposite to that in which the positive are measured. *Syn.* Signed numbers, algebraic numbers. See POSITIVE—positive number.

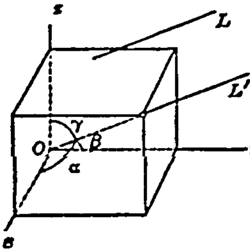
directed set. See MOORE-SMITH CONVERGENCE.

DI-REC′TION, *adj., n.* The relation between two points which is independent of the distance between them.

characteristic directions on a surface. See CHARACTERISTIC.

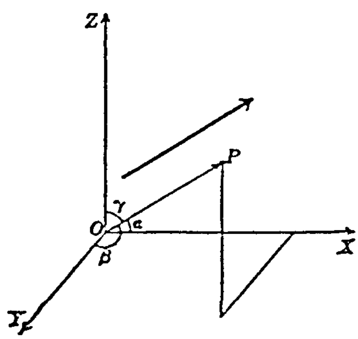
direction angle of a line in the plane. The smallest positive (or zero) angle that the line makes with the positive x-axis. See below, direction of a line in a plane.

direction angles. The three positive angles which a line makes with the positive directions of the coordinate axes. There are two such sets for an undirected line, one for each direction which can be assigned to the line. Direction angles are not independent (see PYTHAGOREAN—Pythagorean relation between direction cosines). In the figure, direction angles of the line L are the angles, α, β and γ, which the parallel line L' makes with the coordinate axes.



direction components of the normal to a surface. See below, direction cosines of the normal to a surface.

direction cosines. The cosines of the *direction angles.* They are usually denoted by l, m, and n, where, if α, β, γ are the direction angles with respect to the x-axis, y-axis, and z-axis, respectively, $l=\cos\alpha$, $m=\cos\beta$, and $n=\cos\gamma$. *Direction cosines* are not independent. When two of them are given, the third can be found, except for sign, by use of the *Pythagorean relation*, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. See below, direction numbers.



direction cosines of the normal to a surface. For a surface S given in parametric representation, $x=x(u,v)$, $y=y(u,v)$, $z=z(u,v)$, direction components of the normal

at a regular point are any three numbers having the ratio $A:B:C$, where

$$A=\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}, \quad B=\begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix},$$

$$C=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

The positive direction of the normal is taken to be the direction for which the direction cosines are $X=A/H$, $Y=B/H$, $Z=C/H$, where $H=\sqrt{A^2+B^2+C^2}$. Thus the orientation of the normal depends on the choice of parameters.

direction of a curve (at a point). The direction of the tangent to the curve at the point. See below, direction of a line.

direction of a line. For a line in the plane, its inclination; *i.e.*, the angle it makes with the x-axis (which is defined as the smallest positive angle obtainable by revolving the positive x-axis, counterclockwise, until it is parallel to the line). For a line in space, its direction angles.

direction numbers (or ratios) of a line in space. Any three numbers, not all zero, proportional to the *direction cosines* of the line. *Syn.* Direction components. If a line passes through the points (x_1, y_1, z_1) and (x_2, y_2, z_2), its direction numbers are proportional to x_2-x_1, y_2-y_1, z_2-z_1, and its direction cosines are

$$\frac{x_2-x_1}{D}, \quad \frac{y_2-y_1}{D}, \quad \frac{z_2-z_1}{D},$$

where

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2},$$

the distance between the points.

principal direction of strain. See STRAIN.

principal direction on a surface. At an ordinary point of a surface there are directions in which the radius of normal curvature attains its absolute maximum and its absolute minimum. These directions are at right angles to each other (unless the radius of normal curvature is the same for all directions at the point), and are called the **principal directions** on the surface at the point. See CURVATURE—principal curvature of a surface at a point, and UMBILICAL—umbilical point on a surface.

DI-REC'TION-AL, *adj.* **directional derivative.** The rate of change of a function with respect to arc length as a point moves in a given direction (*i.e.*, along a given curve). This is equal to the sum of the directed projections, upon the tangent line to the path, of the rates of change of the function in directions parallel to the three axes. Explicitly, for a function $u=F(x, y, z)$, the directional derivative in the direction of a curve whose parametric equations are $x=x(s)$, $y=y(s)$, $z=z(s)$, where s is arc length, is given by:

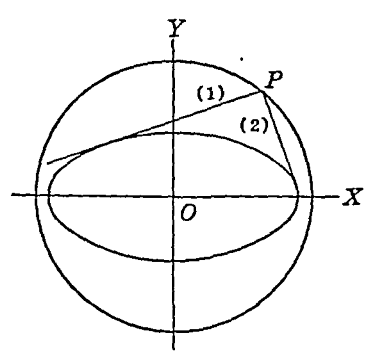
$$\frac{du}{ds}=F_x(x, y, z)\frac{dx}{ds}+F_y(x, y, z)\frac{dy}{ds}+F_z(x, y, z)\frac{dz}{ds}$$
$$=lF_x(x, y, z)+mF_y(x, y, z)+nF_z(x, y, z),$$

where l, m and n are the direction cosines of the tangent to the curve. For a function $u=f(x, y)$ of two variables, this can be written as

$$f_x(x, y)\cos\theta+f_y(x, y)\sin\theta,$$

where θ is the angle which the tangent to the curve (directed in the direction of motion) makes with the directed x-axis. See CHAIN—chain rule.

DI-REC'TOR, *adj.* **director circle of an ellipse (or hyperbola).** The locus of the intersection of pairs of perpendicular tangents to the ellipse (or hyperbola). In the figure, the circle is the director circle of the



ellipse, being the locus of points P which are the intersections of perpendicular tangents like (1) and (2).

director cone of a ruled surface. A cone formed by lines through a fixed point in

space and parallel to the rectangular generators of the given ruled surface. See INDICATRIX—spherical indicatrix of a ruled surface.

DI-REC′TRIX, *n.* **directrix of a conic.** See CONIC.

directrix of a cylindrical surface. See CYLINDRICAL—cylindrical surface.

directrix of a ruled surface. A curve through which a line generating the surface always passes. See CONICAL—conical surface, CYLINDRICAL—cylindrical surface, and PYRAMIDAL—pyramidal surface.

directrix planes of a hyperbolic paraboloid. The two lines of intersection of $z=0$ with the hyperbolic paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z,$$

each taken with the z-axis, determine two planes which are called the directrix planes of the hyperbolic paraboloid.

DIRICHLET. Dirichlet characteristic properties of the potential function

$$\iiint \rho/r\, dV.$$

Assume that ρ and its first partial derivatives are piecewise continuous and that the set of points at which ρ is not zero may be enclosed in a sphere of finite radius. The **Dirichlet properties** of the potential function $U=\iiint \rho/r\, dV$ are: (1) U is of class C^1 throughout space, (2) U is of class C^2 except on surfaces of discontinuity of ρ, $\partial\rho/\partial x$, $\partial\rho/\partial y$, and $\partial\rho/\partial z$; (3) at points external to the body ($\rho=0$) U satisfies Laplace's equation $\partial^2U/\partial x^2+\partial^2U/\partial y^2+\partial^2U/\partial z^2=0$, while at points internal to the body but not on the boundary, U satisfies the more general Poisson equation $\partial^2U/\partial x^2+\partial^2U/\partial y^2+\partial^2U/\partial z^2=\pm 4\pi\rho$ (the sign is plus in the electrostatic case and plus or minus in the gravitational case depending on the conventions adopted); (4) if $M=\iiint \rho\, dV$ and $R^2=x^2+y^2+z^2$, then as $R\to\infty$, $R(U-M/R)\to 0$, while each of $R^3\,\partial(U-M/R)/\partial x$, $R^3\,\partial(U-M/R)/\partial y$, $R^3\,\partial(U-M/R)/\partial z$ remains bounded. See POTENTIAL—potential function for a volume distribution of charge or mass.

Dirichlet integral. The integral

$$\iint_A \left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] dx\, dy,$$

or its analogue for a function of any number of independent variables. The **Dirichlet principle** states that if the *Dirichlet integral* is minimized in the class of functions continuously assuming a given boundary value function on the boundary of A, then the minimizing function is *harmonic* on the interior of A.

Dirichlet problem. Same as the first boundary value problem of potential theory. See BOUNDARY.

Dirichlet product. For a given domain R and a given nonnegative function $p(x, y, z)$, the Dirichlet product $D[u, v]$ of functions $u(x, y, z)$ and $v(x, y, z)$ is defined by

$$D[u, v]=\iiint_R (\nabla u\cdot\nabla v+puv)\, dx\, dy\, dz,$$

where $\nabla u\cdot\nabla v=\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}$. See above, Dirichlet integral.

Dirichlet's conditions. See FOURIER—Fourier's theorem.

Dirichlet's test for convergence of a series. Let $a_1, a_2, \cdots$ be a sequence for which there is a number K with $\left|\sum_{n=1}^{p} a_n\right| < K$ for all p. Then $\sum_{n=1}^{\infty} a_n u_n$ converges if $u_n \geq u_{n+1}$ for all n and $\lim_{n\to\infty} u_n=0$. This test is easily deduced from *Abel's inequality*.

Dirichlet's test for uniform convergence of a series. If $a_n(x)$ are functions for which there is a K for which $\left|\sum_{n=1}^{p} a_n(x)\right| < K$ (where K is independent of p and x), and if $u_n(x) \geq u_{n-1}(x)$ and $u_n(x) \to 0$ *uniformly* as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n(x)u_n(x)$ converges uniformly. Sometimes called *Hardy's test*.

DIS′CON-NECT′ED, *adj.* **disconnected set.** A set which can be divided into two sets U and V which have no points in common and which are such that no accumulation point of U belongs to V and no accumulation point of V belongs to U. A set is said to be totally disconnected if no

subset of more than one point is connected; *e.g.*, the set of rational numbers is totally disconnected. A set is **extremally disconnected** if the closure of each open set is open, or (equivalently) if the closures of two disjoint open sets are disjoint. An extremally disconnected Hausdorff space is *totally disconnected.*

DIS-CON'TI-NU'I-TY, *n.* (1) The property of being noncontinuous. (2) A point at which a given function is not continuous (also called a **point of discontinuity**). A discontinuity may mean a point at which the function is defined but is not continuous, but sometimes is also used for a point at which the function is not defined (particularly for such a point as 0 for the function $y=1/x$). Points of discontinuity of a real-valued function are classified as follows. If the function can be made continuous at the point by being given a new value at the point, then the discontinuity is **removable** (this is the case if the limits from the right and left exist and are equal); *e.g.*, $y=x\sin 1/x$ has a removable discontinuity at the origin. It approaches zero as x approaches zero either from the left or right, although it is not defined for $x=0$. A **nonremovable discontinuity** is any discontinuity which is not removable. An **ordinary discontinuity** (or **jump discontinuity**) is a discontinuity at which the limits of the function from the right and left exist, but are not equal; *e.g.*, the limits (at $x=0$) from the right and left of $y=1/(1+2^{1/x})$ are 0 and 1, respectively. The difference between the right and left limits is called the **jump** of the function (sometimes a jump discontinuity is called simply a jump). A **finite discontinuity** is a discontinuity (removable, ordinary, or neither) such that there is an interval about the point in which the function is bounded; *e.g.*, $y=\sin 1/x$ has a finite discontinuity at $x=0$, which is neither removable nor ordinary. An **infinite discontinuity** of a function $f(x)$ is a discontinuity such that $|f(x)|$ can have arbitrarily large values arbitrarily near the point. See SINGULAR—singular point of an analytic function.

DIS'CON-TIN'U-OUS, *adj.* **discontinuous function.** A function that is not continuous. The term is usually used when speaking of the nature of a function at a given point, or set of points, although it is sometimes used loosely in the sense that a function is discontinuous on an interval when it is discontinuous at some point or points on the interval. See DISCONTINUITY.

DIS'COUNT, *adj., n.* **bank discount.** A discount equal to the simple interest on the obligation; interest paid in advance on a note or other obligation (strictly speaking the interest is made part of the face of the note and paid when the note is paid). *E.g.*, a note for \$100, discounted by the *bank rule* at 6%, would leave \$94 (paying the face value of \$100 at the end of the year is equivalent to paying 6.38% interest, or if the interest is to be 6%, the *true discount* would be \$5.66 and the discount rate 5.66%, not 6%).

bond discount. The difference between the redemption value and the purchase price when the bond is bought below par.

cash discount, or discount for cash. A reduction in price made by the seller because the buyer is paying cash for the purchase.

chain discount. Same as DISCOUNT SERIES.

commercial discount. A reduction in the price of goods, or in the amount of a bill or debt, often given to secure payment before the due date. This discount may be computed either by means of discount rate or by means of interest rate; the former is used in discounting prices and the latter, usually, in discounting interest bearing contracts. The discount in the latter case is the face of the contract minus its present value at the given rate. When simple interest is used, *present value* is $S/(1+ni)$; when compound interest is used, it is $S/(1+i)^n$, where n in both cases is the number of interest periods and S the face of the contract. See below, discount rate, simple discount, compound discount.

compound discount. Discount under compound interest; the difference between face value and the present value at the given rate after a given number of years:

$$D=S-S/(1+i)^n,$$

or, in terms of *discount rate,*

$$D=S-A, \quad \text{where} \quad A=S(1-d)^n.$$

discount factor. The factor which, when multiplied into a sum, gives the present

value over a period of n years; *i.e.*, gives the principal which would amount to the sum at the end of n years at the given interest rate. For compound interest, this factor is $(1+i)^{-n}$, where i is the interest rate. In terms of the *discount rate*, d, this factor is $(1-d)^n$. See below, discount rate.

discount on a note. The difference between the selling value and the present value of a note.

discount on stocks. The difference between the selling value and the face value of stocks, when the former is lower than the latter.

discount problem under compound interest. Finding the present value of a given sum, at a given rate of compound interest; *i.e.*, solving the equation $S=P(1+r)^n$ for P.

discount rate. The percentage used to compute the discount. It is never the same as the interest rate on the contract. See above, bank discount, and below, true discount. If d is the discount rate and i the interest rate, the discount on a sum, S, for one interest period is $S-S/(1+i)$ or Sd, where

$$d=1-1/(1+i)=i/(1+i).$$

discount series. A sequence of discounts consisting of a discount, a discount upon the discounted face value, a discount upon the discounted, discounted face value, etc. The successive discount rates may or may not be the same. *E.g.*, if \$100 is discounted at a discount rate of 10%, the new principal is \$90; if this principal is discounted 5%, the new principal is \$85.50, and the discounts \$10 and \$4.50 are called a discount series. *Syn.* Chain discount.

simple discount. Discount proportional to the time (on the basis of simple interest). If S is the amount due in the future (after n years), P the present value, and i the interest rate, the discount D is equal to $S-P$, where $P=S/(1+ni)$.

time discount. Discount allowed if payment is made within a prescribed time; usually called *cash discount*. A vendor who does credit business often prices his goods high enough to make his credit sales cover losses due to bad accounts, then *discounts* for cash or for cash within a certain period.

trade discount. A reduction from the list price to adjust prices to prevailing prices or to secure the patronage of certain purchasers, especially purchasers of large amounts.

true discount. The reduction of the face value of an agreement to pay, by the simple interest on the reduced amount at a given rate; *e.g.*, the *true discount* on \$100 for one year at 6% is \$5.66, because 6% of \$94.24 is \$5.66. The formula for true discount is: $D=S-S/(1+ni)$, where S is face value, n time in years, and i interest rate.

DIS-CREP'ANCE, *n.* Same as INTERACTION.

DIS-CRETE', *adj.* **discrete set.** A set of numbers, or points, that has no limit points. See ISOLATED—isolated set.

discrete variable. A variable that is not continuous; a variable that takes on only certain disconnected values on any interval of the continuum; a variable whose possible values form a discrete set, *e.g.*, the integers.

DIS-CRIM'I-NANT, *adj., n.* **discriminant function.** (*Statistics.*) A linear combination of a set of n variables that will classify (into two different classes) the events or items for which the measurements of the n variables are available, with the smallest possible proportion of misclassifications. Useful, for example, in the taxonomic problems of classifying individuals of a plant into the various species.

discriminant of a differential equation. For a differential equation of type $F(x, y, p)=0$, where $p=dy/dx$, the p-discriminant is the result of eliminating p between the equations $F(x, y, p)=0$ and $\frac{\partial F(x, y, p)}{\partial p}=0$. If the solution of the differential equation is $u(x, y, c)=0$, the c-discriminant is the result of eliminating c between the equations $u(x, y, c)=0$ and $\frac{\partial u(x, y, c)}{\partial c}=0$. The curve whose equation is obtained by setting the p-discriminant equal to zero contains all *envelopes* of solutions, but also may contain a *cusp locus*, a *tac-locus*, or a *particular solution* (in general, the equation of the tac-locus will be squared and the equation of the particular

solution will be cubed). The curve whose equation is obtained by setting the c-discriminant equal to zero contains all *envelopes* of solutions, but also may contain a *cusp locus*, a *node locus*, or a *particular solution* (in general, the equation of the node locus will be squared and the equation of the cusp locus will be cubed). In general, the cusp locus, node locus, and tac-locus are not solutions of the differential equation. *E.g.*, the differential equation $(dy/dx)^2(2-3y)^2=4(1-y)$ has the general solution $(x-c)^2=y^2(1-y)$ and the p-discriminant and c-discriminant equations are, respectively,

$$(2-3y)^2(1-y)=0 \quad \text{and} \quad y^2(1-y)=0.$$

The line $1-y=0$ is an envelope; $2-3y$ is a tac-locus; $y=0$ is a node locus.

discriminant of a polynomial equation, $x^n+a_1x^{n-1}+\cdots+a_n=0$. The product of the squares of all the differences of the roots taken in pairs. The discriminant is equal to the *resultant* of the equation and its derived equation, except possibly in sign. The discriminant is $(-1)^{n(n-1)/2}$ times this resultant. If the leading coefficient is a_0 instead of 1, the factor a_0^{2n-2} is introduced in the discriminant and the discriminant is $(-1)^{n(n-1)/2}/a_0$ times the resultant. The discriminant is zero if and only if the polynomial equation has a double root. For a **quadratic equation,** $ax^2+bx+c=0$, the discriminant is b^2-4ac. If a, b, and c are real, the discriminant is zero when and only when the roots are equal, and negative or positive according as the roots are imaginary or real. *E.g.*, the discriminant of $x^2+2x+1=0$ is 0 and the two roots are equal; the discriminant of $x^2+x+1=0$ is -3 and the roots are imaginary; the discriminant of $x^2-3x+2=0$ is 1 and the roots, 1 and 2, are real and unequal. See QUADRATIC—quadratic formula. For a real **cubic equation,** $x^3+ax^2+bx+c=0$, the discriminant is equal to

$$a^2b^2+18abc-4b^3-4a^3c-27c^2.$$

This discriminant is positive if the equation has three real, distinct roots; it is negative if there is a single real root and two conjugate imaginary roots; and it is zero if the roots are all real and at least two of them are equal. See RESULTANT—resultant of a set of polynomial equations.

discriminant of a quadratic equation in two variables. If the equation is

$$ax^2+bxy+cy^2+dx+ey+f=0,$$

then the discriminant is the quantity $\Delta=(4acf-b^2f-ae^2-cd^2+bde)$, which can be written

$$\Delta=\tfrac{1}{2}\begin{vmatrix} 2a & b & d \\ b & 2c & e \\ d & e & 2f \end{vmatrix}.$$

The discriminant is also equal to the product of $-(b^2-4ac)$ and the constant term in the equation obtained by translating the axes so as to remove the first degree terms—namely,

$$a'x^2+b'xy+c'y^2-\Delta/(b^2-4ac)=0.$$

The discriminant and the invariant, (b^2-4ac), provide the following criteria concerning the locus of the general quadratic in two variables. If $\Delta\neq0$ and $b^2-4ac<0$, the locus of the general quadratic is a real or imaginary ellipse; if $\Delta\neq0$ and $b^2-4ac>0$, a hyperbola; if $\Delta\neq0$ and $b^2-4ac=0$, a parabola. If $\Delta=0$ and $b^2-4ac<0$, the locus is a point ellipse; if $\Delta=0$ and $b^2-4ac>0$, two intersecting lines; and if $\Delta=0$ and $b^2-4ac=0$, two parallel or coincident lines or no (real) locus. The discriminant Δ is defined differently by different writers, but all the forms are the same except for multiplication by some constant.

discriminant of a quadratic form. The determinant with a_{ij} in row i and column j, where the quadratic form $Q=\sum_{i,j=1}^{n} a_{ij}x_ix_j$ is written so that $a_{ij}=a_{ji}$ for all i and j. If Δ_m is the discriminant of the quadratic form obtained from Q by discarding all terms but those involving only $x_1, x_2, \cdots, x_m$, then there is a linear transformation of the form $x_i=y_i+\sum_{j=i}^{n} b_{ij}y_j$ such that $\sum_{i,j=1}^{n} a_{ij}x_ix_j=\sum_{i=1}^{n}\alpha_iy_i^2$, where $\alpha_1=\Delta_1$, $\alpha_2=\Delta_2/\Delta_1$, $\alpha_3=\Delta_3/\Delta_2, \cdots, \alpha_n=\Delta_n/\Delta_{n-1}$. See TRANSFORMATION—congruent transformation, and INDEX—index of a quadratic form.

DIS-JOINT′, *adj.* Two sets are disjoint if there is no point which belongs to each of

the sets (*i.e.*, if the intersection of the sets is the null set). A system of more than two sets is **pairwise disjoint** (sometimes simply **disjoint**) if each pair of sets belonging to the system is disjoint.

DIS-JUNC'TION, *n.* **disjunction of propositions.** The proposition formed from two given propositions by connecting them with the word *or*, thereby asserting the truth of one or both of the given propositions. The disjunction of two propositions is false if and only if both the propositions are false. *E.g.*, the disjunction of "$2 \cdot 3=7$" and "Chicago is in Illinois" is the true statement "$2 \cdot 3=7$ *or* Chicago is in Illinois." The disjunction of "Today is Tuesday" and "Today is Christmas" is the statement "Today is Tuesday *or* today is Christmas," which is true unless today is neither Tuesday nor Christmas. The disjunction of propositions p and q is usually written $p \vee q$ and read "p *or* q." See CONJUNCTION. *Syn.* Alternation.

DIS-PER'SION, *n.* (*Statistics.*) The variation, scatteration, of the data; the lack of tendency to concentrate or congregate.

measure of dispersion. (*Statistics.*) Usually taken as the STANDARD DEVIATION.

DIS'PRO-POR'TION-ATE, *adj.* **disproportionate subclass numbers.** See SUBCLASS.

DIS-SIM'I-LAR, *adj.* **dissimilar terms.** Terms that do not contain the same powers or the same unknown factors. *E.g.*, $2x$ and $5y$, or $2x$ and $2x^2$, are *dissimilar terms*. See ADDITION—addition of similar terms in algebra.

DIS'TANCE, *n.* **angular distance between two points.** The angle between the two lines drawn from the point of observation (point of reference) through the two points. *Syn.* Apparent distance.

distance between two parallel lines. The length of a perpendicular joining them; the distance from one of them to a point on the other.

distance between two parallel planes. The length of the segment which they cut off on a common perpendicular; the distance from one of them to a point on the other.

distance between two points. The length of the line segment joining the points. In analytic geometry it is found by taking the square root of the sum of the squares of the differences of the corresponding rectangular Cartesian coordinates of the two points. In the plane this is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$, where the points are (x_1, y_1) and (x_2, y_2); in space it is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2},$$

where the points are (x_1, y_1, z_1) and (x_2, y_2, z_2).

distance between two skew lines. The length of the line segment joining them and perpendicular to both.

distance from a line to a point. The perpendicular distance from the line to the point. It can be found by substituting the coordinates of the point in the *normal form* of the equation of the line (if the point and line are both in the (x, y)-plane; see LINE—equation of a line) or by finding the coordinates of the foot of the perpendicular from the point to the line and then finding the distance between these two points.

distance from a plane to a point. The length of the perpendicular from the point to the plane. It may be obtained by substituting the coordinates of the point in the normal form of the equation of the plane. See PLANE—equation of a plane.

distance-rate-time formula. The formula which states that the distance passed over by a body, moving at a fixed rate for a given time, is equal to the product of the rate and time, written $d=rt$.

distance from a surface to a tangent plane. For a surface S: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, the distance between the point corresponding to $(u+du, v+dv)$ and the plane tangent to S at (u, v) is given by $\frac{1}{2}(dx\ dX+dy\ dY+dz\ dZ)+e=\frac{1}{2}(D\ du^2+2D'\ du\ dv+D''\ dv^2)+e=\frac{1}{2}\Phi+e$, where X, Y, Z are the direction cosines of the normal to S, e denotes terms of the third and higher orders in du and dv, and

$$D=X\frac{\partial^2 x}{\partial u^2}+Y\frac{\partial^2 y}{\partial u^2}+Z\frac{\partial^2 z}{\partial u^2}$$

$$=\Sigma X\frac{\partial^2 x}{\partial u^2}=-\Sigma\frac{\partial X}{\partial u}\frac{\partial x}{\partial u},$$

$$D'=\Sigma X\frac{\partial^2 x}{\partial u\,\partial v}=-\Sigma\frac{\partial X}{\partial u}\frac{\partial x}{\partial v}=-\Sigma\frac{\partial X}{\partial v}\frac{\partial x}{\partial v},$$

$$D''=\Sigma X\frac{\partial^2 x}{\partial v^2}=-\Sigma\frac{\partial X}{\partial v}\frac{\partial x}{\partial v}.$$

See SURFACE—fundamental quadratic form of a surface.

Minkowski distance function. See MINKOWSKI.

polar distance. Same as CODECLINATION.

zenith distance of a star. See ZENITH.

DIS'TRI-BU'TION, *n.* (*Statistics.*) The relative arrangement of a set of numbers (elements); a set of values of a variable and the frequencies of each value. Sometimes called a *frequency distribution* to distinguish it from an arrangement according to some other criterion, such as time or location.

binomial distribution. See BINOMIAL—binomial distribution.

distribution function. (*Statistics.*) A function giving the cumulative frequency corresponding to the various values of a variable. The cumulative frequency is the cumulative frequency from lowest values up to given values of the variable. Mathematically, $F(x_k)=\sum_{i=1}^{k} f(x_i)$ is the distribution function of the discontinuous variable x with n values that range from x_1 to x_n. The total frequency is yielded by summation over the n values. For a continuous variable, $F(b)=\int_{-\infty}^{b} f(x)\,dx$ is the distribution function and yields the **frequency cumulated** from $-\infty$ to b, where $f(x)$ is the frequency function. Integration over the entire permissible range gives the total frequency. It is not yet universal practice to confine the term *frequency function* to $f(x)$ and *distribution function* to $F(x)$, although such usage is quite common. Also common is the use of either term in either sense. It is customary to define the distribution function so that the integral over the entire permissible range of the variable equals 1. In this form $F(x)$ is also called the **probability distribution function** and $f(x)$ becomes the **probability density function.** See FREQUENCY—frequency function, and PROBABILITY—probability density function.

F distribution. The *random sampling distribution* of the ratio of two independent estimates of the variance of a normal distribution:

$$F=\frac{s_1^2}{s_2^2}=\frac{n_2x_1^2}{n_1x_2^2},$$

where there are n_1 and n_2 degrees of freedom in the first and second independent estimates, respectively;

$$dp=\frac{\Gamma\left(\frac{n_1+n_2}{2}\right) n_1^{n_1/2} n_2^{n_2/2} F^{(n_1-2)/2}}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)(n_2+n_1F)^{(n_1+n_2)/2}}\,dF$$

is the density function of F, with n_1 and n_2 degrees of freedom. Student's "t," when squared, is the F ratio with $n_1=1$ and n_2 equal to the degrees of freedom associated with Student's "t." *Chi-square* is equal to Fn_1, when $n_2\to\infty$, with n_1 degrees of freedom. When $n_1=1$ and $n_2\to\infty$, F is equal to the square of the normal deviate whose mean and variance are 0 and 1, respectively. *Fisher's z* is equal to $\frac{1}{2}\log_e F$. See FISHER.

frequency distribution. See FREQUENCY.

Gibrat distribution. See GIBRAT.

normal distribution. (*Statistics.*) A distribution which follows the normal frequency curve. See FREQUENCY—normal frequency curve.

Pearson distribution. See PEARSON.

Poisson distribution. See POISSON.

relative distribution function. See PROBABILITY—probability density function.

skew distribution. (*Statistics.*) A nonsymmetrical distribution. A distribution is *skewed to the left* (*right*) if the longer tail is on the left (right)—also called *negative* (*positive*) *skewness.* More precisely, it is skewed to the right (left) if the *third moment* about the mean is positive (negative).

symmetrical distribution. (*Statistics.*) A distribution that is symmetrical about the *median*; a distribution such that one side is a reflection of the other through the median.

truncated distribution. A distribution which is arbitrarily cut so that there are no values of the variable x for which $x>a$ (or $x<a$). The distribution is said to be *truncated* at the value a.

DIS-TRIB'U-TIVE, *adj.* An operation is distributive relative to a rule of combination if performing the operation upon the combination of a set of quantities is equivalent to performing the operation upon each member of the set and then

combining the results by the same rule of combination. *E.g.*,

$$\frac{d(u+v)}{dx}=\frac{du}{dx}+\frac{dv}{dx},$$

the rule of combination being addition. The function $\sin x$ is not distributive, since $\sin(x+y)\neq\sin x+\sin y$. See FIELD; and below, distributive law of arithmetic and algebra.

distributive law of arithmetic and algebra. The law which states that

$$a(b+c)=ab+ac$$

for any numbers a, b, and c; *e.g.*, $2(3+5)=2\cdot3+2\cdot5$, each expression being equal to 16. This law can be extended to state that the product of a monomial by a polynomial is equal to the sum of the products of the monomial by each term of the polynomial; *e.g.*, $2(3+x+2y)=6+2x+4y$. When two polynomials are multiplied together, one is first treated as a monomial and multiplied by the individual terms of the other, then the results multiplied out according to the above law (or one may multiply each term of one polynomial by each term of the other and add the results). *E.g.*,

$$(x+y)(2x+3)=x(2x+3)+y(2x+3)$$
$$=2x^2+3x+2xy+3y.$$

DI-VER'GENCE, *adj., n.* **divergence of a sequence or series.** The property of being divergent; the property of not being convergent. See DIVERGENT.

divergence of a vector function, $F(x, y, z)$. $\nabla\cdot F$, where ∇ is the operator

$$i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}$$

and

$$\nabla\cdot F=i\cdot\frac{\partial F}{\partial x}+j\cdot\frac{\partial F}{\partial y}+k\cdot\frac{\partial F}{\partial z}$$
$$=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z},$$

where $F=i\,F_x+j\,F_y+k\,F_z$.

If, for instance, F is the velocity of a fluid at the point $P:(x, y, z)$, then $\nabla\cdot F$ is the rate of change of volume per unit volume of an infinitesimal portion of the fluid containing P. See below, divergence of a tensor.

divergence of a tensor. The divergence of a contravariant tensor T^i of order one (*i.e.*, a contravariant vector field) is $T^i_{,i}$, or $\frac{1}{\sqrt{g}}\frac{\partial(T^i\sqrt{g})}{\partial x^i}$, where the *summation convention* applies, g is the determinant having g_{ij} in the ith row and jth column (g_{ij} being the *fundamental metric tensor*), and $T^i_{,j}$ is the covariant derivative of T^i. The divergence of a covariant tensor T_i of order one (*i.e.*, a *covariant vector field*) is $g^{ij}T_{i,j}$, or $T_{,i}^{\ i}$, where $T^i=g^{ij}T_j$ and g^{ij} is $1/g$ times the cofactor of g_{ji} in g.

divergence theorem. Same as GREEN'S THEOREM (in space).

DI-VER'GENT, *adj.* **divergent sequence.** A sequence which does not converge. It is said to be **properly divergent, or to oscillate,** in the sense described below for the sequence of partial sums of a divergent series.

divergent series. A series which does not converge. The sequence of partial sums $S_1, S_2, \ldots$ of a *divergent series* (S_n is the sum of the first n terms of the series) is a *divergent sequence.* The series is **properly divergent** if the partial sums become arbitrarily large for large values of n in the sense that, for any number N, $S_n>M$ for all but a finite number of values of n, or if they become arbitrarily small (algebraically) in the sense that, for any number M, $S_n<M$ for all but a finite number of values of n. In these two cases, one writes $\lim_{n\to\infty} S_n=+\infty$ and $\lim_{n\to\infty} S_n=-\infty$. All other types of divergent series are called **oscillating divergent series,** or just **oscillating series** (the latter, however, is sometimes used of convergent series whose sums oscillate about the limit as they approach it—such as, $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$). The series $1+2+3+\cdots$, $1+\frac{1}{2}+\frac{1}{3}+\cdots$, and $-1-1-1-1\cdots$ are *properly divergent,* while $1-1+1-1+\cdots$ and $1-2+3-4+\cdots$ are *oscillating divergent series.* For the last example, the partial sums are $1, -1, 2, -2, 3, -3, 4, -4, \cdots$. This sequence is divergent in the sense that, for any number M, $|S_n|>M$ for all but a finite number of values of n; one writes: $\lim_{n\to\infty} S_n=\infty$.

summation of divergent series. See SUMMATION—summation of divergent series.

DI-VIDE', *v.* To perform a division.

DIV'I-DEND, *n.* (1) A quantity which is to be divided by another quantity. (2) *In finance*, profits of a stock company or any joint enterprise which are to be distributed among the shareholders. (3) The amount of such profits as noted in (2), which accrue to each shareholder.

dividend on a bond. The periodic, usually semiannual, interest paid on a bond. The **dividend date** is the date upon which the dividend is due; the interest rate named in the bond is the **dividend rate** or **bond rate.** An **accrued dividend** is a partial dividend; the interest on the face value of a bond from the nearest preceding dividend date to the purchase date. In bond market parlance, **accrued interest** is used synonymously with *accrued dividends.*

dividend on stock. The portion of the profits of the business which is paid on each share of stock.

DI-VID'ERS, *n.* An instrument like a compass, but with a point on each leg.

DI-VIS'I-BIL'I-TY, *n.* **special criteria for divisibility in arithmetic.** A number is divisible by 3 (or 9) when, and only when, the sum of the digits is divisible by 3 (or 9); *e.g.*, 35,712 is divisible by both 3 and 9, since the sum of the digits is 18. A number is divisible by 2 if the last digit is divisible by 2. A number is divisible by 4 if the number consisting of the last two digits on the right is divisible by 4. A number is divisible by 8 if the number formed by the last three digits is divisible by 8. A number is divisible by 5 if it ends in 0 or 5.

DI-VI'SION, *n.* Division is the *inverse operation* to multiplication. The result of dividing one number (the **dividend**) by another (the **divisor**) is called their **quotient.** The quotient a/b of two numbers a and b is that number c such that $b \cdot c = a$, provided c exists and has only one possible value (if $b=0$, then c does not exist if $a \neq 0$, and c is not unique if $a=0$; *i.e.*, $a/0$ is meaningless for all a, and **division by zero is meaningless**); the quotient a/b can also be defined as the product of a and the inverse of b (see GROUP). *E.g.*, $6/3=2$, because $3 \cdot 2=6$; $(3+i)/(2-i)=1+i$, because $3+i=(2-i)(1+i)$. The **division of a fraction by an integer** is accomplished by dividing the numerator (or multiplying the denominator) by the integer ($4/5 \div 2 = 2/5$ or $4/10$, because $2 \cdot 2/5 = 4/5$ and $2 \cdot 4/10 = 4/5$); **division by a fraction** is accomplished by inverting the fraction and multiplying by it, or by writing the quotient as a complex fraction and simplifying ($7/5 \div 2/3 = (3/2)(7/5) = 21/10$, because $7/5 = (2/3)(3/2)(7/5)$; see FRACTION—complex fraction); **division of mixed numbers** is accomplished by reducing the mixed numbers to fractions and dividing these results ($1\frac{2}{3} \div 3\frac{1}{2} = 5/3 \div 7/2 = 10/21$).

division by a decimal. See above, DIVISION. Accomplished by multiplying dividend and divisor by a power of 10 that makes the divisor a whole number (*i.e.*, moving the decimal point to the right in the dividend as many places as there are decimal places in the divisor), then dividing as with whole numbers, placing a decimal point in the quotient in the place arrived at before using the first digit after the decimal place in the dividend. *E.g.*,

$$28.7405 \div 23.5 = 287.404 \div 235 = 1.223.$$

division modulo p. If, in the process of performing the division transformation, $f(x)=q(x) \cdot d(x)+r(x)$, any of the coefficients are increased or diminished by multiples of p, the process is called *division modulo p* and is written $f(x)=q(x) \cdot d(x)+r(x) \pmod p$. This definition applies only when each coefficient is an integer. Each coefficient is usually written as one of the integers $0, 1, 2, \cdots, p-1$, two integers being regarded as equal (or equivalent) if they differ by a multiple of p.

division in a proportion. Passing from a proportion to the statement that the first antecedent minus its consequent is to its consequent as the second antecedent minus its consequent is to its consequent, *i.e.*, from $a/b=c/d$ to $(a-b)/b=(c-d)/d$. See COMPOSITION—composition in a proportion.

division by use of logarithms. See LOGARITHM.

harmonic division of a line. See HARMONIC.

point of division. See POINT—point of division.

ratio of division or **division ratio.** See POINT—point of division.

short division and long division. (1) Division is called short (or long) according as the process can (or cannot) be carried out mentally. It is customary to discriminate between long and short division solely upon the basis of the complexity of the problem. When the steps in the division must be written down, it is called **long division**; otherwise it is **short division.** (2) Division is short (or long) if the divisor contains one digit (or more than one digit); in *algebra*, if the divisor contains one term (or more than one term).

synthetic division. See SYNTHETIC.

the division transformation. The relation dividend = quotient × divisor + remainder. Rarely used.

DI-VI'SOR, *n.* The quantity by which the dividend is to be divided. See DIVISION.

common divisor of two or more quantities. A quantity which is a factor of each of these quantities. It is sometimes called a **common measure.** A common divisor of 10 and 15 is 5.

greatest common divisor of two or more quantities. The greatest (largest) quantity which is a common divisor of these quantities. It is written G.C.D. The G.C.D. of 30 and 42 is 6. *Tech.* the G.C.D. of two quantities is a common divisor of the two quantities that is divisible by every one of their common divisors. *Syn.* Greatest common measure.

normal divisor of a group. See INVARIANT—invariant subgroup.

DO-DEC'A-GON, *n.* A polygon having twelve sides.

regular dodecagon. See REGULAR—regular polygon.

DO'DEC-A-HE'DRON, *n.* A polyhedron having twelve faces.

regular dodecahedron. A dodecahedron whose faces are regular pentagons and whose polyhedral angles are congruent. See figure under POLYHEDRON—regular polyhedron.

DO-MAIN', *n.* (1) A *field*, as the *number domain* of all rational numbers, or of all real numbers (see FIELD). (2) Any *open connected set* that contains at least one point. Also used for any open set containing at least one point. Sometimes called a **region.** (3) The **domain of a function** is the set of values which the independent variable may take on, or the *range of the independent variable.* See FUNCTION. (4) See below, integral domain.

domain of dependence for a partial differential equation. See DEPENDENCE.

integral domain. A *commutative ring with unit element* which has no proper divisors of zero (proper divisors of zero are nonzero elements x and y for which $x \cdot y = 0$, where 0 is the additive identity). The assumption that there are no proper divisors of zero is equivalent to the cancellation law: $x = y$ whenever $xz = yz$ and $z \neq 0$. The set of ordinary integers (positive, negative, and 0) and the set of all algebraic integers are integral domains. See ALGEBRAIC—algebraic number, and RING.

DOM'I-NANT, *adj.* **dominant strategy.** See STRATEGY.

dominant vector. A vector $a = (a_1, a_2, \cdots, a_m)$ such that, relative to a second vector $b = (b_1, b_2, \cdots, b_m)$, the inequality $a_i \geqq b_i$ holds for each i ($i = 1, 2, \cdots, m$). If the strict inequality $a_i > b_i$ holds for each i, the dominance is said to be **strict.**

DOT, *n.* **dot product.** See MULTIPLICATION—multiplication of vectors.

DOU'BLE, *adj.* **double-angle formulas.** See TRIGONOMETRY—double angle formulas of trigonometry.

double integral. See INTEGRAL—iterated integral, multiple integral.

double law of the mean. See MEAN—mean value theorem for derivatives.

double ordinate. See ORDINATE.

double point. See POINT—multiple point.

double root of an algebraic equation. A root that is repeated, or occurs exactly twice in the equation; a root such that $(x-r)^2$, where r is the root, is a factor of the left member of the equation when the right is zero, but $(x-r)^3$ is not such a factor. *Syn.* Repeated root, root of multiplicity two, coincident roots, equal roots. See MULTIPLE—multiple root of an equation.

double tangent. (1) A tangent which has two *noncoincident* points of tangency with

the curve. (2) Two coincident tangents, as the tangents at a cusp. See POINT—double point.

DOU'BLET, *n.* See POTENTIAL—concentration method for the potential of a complex.

DRAFT, *n.* An order written by one person and directing another to pay a certain amount of money.

after-date draft. An accepted draft, for which the time during which discount is reckoned (if there be any) begins on the date of the draft.

after-sight draft. An accepted draft for which the time during which discount is reckoned begins with the date of acceptance.

bank draft. A draft drawn by one bank upon another.

commercial draft. A draft made by one firm on another to secure the settlement of a debt.

DRAG, *n.* If the total force F that is applied to a body B gives B a motion with velocity vector $\mathbf{v}$, then the component of F in the direction opposite to $\mathbf{v}$ is called *drag.* In exterior ballistics, the drag F_v is given approximately by the formula

$$F_v = \rho d^2 v^2 K,$$

where ρ is the density of air, d is the diameter of the shell, v is the speed at which it is traveling, and K is the constant called the **drag coefficient** of the shell. See LIFT.

axial drag. In exterior ballistics, *axial drag* is the component, in the direction opposite to that of the advancing axis of a shell, of the total force acting on the shell. It is found that the axial drag F_a is given approximately by the formula

$$F_a = \rho d^2 v_a{}^2 K_a,$$

where ρ is the density of air, d is the diameter of the shell, v_a is the component of the velocity in the direction of the axis of the shell, and K_a is a constant. The constant K_a is called the **axial-drag coefficient** of the shell; it depends mostly on the shape of the shell, but also somewhat on its size.

DRAWING TO SCALE. See SCALE—drawing to scale.

DU'AL, *adj.* **dual formulas.** Formulas related in the same way as dual theorems.

dual theorems. See PRINCIPLE—principle of duality of projective geometry, and principle of duality in a spherical triangle. Sometimes called RECIPROCAL THEOREMS.

DU-AL'I-TY, *n.* **Poincaré duality theorem.** The p-dimensional Betti numbers B_G^p of an orientable manifold which is homeomorphic to the set of points of an n-dimensional simplicial complex satisfy

$$B_G^p = B_G^{n-p},$$

where G is the group for which chains and homology groups are defined. Poincaré proved the theorem for the case G is the group of rational numbers; the proof for G the group of integers *mod 2* was given by Veblen and the proof for G the group of integers *mod p* (p a prime) was given by Alexander. See BETTI—Betti number; HOMOLOGY—homology group.

principle of duality of projective geometry. The principle that if one of two dual theorems is true the other is also. In a *plane*: point and line are called **dual elements**; the drawing of a line through a point and the marking of a point on a line are known as **dual operations,** as are also the drawing of two lines through a point and the marking of two points on a line, or the bringing of two lines to intersect in a point and the joining of two points by a line; figures which can be obtained from one another by replacing each element by the dual element and each operation by the dual operation are called **dual figures,** as three lines passing through a point and three points lying on a line (three *concurrent* lines and three *collinear* points). Theorems which can be obtained from one another by replacing each element in one by the dual element and each operation by the dual operation are called **dual theorems.** In *space*: the point and plane are dual elements (called **space duals**), the definitions of dual operations, figures, and theorems being analogous to those in the plane. Some writers state dual theorems in such terms that they are interchanged merely by interchanging the words point and line (or point and plane), as, for instance, two points determine a line—two lines determine a point, or two points on a

line—two lines on a point. *E.g.*, the two following statements are plane duals: (a) one and only one line is determined by a point and the point common to two lines; (b) one and only one point is determined by a line and the line common to two points.

principle of duality in a spherical triangle. In any formula involving the sides and the supplements of the angles opposite the sides, another true formula may be obtained by interchanging each of the sides with the supplement of the angle opposite it. The new formula is called the dual formula.

DU'EL, *n.* A two-person zero-sum game involving the timing of decisions. Delay of action increases accuracy but also increases the likelihood that the opponent will have acted first. A duel is a **noisy duel** if each player knows at all times whether or not the opponent has taken action; it is a **silent duel** if the players never know whether or not their opponent has taken action.

DUHAMEL'S THEOREM. If the sum of n *infinitesimals (each a function of* n*)* approaches a limit as n increases (becomes infinite), then the same limit is approached by the sum of the infinitesimals formed by adding to each of these infinitesimals other infinitesimals which are *uniformly* of higher order than the ones to which they are added. *E.g.*, the sum of n terms, each equal to $1/n$, is equal to 1 for all n. Hence this sum approaches (is) 1 as n increases. The sum of n terms, each equal to $1/n + 1/n^2$, must, by *Duhamel's Theorem*, also approach 1 as n increases. This is seen to be true, from the fact that this sum is $1+1/n$, which certainly approaches 1 as n increases. See INTEGRAL—definite integral. *Tech.* If

$$\lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i(n) = L,$$

then

$$\lim_{n\to\infty} \sum_{i=1}^{n} [\alpha_i(n)+\beta_i(n)] = L,$$

provided that for any $\epsilon > 0$ there exists an N such that $|\beta_i(n)/\alpha_i(n)| < \epsilon$ for all i's and for all $n > N$. (When β_i/α_i satisfied this restriction the ratios β_i/α_i $(i = 1, 2, 3, \cdots)$ are said to *converge uniformly* to zero). This is a sufficient, but not a necessary, condition for the two limits to be the same. All that is necessary is that the sum of the n betas approach zero as n becomes infinite, which can happen, for example, when any finite number of the betas are larger than the alphas, provided each of them approaches zero and the other betas are such that all β_i/α_i formed from them converge uniformly to zero. See UNIFORM—uniform convergence of a set of functions.

DU'O-DEC'I-MAL, *adj.* **duodecimal system of numbers.** A system of numbers in which twelve is the base, instead of ten. *E.g.*, in the *duodecimal system*, 24 would mean two twelves plus four, which would be 28 in the decimal system. See BASE—base of a system of numbers.

DUPIN. Dupin indicatrix of surface at a point. If the tangents to the lines of curvature at the point P of the surface S are taken as ξ, η coordinate axes, and ρ_1 and ρ_2 are the corresponding radii of principal curvature of S at P, then the **Dupin indicatrix** of S at P is $\dfrac{\xi^2}{|\rho_1|} + \dfrac{\eta^2}{|\rho_2|} = 1$, or $\dfrac{\xi^2}{\rho_1} + \dfrac{\eta^2}{\rho_2} = \pm 1$, or $\xi^2 = |\rho_1|$, according as the total curvature of S at P is positive, negative, or zero $(1/\rho_2 = 0)$. The curve of intersection of S and a nearby plane parallel to the tangent plane at P is approximately similar to the Dupin indicatrix of S at P, or, if the curvature of S is negative at P, to one of the hyperbolas constituting the Dupin indicatrix. Accordingly, a point of the surface is said to be an **elliptic, hyperbolic,** or **parabolic point** according as the total curvature is positive, negative, or zero there.

DU'PLI-CA'TION, *n.* **duplication of the cube.** Finding the edge of a cube whose volume is twice that of a given cube, using only straightedge and compasses; the problem of solving the equation, $y^3 = 2a^3$, for y, using only straightedge and compasses. This is impossible, since the cube root of 2 cannot be expressed in terms of radicals of index 2, and square

roots are the only kind of irrationals that can be evaluated by means of straightedge and compasses alone.

DU'TY, *n.* A tax levied by a government on imported (sometimes on exported) merchandise at the time of entering (or leaving) the country.

ad valorem duty. A duty which is a certain per cent of the value of the goods.

DY'AD, *n.* The juxtaposition of two vectors, without either scalar or vector multiplication being indicated, as $\mathbf{AB} = \mathbf{\Phi}$. A dyad is thought of as an operator which may operate on a vector by either *scalar* or *vector multiplication*, and in either order: $\mathbf{\Phi} \cdot \mathbf{F} = \mathbf{A}(\mathbf{B} \cdot \mathbf{F})$, $\mathbf{F} \cdot \mathbf{\Phi} = (\mathbf{F} \cdot \mathbf{A})\mathbf{B}$, $\mathbf{\Phi} \times \mathbf{F} = (\mathbf{A} \times \mathbf{B}) \times \mathbf{F}$, $\mathbf{F} \times \mathbf{\Phi} = \mathbf{F} \times (\mathbf{A} \times \mathbf{B})$. The first vector is called the **antecedent,** the second the **consequent.** The sum of two or more dyads is called a **dyadic.** If the order of the factors in each term of a dyadic is changed, *i.e.*, $\mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \mathbf{A}_3\mathbf{B}_3$ is written $\mathbf{B}_1\mathbf{A}_1 + \mathbf{B}_2\mathbf{A}_2 + \mathbf{B}_3\mathbf{A}_3$, the two dyadics are called **conjugate dyadics.** Two dyadics $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are defined to be equal if $\mathbf{r} \cdot \mathbf{\Phi}_1 = \mathbf{r} \cdot \mathbf{\Phi}_2$ and $\mathbf{\Phi}_1 \cdot \mathbf{r} = \mathbf{\Phi}_2 \cdot \mathbf{r}$ for all $\mathbf{r}$. If a dyadic is equal to its conjugate, it is said to be **symmetric.** If it is equal to the negative of its conjugate, it is said to be **anti-symmetric.** The (direct) product of dyads **AB** and **CD** is defined as the dyad $(\mathbf{B} \cdot \mathbf{C})\mathbf{AD}$.

DY-AD'IC, *n.* See DYAD.

DY-NAM'ICS, *n.* A branch of mechanics studying the effects of forces on rigid and deformable bodies. It is usually treated under two heads—statics and kinematics. See STATICS, KINEMATICS, and KINETICS.

DYNE, *n.* The unit of force in the c.g.s. system of units (centimeter-gram-second system). See FORCE—unit force.

E

e. The base of the natural system of logarithms; the limit of $(1+1/n)^n$ as n increases without limit. Its numerical value is $2.7182818284 \cdots$. The binomial form, $(1+1/n)^n$, occurs in the process of deriving the formula for the derivative of $\log x$ with respect to x. Its limit can be approximated by expanding by the *binomial theorem* and adding the limits of successive terms, giving $e = 1 + 1/1! + 1/2! + 1/3! + \cdots$.

EC-CEN'TRIC, *adj.* **eccentric angle and circles of an ellipse.** See ELLIPSE.

eccentric angle and circles of an hyperbola. See HYPERBOLA—parametric equations of the hyperbola.

eccentric, or excentric, configurations. Configurations with centers which are not coincident. The term is used mostly with reference to two circles.

EC'CEN-TRIC'I-TY, *n.* **eccentricity of a parabola, ellipse, or hyperbola.** See CONIC.

E-CLIP'TIC, *n.* The great circle in which the plane of the earth's orbit cuts the celestial sphere; the path in which the sun appears to move.

EDGE, *n.* A line which is the intersection of two plane faces of a solid. See POLYHEDRON. An **edge of a polyhedral angle** is the intersection of two faces of the polyhedral angle. See ANGLE—polyhedral angle.

lateral edge of a prism. See PRISM.

EDVAC. A computing machine built at the University of Pennsylvania for the Ballistic Research Laboratories, Aberdeen Proving Ground. EDVAC is an acronym for *Electronic Discrete Variable Automatic Computer.*

EF-FEC'TIVE, *adj.* **effective interest rate.** See INTEREST.

EF-FI'CIEN-CY, *n.* (*Statistics.*) The statistic "t" is an **efficient estimate** of the parameter T in a frequency function $f(x, T)$ if: (1) $\sqrt{N}(t-T)$ is asymptotically normally distributed with zero mean and finite variance, σ^2; (2) if "t_1" is any other statistic such that $\sqrt{N}(t_1-T)$ is asymptotically normally distributed with zero mean and finite variance, $\sigma_1{}^2$, then $\sigma^2 \leq \sigma_1{}^2$. The **efficiency** (relative to t_1) is defined numerically as $\sigma^2/\sigma_1{}^2$. A statistic "t" is **inefficient** if either condition (1) or (2) is not satisfied, although the latter condition is the usual criterion.

EIGENFUNCTION, *n.* **eigenfunction of a homogeneous integral equation.** A solution of the equation. A necessary and sufficient condition that the equation

$$y(x)=\lambda\int_a^b K(x,t)y(t)\,dt$$

have a solution other than $y(x)\equiv 0$ is that λ be an *eigenvalue* of the kernel $K(x, t)$. *E.g.*, $y(x)=1$ is an *eigenfunction* and $\lambda=1/(b-a)$ an *eigenvalue* of the kernel $K(x, t)=1$. See HILBERT—Hilbert-Schmidt theory of integral equations with symmetric kernels. *Syn.* Characteristic function, fundamental function, autofunction.

EIGENVALUE, *n.* **eigenvalue for integral equations.** An *eigenvalue* of a kernel $K(x, t)$ is a real or complex number λ that satisfies the equation $D(\lambda)=0$, where $D(\lambda)$ is the *Fredholm determinant* of the kernel $K(x, t)$. A number λ is an *eigenvalue* of a kernel $K(x, t)$ if and only if there is a function $y(x)\not\equiv 0$ such that

$$y(x)=\lambda\int_a^b K(x,t)y(t)\,dt.$$

Syn. Characteristic number, value, or constant; fundamental number. See EIGENFUNCTION—eigenfunction of a homogeneous integral equation.

eigenvalue of a matrix. A root of the characteristic equation of the matrix. *Syn.* Characteristic root, characteristic number, latent root. A matrix A has the eigenvalue λ if and only if there is a vector $\mathbf{x}$ of components $x_1, x_2, \cdots, x_n$ such that $A\mathbf{x}=\lambda\mathbf{x}$, where multiplication is matrix multiplication and $\mathbf{x}$ is considered to be a one-column matrix. Such a vector $\mathbf{x}$ is called an **eigenvector** of the matrix. The set of all eigenvalues of a matrix is called the **spectrum** of the matrix.

EIGENVECTOR, *n.* **eigenvector of a matrix.** See EIGENVALUE—eigenvalue of a matrix.

E-LAS'TIC, *adj.* **elastic bodies.** Bodies possessing the property of recovering their size and shape when the forces producing deformations are removed.

elastic constants. See HOOKE—generalized Hooke's law, MODULUS—Young's modulus, POISSON'S RATIO, and LAMÉ'S CONSTANTS.

E'LAS-TIC'I-TY, *n.* (1) The property possessed by substances of recovering their size and shape when the forces producing deformations are removed. (2) The mathematical theory concerned with the study of the behavior of elastic bodies. It deals with the calculation of stresses and strains in elastic substances subjected to the action of prescribed forces or deformations. The theory of elasticity of small displacements is called the **linear theory.** The **first fundamental problem of elasticity** is the problem of the determination of the state of stress and deformation in the interior of a body when its surface is deformed in a known way. The **second fundamental problem of elasticity** is the problem of the determination of the state of stress and deformation in the interior of a body when its surface is subjected to a specified distribution of external forces.

volume elasticity, or bulk modulus. The quotient of the increase in pressure and the change in unit volume; the negative of the product of the volume and the rate of change of the pressure with respect to the volume, *i.e.*, $E=-V\,dp/dV$.

Young's modulus of elasticity. A measure of the elasticity of stretching or compression; the ratio of the stress to the resulting strain.

E-LEC'TRO-MO'TIVE, *adj.* **electromotive force.** Denoted by E. M. F. (1) That which causes current to flow. (2) The energy added per unit charge due to the mechanical (or chemical) action producing the current. (3) The open circuit difference in potential between the terminals of a cell or generator.

E-LEC'TRO-STAT'IC, *adj.* **electrostatic intensity.** The force a unit positive charge would experience if placed at the point in question assuming that this is done without altering the positions of the other charges in the universe. This assumption should be regarded as a convenient mathematical fiction rather than as a physical possibility. If $e\mathbf{E}$ is the electrostatic force experienced by a charge e when placed at the point P, then the vector $\mathbf{E}$ is the electrostatic intensity at P. Dimensionally, $\mathbf{E}$ is force per unit charge. The electric intensity due to a single charge e is given by the

expression $er^{-2}\rho_1$. See CHARGE—point charge, and COULOMB—Coulomb's law for point-charges. Here r is the distance from the charge point to the field point and ρ_1 is the unit vector pointing from charge point to field point.

electrostatic potential. See POTENTIAL.

electrostatic potential of a complex of charges. The scalar point function $\Sigma e_i/r_i$ $(=e_1/r_1+e_2/r_2+\cdots+e_n/r_n)$. Here $r_1, r_2, \cdots, r_n$ are the distances from the charges $e_1, e_2, \cdots, e_n$ to the field point—the point, supposedly free of charge, at which we are computing the potential. Thus the potential in this case is a point function which is defined at all points excepting the charge points. In rectangular Cartesian coordinates, $e_i/r_i=e_i[(x-x_i)^2+(y-y_i)^2+(z-z_i)^2]^{-1/2}$, where x, y, z and x_i, y_i, z_i are the coordinates of the field point and of the ith charge point. The potential at a field point P is equal to the work done by the field in repelling a unit positive charge from P to infinity or to the work that must be done against the field to bring the unit charge from infinity to rest at the point P. This work is given by the line integral of the tangential component of the electric intensity E—namely, $\int_p^\infty \mathbf{E}\cdot\tau\, ds$, where τ is the unit tangent vector to the curve employed and s is the arc length. See COULOMB—Coulomb's law for point charges; FORCE—field of force; and above, electrostatic intensity. This line integral is independent of the path. The negative *gradient* of this potential function considering the charge points as fixed and the field point as variable is equal to the *electric intensity*. This potential function satisfies Laplace's partial differential equation at field points and is of the order of $\frac{1}{r}$ for r infinite.

electrostatic unit of charge. A charge of such magnitude that when placed one centimeter away from a duplicate charge will repel it with a force of one dyne. Evidently, if force, distance, and charge are measured in dynes, centimeters, and electrostatic units, respectively, the constant k in Coulomb's law for point charges will assume the value unity. See COULOMB. In this definition, the charges are to be in free space—otherwise the dielectric constant of the medium must also be involved.

Gauss's fundamental theorem of electrostatics. See GAUSS.

superposition principle for electrostatic intensity. The principle that the electrostatic intensity due to a complex (or set) of charges e_1 at P_1, e_2 at P_2, etc., is equal to the vector sum of the electric intensities due to the separate charges: $\Sigma e_i r_i^{-2}\rho_i$.

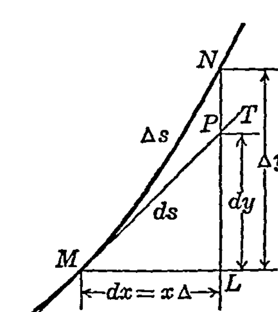
EL'E-MENT, *n.* See CONE, CYLINDER, CYLINDRICAL—cylindrical surface, DETERMINANT.

element of an analytic function of a complex variable. See ANALYTIC—analytic continuation of an analytic function of a complex variable.

element of integration. The expression following the integral sign (or signs) in a definite integral (or multiple integral). If the integral is being used to determine area (or volume, mass, etc.), the element is called the element (or **differential**) of area (or volume, mass, etc.). It can then be interpreted as an approximation to the area (or volume, mass, etc.) of small pieces, the limit of whose sum as the pieces decrease in size in a suitable way is the value of the area (or volume, mass, etc.). See INTEGRAL—definite integral, multiple integral. Following are some particular examples of elements of integration: The **element of arc length** (or **linear element**) of a curve is an approximation to the length of the curve (see LENGTH), between two points, which (for a curve in the plane) is equal to

$$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{1+(dy/dx)^2}\,dx$$
$$=\sqrt{(dx/dy)^2+1}\,dy,$$

where dy/dx is to be determined in terms of x before integrating, and dx/dy in terms of y, from the equation of the curve. It



can be seen from the figure that $ds=MP$ is an approximation of the arc-length $MN=\Delta s$, which results from an increase of Δx in the independent variable. **In polar form:**

$$ds=\sqrt{\rho^2+(d\rho/d\theta)^2}\,d\theta.$$

If the equation of a space curve is in the parametric form, $x=f(t)$, $y=g(t)$, $z=h(t)$, the element of length is

$$\sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2}\,dt.$$

The **element of plane area** (denoted by dA) for an area bounded by the curve $y=f(x)$, the x-axis, and the lines $x=a$ and $x=b$, is usually taken as $f(x)\,dx$. The area is then equal to

$$\int_a^b f(x)\,dx.$$

In polar coordinates, dA is taken as $\frac{1}{2}r^2d\theta$ or $\frac{1}{2}\rho^2d\theta$. Then

$$A=\tfrac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2\,d\theta$$

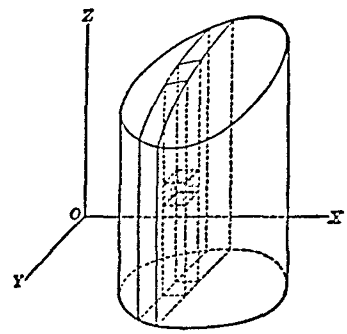
is the area bounded by the two rays $\theta=\theta_1$, $\theta=\theta_2$ and by the given curve for which ρ is expressed as a function of θ. In double integration, the element of area in rectangular Cartesian coordinates is $dxdy$, and in polar coordinates it is $\rho d\rho d\theta$ (also see SURFACE—surface area, surface of revolution). The **element of volume** can be taken as $A(h)dh$, where $A(h)$ is the area of a cross-section perpendicular to the h axis (for a particular example of this, see REVOLUTION—solid of revolution). For triple integration in Cartesian coordinates, the element of volume is $dxdydz$. The volume then equals

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} dx\,dy\,dz,$$

where z_1 and z_2 are constants, y_1 and y_2 may be functions of z, and x_1 and x_2 may be functions of y, or z, or of both y and z, these functions depending upon the particular shape of the surface that bounds the volume. The order of integration may, of course, be changed (the proper change in limits being made) to best suit the volume under consideration. The figure shows an element of volume in rectangular coordinates and illustrates the process of finding the volume by an integral of the form

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} dz\,dy\,dx.$$

In cylindrical coordinates, the element of volume is $dv=r\,dr\,d\theta\,dz$, and in polar (spherical) coordinates it is $dv=r^2\sin\theta\,dr\,d\theta\,d\phi$.



The **element of mass** is $dm=\rho dV$, where dV is an element of arc, area, or volume and ρ is the density (mass per unit length, area, or volume). Also see AREA; VOLUME; MOMENT—moment of a mass, moment of inertia; PRESSURE—fluid pressure; and WORK.

elements of geometry, calculus, etc. The fundamental assumptions and propositions of the subject.

geometrical element. (1) A point, line, or plane. (2) Any of the parts of a configuration, as the sides and angles of a triangle.

EL-E-MEN'TA-RY, *adj.* **elementary divisor of a matrix.** See INVARIANT—invariant factor of a matrix.

elementary operations on determinants or matrices. The operations: (I) Interchange of two rows, or of two columns; (II) addition to a row of a multiple of another row, or addition to a column of a multiple of another column; (III) multiplication of a row or of a column by a nonzero constant. Operation (II) leaves the value of a determinant unchanged, (I) leaves the numerical value unchanged but changes the sign, and (III) is equivalent to multiplying the determinant by the constant. See also EQUIVALENT—equivalent matrices.

elementary symmetric functions. See SYMMETRIC.

EL'E-VA'TION, *n.* **(elevation of a given point).** The height of the point above a given plane, above sea level unless otherwise indicated.

angle of elevation. See ANGLE—angle of elevation.

E-LIM'I-NANT, *n.* See RESULTANT.

E-LIM'I-NA'TION, *n.* **elimination of an unknown** from a set of simultaneous equations. The process of deriving from these equations another set of equations which does not contain the unknown that was to be eliminated and is satisfied by any values of the remaining unknowns which satisfy the original equations. This can be done in various ways. **Elimination by addition or subtraction** is the process of putting a set of equations in such a form that when they are added or subtracted in pairs one or more of the variables disappears, then adding or subtracting them as the case may require to secure a system (or perhaps one equation) containing at least one less variable. *E.g.*, (a) given $2x+3y+4=0$ and $x+y-1=0$, x can be eliminated by multiplying the latter equation by 2 and subtracting the result from the first equation, giving $y+6=0$; (b) given

(1) $4x+6y-z-9=0,$

(2) $x-3y+z+1=0,$

(3) $x+2y+z-4=0,$

y can be eliminated by multiplying (2) by 2 and adding the result to (1), and (3) by -3 and adding to (1). The results are $6x+z-7=0$ and $x-4z+3=0$. **Elimination by comparison** is the process of putting two equations in such forms that their left (or right) members are identical and the other members do not contain one of the variables, then equating the right (or left) members. *E.g.*, $x+y=1$ and $2x+y=5$ can be written $x+y=1$ and $x+y=5-x$, respectively. Hence $5-x=1$. **Elimination by substitution** is the process of solving one of a set of equations for one of the unknowns (in terms of the other unknowns), then substituting this expression in place of this unknown in the other equations. *E.g.*, in solving $x-y=2$ and $x+3y=4$, one might solve the first equation for x, getting $x=y+2$, and substitute in the second, getting

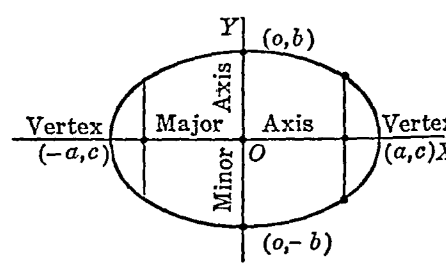
$$y+2+3y=4 \quad \text{or} \quad y=\tfrac{1}{2}.$$

See RESULTANT—resultant of a set of polynomial equations.

EL-LIPSE', *n.* A sort of elongated circle, like a longitudinal section of a football; any plane section of a circular conical surface, which is a closed curve (*i.e.*, not a parabola, hyperbola, or straight lines); the plane curve which is the set of all points which are such that the sum of the distances of one of the points from two fixed points (called the **foci**) is constant; a conic whose eccentricity is less than unity. The ellipse is symmetric with respect to two lines, called its **axes**. *Axes* usually refer to the segments cut off on these lines by the ellipse, and are called the **major** (longer) and **minor** (shorter) axes. If the major and minor axes lie on the x-and y-axes, respectively, the center is then at the origin and the equation of the ellipse, in Cartesian coordinates, is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

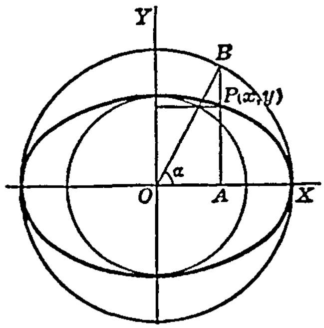
where a and b are the lengths of the **semimajor** and **semiminor** axes. This is the **standard form** of the equation of the ellipse, and is the equation of the ellipse in the position illustrated. The distance from an



end of the minor axis to a focus is a. If c is the distance from the center to a focus, then the ratio c/a is called the **eccentricity** of the ellipse (see CONIC). Two ellipses are said to be **similar** if they have the same eccentricity. The intersection of the axes is called the **center** of the ellipse, the points where the ellipse cuts its major axis are called its **vertices**, and the chords through its foci and perpendicular to its major axis are called the **latera recta** (plural of **latus rectum**). If the center of the ellipse is at the point (h, k) and its axes are parallel to the coordinate axes, its Cartesian equation is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

If the 1 in the right member of this equation is replaced by 0, the equation is said to be the equation of a **point ellipse,** since the equation has the form of the equation of an ellipse but is satisfied by the coordinates of only one point. If 1 is replaced by -1, the equation is said to be the equation of an **imaginary ellipse,** since no point with real coordinates satisfies the equation. When the ellipse has its center at the origin and its axes on the coordinate axes, the **parametric equations** are



$$x = a \cos \alpha, \quad y = b \sin \alpha,$$

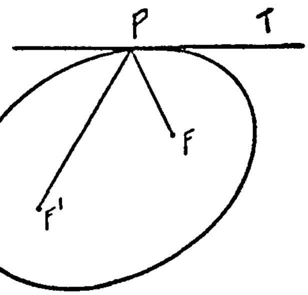
where a and b are the lengths of the semimajor and semiminor axes, and α is the angle (at the origin) in the right triangle whose legs are the abscissa, OA, of the point $P(x, y)$ on the ellipse, and the ordinate, AB, to the circle with radius a and center at the origin. The angle, α, is called the **eccentric angle** of the ellipse. The two circles in the figure are called the **eccentric circles** of the ellipse. A circle is an extreme case of an ellipse with eccentricity zero, with its major and minor axes equal, and with coincident foci. See CONIC, and DISCRIMINANT—discriminant of a quadratic equation in two variables.

area of an ellipse. The product of π and the lengths of the semi-major and semiminor axes (*i.e.*, πab). This reduces to the formula for the area of a circle (πr^2) when the major and minor axes of the ellipse are equal, *i.e.*, when the ellipse is a circle.

diameter of an ellipse. The locus of the midpoints of a set of parallel chords. Any diameter must pass through the center of the ellipse and always belongs to a set of parallel chords defining some other diameter. Two diameters in this relation to each other are called **conjugate diameters.**

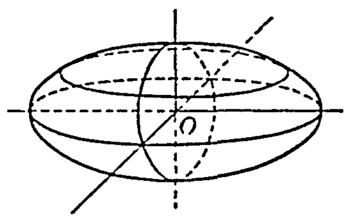
director circle of an ellipse. See DIRECTOR.

focal property of the ellipse. Lines drawn from the foci of an ellipse to any point on the ellipse make equal angles with the tangent (and normal) to the ellipse at the point (see figure). Hence, if the ellipse is constructed from a strip of polished metal, rays



of light emanating from one focus will come together at the other focus. This is sometimes called the *optical* or *reflection property* of the ellipse. When reflection of sound instead of light is being considered it is called the *acoustical property* of the ellipse.

EL-LIP'SOID, *n.* A surface whose plane sections are all either ellipses or circles. An ellipsoid is symmetrical with respect to three mutually perpendicular lines (called the axes), and with respect to the three planes determined by these lines. The intersection of these lines is called the **center.**



Any chord through the center is called a **diameter.** The standard equation of the ellipsoid, with center at the origin and intercepts on the axes, a, $-a$, b, $-b$, and c, $-c$, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

If $a > b > c$, a is called the semimajor axis, b the semimean axis, and c the semiminor axis. If $a = b = c$, the equation becomes the equation of a sphere. If the 1 in the right

member of the above equation is replaced by 0, the equation is said to be the equation of a **point ellipsoid** (since the coordinates of only one point satisfy the equation); if the 1 is replaced by -1, then no real values of the coordinates satisfy the equation and it is said to be the equation of an **imaginary ellipsoid.** The **center of an ellipsoid** is the point of symmetry of the ellipsoid. This point is the intersection of the three principal planes of the ellipsoid. An **ellipsoid of revolution** (or **spheroid**) is an ellipsoid generated by revolving an ellipse about one of its axes (see SURFACE—surface of revolution). This is an ellipsoid whose sections by planes perpendicular to one of its axes are all circles. The axis passing through the centers of these circular sections is called the **axis of revolution.** The largest circular section is called the **equator** of the ellipsoid of revolution. The extremities of the axis of revolution are called the **poles** of the ellipsoid of revolution. The ellipsoid of revolution is said to be **prolate** if the diameter of its equatorial circle is less than the length of the axis of revolution, and **oblate** if this diameter is greater than the length of the axis of revolution.

confocal ellipsoids. See CONFOCAL—confocal quadrics.

similar ellipsoids. See SIMILAR.

volume of an ellipsoid. If a, b, and c are the semi-axes, the volume is $\frac{4}{3}\pi abc$. When $a=b=c$ the ellipsoid is a sphere and this formula becomes $\frac{4}{3}\pi a^3$.

EL'LIP-SOI'DAL, *adj.* **ellipsoidal coordinates.** See COORDINATE.

EL-LIP'TIC, or EL-LIP'TI-CAL, *adj.* **elliptic conical surface.** A conical surface whose directrix is an ellipse. When the vertex is at the origin and the axis coincident with the z-axis in a system of rectangular Cartesian coordinates, its equation is

$$x^2/a^2+y^2/b^2-z^2/c^2=0.$$

When $a=b$, this is a **right circular cone.**

elliptic coordinates of a point. Coordinates in the plane determined by confocal conics (ellipses and hyperbolas) or coordinates in space determined by confocal quadrics (in the latter case, usually called *ellipsoidal coordinates*). See CONFOCAL—confocal quadrics, and CURVILINEAR—curvilinear coordinates of a point in space.

elliptic cylinder. See CYLINDER.

elliptic function. The inverse $x=\phi(y)$ of an elliptic integral y with limits of integration x_0 and x. See below, Jacobian elliptic functions, and Weierstrassian elliptic functions. An **elliptic function of a complex variable** is defined as a doubly periodic single-valued function $f(z)$ of the complex variable z such that $f(z)$ has no singularities other than poles in the finite plane. A doubly periodic function cannot be an *entire* function, unless it is a constant.

elliptic modular function. See MODULAR.

elliptic integral. Any integral of the type

$\int R(x, \sqrt{S})\,dx$, where $S=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4$ has no multiple roots (a_0 and a_1 are not both zero), and $R(x, \sqrt{S})$ is a *rational function* of x and $\sqrt{S}$. Integrals of the form

$$I_1=\int_0^x \frac{dt}{(1-t^2)^{1/2}(1-k^2t^2)^{1/2}}=\int_0^\phi \frac{d\psi}{(1-k^2\sin^2\psi)^{1/2}},$$

$$I_2=\int_0^x \frac{(1-k^2t^2)^{1/2}}{(1-t^2)^{1/2}}\,dt=\int_0^\phi (1-k^2\sin^2\psi)^{1/2}\,d\psi,$$

$$I_3=\int_0^x \frac{dt}{(t^2-a)(1-t^2)^{1/2}(1-k^2t^2)^{1/2}}=\int_0^\phi \frac{d\psi}{(\sin^2\psi-a)(1-k^2\sin^2\psi)^{1/2}},$$

where $\sin\phi=x$, were called (by Legendre) **incomplete elliptic integrals of the first, second, and third kinds,** respectively. The **modulus** of one of these elliptic integrals is k and the **complementary modulus** is $k'=(1-k^2)^{1/2}$, it being usual to take $0<k^2<1$. The integrals are said to be **complete** if $x=1$ ($\phi=\frac{1}{2}\pi$). Also, $I_1=\beta$,

$$I_2=\int_0^\beta \mathrm{dn}^2\,t\,dt,$$

and

$$I_3=\int_0^\beta (\mathrm{sn}^2\,t-\mathrm{sn}^2\,\alpha)^{-1}\,dt,$$

where $x=\mathrm{sn}\,\beta$, $a=\mathrm{sn}^2\,\alpha$, and sn t and dn t are *Jacobian elliptic functions*. The incomplete elliptic integral of the second kind is sometimes taken to be of the form

$$\int_0^x t^2(1-t^2)^{-1/2}(1-k^2t^2)^{-1/2}\,dt.$$

Elliptic integrals are so named because they were first encountered in the problem of finding the circumference of an ellipse.

Jacobian elliptic functions. The functions sn z, cn z, dn z defined by $y=\operatorname{sn}(z, k)=\operatorname{sn} z$ if

$$z=\int_0^y (1-t^2)^{-1/2}(1-k^2t^2)^{-1/2}\,dt,$$

and $\operatorname{sn}^2 z+\operatorname{cn}^2 z=1$, $k^2\operatorname{sn}^2 z+\operatorname{dn}^2 z=1$, where the sign of cn z and dn z are chosen so that cn $(0)=$ dn $(0)=1$. The number k is the **modulus** of the functions and $k'=\sqrt{1-k^2}$ is the **complementary modulus.** If

$$K=\int_0^1 (1-t^2)^{-1/2}(1-k^2t^2)^{-1/2}\,dt$$

and

$$K'=\int_0^1 (1-t^2)^{-1/2}(1-k'^2t^2)^{-1/2}\,dt,$$

then sn z, cn z, and dn z are *doubly periodic* functions with periods $(4K, 2iK')$, $(4K, 2K+2iK')$, and $(2K, 4iK')$, respectively. Also,

$$\frac{d\operatorname{sn} z}{dz}=\operatorname{cn} z\operatorname{dn} z,\quad \frac{d\operatorname{cn} z}{dz}=-\operatorname{sn} z\operatorname{dn} z,$$

$$\frac{d\operatorname{dn} z}{dz}=-k^2\operatorname{sn} z\operatorname{cn} z.$$

Jacobi's notation for these functions was sinam z, cosam z, Δam z. He also wrote tanam z for sn z/cn z. See above, elliptic integrals.

elliptic paraboloid. See PARABOLOID.

elliptic partial differential equation. A real second-order partial differential equation of the form

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+F\left(x_1,\cdots,x_n, u, \frac{\partial u}{\partial x_1},\cdots,\frac{\partial u}{\partial x_n}\right)=0,$$

such that the quadratic form $\sum_{i,j=1}^{n} a_{ij}x_ix_j$ is nonsingular and definite; *i.e.*, by means of a real linear transformation this quadratic form can be reduced to a sum of n squares all of the same sign. Typical examples are the Laplace and Poisson equations. See INDEX—index of a quadratic form.

elliptic point on a surface. A point whose *Dupin indicatrix* is an ellipse.

elliptic type of Riemann surface. See TYPE—type of a Riemann surface.

Weierstrassian elliptic functions. The function $p(z)$ defined by $y=p(z)$ if $z=\int_y^\infty S^{-1/2}\,dt$, where $S=4t^3-g_2t-g_3=4(t-e_1)(t-e_2)(t-e_3)$, and the function $p'(z)=\sqrt{4p^3-g_2p-g_3}$. These are *doubly periodic* functions with periods $2\omega_1$, $2\omega_2$, where $\omega_1=K(e_1-e_3)^{-1/2}$ and $\omega_2=iK'(e_1-e_3)^{-1/2}$, and K and K' are as defined above under *Jacobian elliptic functions.* Any *elliptic function* $f(z)$ can be expressed as a product of $p'(z)$ and a rational function of $p(z)$, where $p(z)$ and $p'(z)$ have the same periods as $f(z)$. Also, $p(z)=e_3+(e_1-e_3)[\operatorname{sn}\{z(e_1-e_3)^{1/2}\}]^{-2}$, where sn z is a *Jacobian elliptic function,* and

$$p(z)=\frac{1}{z^2}+\sum_{m,n}\left\{\frac{1}{(z-\Omega_{m,n})^2}-\frac{1}{\Omega_{m,n}{}^2}\right\},$$

where $\Omega_{m,n}=2m\omega_1+2n\omega_2$ and the summation is over all integral values of m and n except $m=n=0$.

E'LON-GA'TION, *n.* (1) The limit of the ratio of the increment Δl in length l of a vector, joining two points of a body (this increment resulting from the body being subjected to a deformation), to its undeformed length l as l is allowed to approach zero. In symbols, $e=\lim_{l\to 0}\Delta l/l$. This limit has in general different values depending on the direction of the vector in the deformed medium. (2) The change in length per unit length of a vector in a deformed medium.

elongations and compressions. Same as ONE-DIMENSIONAL STRAINS. See STRAIN.

EM-PIR'I-CAL, *adj.* **empirical formula, assumption, or rule.** A statement whose reliability is based upon a limited number of observations (such as laboratory experiments) and is not necessarily supported by any established theory or laws; formulas based upon immediate experience rather than logical (or mathematical) conclusions.

empirical curve. A curve that is drawn to approximately fit a set of statistical data. It is usually assumed to represent, approximately, additional data of the same kind. See METHOD—method of least squares, and GRAPHING—statistical graphing.

END, *adj.* **end point of a curve.** A point at which a branch of the curve ends.

end point of an interval. See INTERVAL.

EN'DO-MOR'PHISM, *n.* See HOMOMORPHISM.

EN-DORSE', *v.* Same as INDORSE.

EN-DOW'MENT, *adj.* **endowment insurance.** See INSURANCE—life insurance.

EN'ER-GY, *n.* The capacity for doing work.

conservation of energy. A principle asserting that energy can neither be created nor destroyed. In mechanics this principle asserts that in a conservative field of force the sum of the kinetic and potential energies is a constant.

energy integral. (1) An integral that arises in the solution of the particular differential equation of motion, $d^2s/dt^2 = \pm k^2s$, describing simple harmonic motion. The integral is $v^2/2 = \pm k^2 \int s\, ds$ and is called the energy integral, because when it is multiplied by m it is equal to the *kinetic energy*, $\frac{1}{2}mv^2$. (2) An integral stating that the sum of the potential and kinetic energies is constant, in a dynamic system in which this is true.

kinetic energy. The energy a body possesses by virtue of its motion. A particle of mass m moving with velocity $\mathbf{v}$ has kinetic energy of amount $\frac{1}{2}m\mathbf{v}^2$. In a conservative field of force, the work done by the forces in displacing the particle from one position to another is equal to the change in the kinetic energy. A body rotating about an axis and having angular velocity ω and moment of inertia I about the axis has kinetic energy of amount $\frac{1}{2}I\omega^2$.

potential energy. The energy a body possesses by virtue of its position. A term applicable to conservative fields of force only; potential energy is defined as the negative of the work done in displacing a particle from its standard position to any other position. See ENERGY—conservation of energy.

principle of energy. A principle in mechanics asserting that the increase in kinetic energy is equal to the work done by the force.

ENIAC, *n.* An early large-scale general-purpose computing machine (Electronic Numerical Integrator and Computer) that was built at the University of Penn., demonstrated publicly in 1946, and moved to the Ballistic Research Laboratories, Aberdeen Proving Ground, in 1947.

ENNEPER. equations of Enneper. Integral equations for the coordinate functions of a minimal surface referred to its minimal curves as parametric curves:

$$x = \tfrac{1}{2}\int (1-u^2)\phi(u)\, du + \tfrac{1}{2}\int (1-v^2)\psi(v)\, dv,$$

$$y = \frac{i}{2}\int (1+u^2)\phi(u)\, du - \frac{i}{2}\int (1+v^2)\psi(v)\, dv,$$

$$z = \int u\phi(u) + \int v\psi(v)\, dv,$$

where $\phi(u)$ and $\psi(v)$ are arbitrary analytic functions. See WEIERSTRASS—equations of Weierstrass.

surface of Enneper. See SURFACE.

EN-TIRE', *adj.* **entire function.** A function which can be expanded in a Maclaurin's series, valid for all finite values of the variable; a function of a complex variable which is analytic for all finite values of the variable. *Syn.* Integral function. An entire function $f(z)$ is said to be of **order** ρ for $\theta_1 < \theta < \theta_2$ provided

$$\limsup_{r\to\infty} \frac{\log |f(re^{i\theta})|}{r^{\rho+\epsilon}} = 0,$$

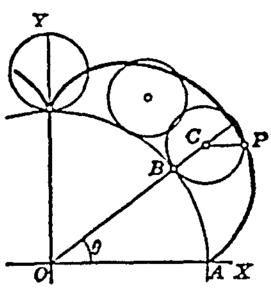
uniformly in θ, $\theta_1 < \theta < \theta_2$, for each $\epsilon > 0$, but not for any $\epsilon < 0$. See LIOUVILLE—Liouville's theorem, PHRAGMÉN-LINDELÖF FUNCTION, and PICARD'S THEOREMS.

entire series. A power series which converges for all values of the variable. The exponential series, $1 + x + x^2/2! + x^3/3! + \cdots + x^n/n! + \cdots$, is an entire series.

E-NU'MER-A-BLE, *adj.* **enumerable set.** Same as COUNTABLE SET. *Enumerably infinite* is used in the same sense as *countably infinite*, to denote an infinite set whose members can be put into one-to-one correspondence with the positive integers.

EN′VE-LOPE, *n.* The envelope of a one-parameter family of curves is a curve that is tangent to (has a common tangent with) every curve of the family. Its equation is obtained by eliminating the parameter between the equation of the curve and the partial derivative of this equation with respect to this parameter. The envelope of the circles $(x-a)^2+y^2-1=0$ is $y=\pm 1$. See DIFFERENTIAL—solution of a differential equation. In particular, the envelope of a one-parameter family of straight lines is a curve that is tangent to every member of the family of lines; *e.g.*, the curve $4(x-2)^3=27y^2$ is the envelope of the family of lines $y=-\frac{1}{2}cx+c+\frac{1}{8}c^3$ and is the result of eliminating c between this equation and the equation $0=-\frac{1}{2}x+1+\frac{3}{8}c^2$. The envelope of a one-parameter family of surfaces is the surface that is tangent to (has a common tangent plane with) each of the surfaces of the family along their characteristics; the locus of the characteristic curves of the family. See CHARACTERISTIC—characteristic of a one-parameter family of surfaces.

EP-I-CY′CLOID, *n.* The plane locus of a point fixed on the circumference of a circle as the circle rolls on the outside of a fixed circle (remaining in the same plane as the fixed circle). If b (OB in the figure) is the radius of the fixed circle with center at the origin, a (BC) is the radius of the rolling



circle, and θ is the angle at the origin subtended by the arc, AB, which has already contacted the rolling circle when the point is in the position $P:(x, y)$, the parametric equations of the curve are:

$$x=(a+b)\cos\theta-a\cos[(a+b)\theta/a],$$
$$y=(a+b)\sin\theta-a\sin[(a+b)\theta/a].$$

The curve has one arch when $a=b$, two arches when $a=b/2$, and n arches when $a=b/n$. It has a cusp of the first kind at every point at which it touches the fixed circle.

EP-I-TRO′CHOID, *n.* A generalization of an epicycloid, in which the describing point may be at any fixed point on the radius of the rolling circle, or this radius extended. Letting h denote the distance from the center of the rolling circle to the describing point, and using a, b, and θ as in the discussion of the epicycloid, the parametric equations of the *epitrochoid* are:

$$x=(a+b)\cos\theta-h\cos[(a+b)\theta/a],$$
$$y=(a+b)\sin\theta-h\sin[(a+b)\theta/a].$$

The cases for $h<a$ and $h>a$ are analogous to the corresponding cases ($b<a$ and $b>a$) in the discussion of the *trochoid.* See figure under TROCHOID.

EP′I-TRO-CHOI′DAL, *adj.* **epitrochoidal curve.** The locus of a point in the plane of a circle which rolls without slipping on another circle in such a way that the planes of the two circles meet under constant angle. All epitrochoidal curves are spherical curves. See SPHERICAL—spherical curve.

EP′SI-LON, *adj., n.* The fifth letter of the Greek alphabet, written, lower case, ϵ; capital, E.

epsilon-chain. A finite succession of points $p_1, p_2, \cdots, p_n$ such that the distance between any two successive points is less than epsilon (ϵ), ϵ being some positive real number. Any two points of a *connected* set can be joined by an ϵ-chain for any $\epsilon>0$, while a compact set is connected if every pair of its elements can be joined by an ϵ-chain for any $\epsilon>0$.

epsilon symbols. The symbols $\epsilon^{i_1 i_2 \cdots i_k}$ and $\epsilon_{i_1 i_2 \cdots i_k}$, which are defined as being zero unless the integers $i_1, i_2 \cdots, i_k$ consist of the integers $1, 2, \cdots, k$ in some order, and to be $+1$ or -1 according as $i_1, i_2, \cdots, i_k$ is obtained from $1, 2, \cdots, k$ by an even or an odd permutation. If $\delta^{j_1 j_2 \cdots j_k}_{i_1 i_2 \cdots i_k}$ is the *generalized Kronecker delta*, then $\epsilon^{i_1 i_2 \cdots i_k}=\delta^{i_1 \cdots i_k}_{1,2,\cdots,k}=\delta^{1,2,\cdots,k}_{i_1 i_2 \cdots i_k}=\epsilon_{i_1 i_2 \cdots i_k}$. The two epsilon symbols are *relative numerical tensor fields* of weight $+1$ and -1, respectively.

E′QUAL, *adj.* In *geometry,* **equal** is used to denote exact agreement with respect to some particular property, but not necessarily actual *congruence. E.g.,* triangles with equal altitudes and equal bases are said to be equal because their areas are the same, but they may not be congruent. Equality and congruence are **synonymous** in some cases; two equal angles are congruent. *Equal* is sometimes, although rarely, used in the same sense as *congruent.* In *analysis,* **equal** is used in describing a relation between two quantities that are alike in any or all senses, the sense in which they are alike being specified; *e.g.,* if two functions of a variable are equal numerically for all values of the variable the relation is an identity; if they are equal for only certain values, the relation is an equation. *Tech.* An *equals relation* is a relation which is *reflexive, symmetric,* and *transitive.* Equality is then defined by means of the particular equals relation which applies to the case at hand. See EQUIVALENCE—equivalence relation.

equal roots of an equation. See MULTIPLE—multiple root of an equation, and DISCRIMINANT—discriminant of a polynomial equation.

E-QUAL′I-TY, *n.* The state of being *equal*; the statement, usually in the form of an equation, that two things are equal.

continued equality. Three or more quantities set equal by means of two or more equality signs in a continuous expression, as $a=b=c=d$, or $f(x, y)=g(x, y)=h(x, y)$. The last expression is equivalent to the equations $f(x, y)=g(x, y)$ and $g(x, y)=h(x, y)$.

equality of two complex numbers. The property of having their real parts equal and their pure imaginary parts equal ($a+bi=c+di$ means $a=c$ and $b=d$); the property of having equal moduli and amplitudes which differ by integral multiples of 2π.

E-QUATE′, *v.* **to equate one expression to another.** To form the algebraic statement of equality which states that the two expressions are equal. The statement may be either an *identity* or a *conditional equation* (commonly called simply an *equation*). *E.g.,* one may *equate* $(x+1)^2$ to x^2+2x+1, getting the identity $(x+1)^2=x^2+2x+1$; or one may *equate* $\sin x$ and $2x+1$; or one may *equate* coefficients in $ax+b$ and $2x+3$, getting $a=2$, $b=3$.

E-QUAT′ED, *adj.* **equated date (for a set of payments).** The date upon which they could all be discharged by a single payment equal to the sum of their values when due, taking into account accumulations of payments due prior to that date and present values, at that date, of future payments. *Syn.* Average date.

equated time. (*Finance.*) The time from the present to the equated date.

E-QUA′TION, *n.* A statement of equality between two quantities. Equations are of two types, **identities** and **conditional equations** (or usually simply *equations*). A *conditional equation* is true only for certain values of the unknown quantities involved (see IDENTITY); *e.g.,* $x+2=5$ is a true statement only when $x=3$; and $xy+y-3=0$ is true when $x=2$ and $y=1$, and for many other pairs of values of x and y; but for still other pairs it is false.

amortization equation. See AMORTIZATION.

auxiliary equation. See DIFFERENTIAL—linear differential equations.

biquadratic equation. A polynomial equation of the fourth degree. *Syn.* Quartic equation.

compatibility equations. See COMPATIBILITY.

cubic equation. See CUBIC.

defective equation. An equation which has fewer roots than some equation from which it has been obtained. Roots may be lost, for instance, by dividing both members of an equation by a function of the variable. If $x^2+x=0$ is divided by x, the result, $x+1=0$, is defective; it lacks the root 0.

difference equation. See DIFFERENCE.

differential equation. See DIFFERENTIAL—differential equations.

differential equations of Bessel, Hermite, Laguerre, Legendre, etc. See the respective names.

equation of continuity. (*Hydrodynamics.*) The equation $\operatorname{div} q=\nabla\cdot q=0$, where q represents the flux of some fluid. If there are no sources or sinks in a fluid, this

equation states that the fluid does not concentrate toward or expand from any point. If this equation holds at every point in a body of liquid, then the lines of vector flux must be closed or infinite. Such a distribution of vectors is called *solenoidal.*

equation of a curve. See various headings under CURVE, and PARAMETRIC—parametric equations of a curve. For a curve in space, the equations of any two surfaces whose intersection is the curve can serve as equations of the curve.

equation of a cylinder. See CYLINDRICAL—cylindrical surface.

equation of a line. See LINE—equation of a straight line.

equation of motion. An equation, usually a differential equation, stating the law by which a particle moves.

equation of the normal line to a plane curve. See NORMAL—normal line to a plane curve.

equation in the p-form. A polynomial equation in one variable, in which the coefficient of the highest degree term is unity and the other coefficients are all integers.

equation of payments. An equation stating the equivalence of two sets of payments on a certain date, each payment in each set having been accumulated or discounted to that certain date, called the comparison, or focal, date.

equation of a plane. See PLANE.

equation of a surface. An equation satisfied by those, and only those, values of its variables which are coordinates of points on the surface. See PARAMETRIC—parametric equations, CYLINDRICAL—cylindrical surface.

equation of value. An equation of payments stating that a set of payments is equivalent to a certain single payment, at a given time. Some authors use only the term equation of value and others use only equation of payments, making no distinction between the cases in which there are two sets of several payments and those in which one set is replaced by a single payment. See above, equation of payments.

exponential equation. An equation in which the unknown letter (or letters) occurs in an exponent. Usually refers to an equation in which the unknown appears only in the exponent, but is sometimes used for equations having the unknown in the exponent and elsewhere; an equation of the form $2^x-5=0$ would always be called an exponential equation.

homogeneous equation. See HOMOGENEOUS.

inconsistent equations. See CONSISTENCY.

indeterminate equation. An equation containing more than one variable, such as $x+2y=4$. The equation is indeterminate because there are, in general, an unlimited number of sets of values which satisfy it. Historically, this kind of equation has been of particular interest when the coefficients are integers and it is required to find expressions for the sets of integral values of the variables that satisfy the given equation. Under these restrictions, the equations are called Diophantine equations.

indeterminate system of linear equations. A system of linear equations having an infinite number of solutions.

integral equations. See various headings under INTEGRAL.

irrational (or radical) equation. An equation containing the unknown, or unknowns, under radical signs or with fractional exponents. The equations

$$\sqrt{x^2+1}=\sqrt{x+2} \quad \text{and} \quad x^{1/2}+1=3$$

are both irrational equations.

linear differential equations. See DIFFERENTIAL—linear differential equations.

linear equation. See LINEAR.

locus of an equation. See LOCUS.

logarithmic equation. An equation containing the logarithm of the unknown. It is usually called logarithmic only when the unknown occurs only in the arguments of logarithms; $\log x + 2\log 2x+4=0$ is a logarithmic equation.

minimal (or minimum) equation. See ALGEBRAIC—algebraic number; CHARACTERISTIC — characteristic equation of a matrix.

multiple root of an equation. See MULTIPLE.

numerical equation. An equation in which the coefficients of the variables and the constant term are numbers, not literal constants. The equation $2x^2+5x+3=0$ is a numerical equation.

parametric equations. See PARAMETRIC.

partial differential equations. See DIFFERENTIAL—partial differential equations.

polynomial equation. A polynomial in

one or more variables, set equal to zero. The **degree** of the equation is the degree of the polynomial (see DEGREE—degree of a polynomial, or equation). The **general equation of the second degree in two variables** is

$$ax^2+by^2+cxy+dx+ey+f=0.$$

(See DISCRIMINANT—discriminant of the general quadratic.) The **general equation of the *n*th degree in one variable** is a polynomial equation of the nth degree whose coefficients are literal constants, such as

$$a_0x^n+a_1x^{n-1}+\cdots+a_n=0.$$

A polynominal equation of nth degree is said to be **complete** if none of its coefficients are zero; it is **incomplete** if one or more coefficients (other than the coefficient of x^n) are zero. A **root** of a polynomial equation in one variable is a value of the variable which reduces the equation to a true equality. Sometimes a solution can be found by factoring the equation; *e.g.*, the equation $x^2+x-6=0$ has roots 2 and -3, since $x^2+x-6=(x-2)(x+3)$. If the equation can not be factored, the method of solution usually consists of some method of successive approximations. Horner's and Newton's methods offer systematic approximations of this type. For the determination of imaginary roots, one can substitute $u+iv$ for the variable, equate the real and imaginary parts to zero, and solve these equations for u and v (usually by some method of successive approximations). See RATIONAL—rational root theorem, QUADRATIC—quadratic formula, CARDAN—Cardan's solution of the cubic, FERRARI'S solution of the quartic, and FUNDAMENTAL—fundamental theorem of algebra.

quadratic equation. See QUADRATIC.

reciprocal equation. See RECIPROCAL.

redundant equation. An equation containing roots that have been introduced by operating upon a given equation; *e.g.*, such roots may be introduced by multiplying both members by the same function of the unknown, or by raising both members to a power. The introduced roots are called **extraneous roots.** The equation $x-1=\sqrt{x+1}$, when squared and simplified, becomes $x^2-3x=0$, which has the roots 0 and 3, hence is *redundant*, because 0 does not satisfy the original equation.

simultaneous equations. See SIMULTANEOUS.

theory of equations. See THEORY.

transformation of an equation. See TRANSFORMATION.

trigonometric equation. See TRIGONOMETRIC.

E-QUA'TOR, *n.* **celestial equator.** The great circle in which the plane of the earth's equator cuts the celestial sphere. See HOUR—hour angle and hour circle.

equator of an ellipsoid of revolution. See ELLIPSOID—ellipsoid of revolution.

geographic equator (earth's equator). The great circle that is the section of the earth's surface by the plane through the center of the earth and perpendicular to the earth's axis. See ELLIPSOID—ellipsoid of revolution.

E'QUI-AN'GU-LAR, *adj.* **equiangular hyperbola.** Same as RECTANGULAR HYPERBOLA. See HYPERBOLA.

equiangular polygon. A polygon having all of its interior angles equal. An equiangular triangle is necessarily equilateral, but an equiangular polygon of more than three sides need not be equilateral.

equiangular spiral. Same as *logarithmic* spiral. Called equiangular because the angle between the tangent and radius vector is a constant. See LOGARITHMIC—logarithmic spiral.

mutually equiangular polygons. See MUTUALLY.

equiangular transformation. See ISOGONAL—isogonal transformation.

E-QUI-A'RE-AL MAP. Same as AREA-PRESERVING MAP. See Map.

E-QUI-CON-TIN'U-OUS, *adj.* **equicontinuous functions.** The functions $\{f_n\}$ are said to be *equicontinuous* over an interval if, for arbitrary $\epsilon>0$, there exists a δ_ϵ (δ dependent on ϵ) such that $|f_i(x_1)-f_i(x_2)|<\epsilon$ for all i, whenever $|x_1-x_2|<\delta_\epsilon$ for points x_1 and x_2 on the interval (a, b). See ASCOLI'S THEOREM.

E'QUI-DIS'TANT, *adj.* **equidistant system of parametic curves.** See PARAMETRIC.

E'QUI-LAT'ER-AL, *adj.* **equilateral hyperbola.** See HYPERBOLA—rectangular hyperbola.

equilateral polygon. A polygon having all of its sides equal. An equilateral triangle is necessarily equiangular, but an equilateral polygon of more than three sides need not be equiangular.

equilateral spherical polygon. A spherical polygon which has all of its sides equal.

mutually equilateral polygons. See MUTUALLY.

E'QUI-LIB'RI-UM, *n.* **equilibrium of forces.** The property of having their resultant force and the sum of their torques about any axis equal to zero. See RESULTANT.

equilibrium of a particle or a body. A particle is in equilibrium when the resultant of all forces acting on it is zero. A body is in equilibrium when it has no acceleration, either of translation or rotation; a rigid body is in equilibrium when its center of mass has no acceleration and the body has no angular acceleration. The conditions for a body to be in equilibrium are: (1) That the resultant of the forces acting on it be equal to zero; (2) that the sum of the moments of these forces about every axis be equal to zero (about each of three mutually perpendicular axes suffices).

E'QUI-PO-TEN'TIAL, *adj.* **equipotential surface.** A surface on which a potential function U maintains a constant value. More generally, if U is any point function, then an equipotential surface relative to U is a surface on which U is constant.

E-QUIV'A-LENCE, *adj., n.* **equivalence class.** If an *equivalence relation* is defined on a set, then the set can be separated into classes by the convention that two elements belong to the same class if and only if they are equivalent. These classes are called *equivalence classes.* Two equivalence classes are identical if they have an element in common. Each element belongs to one of the equivalence classes. *E.g.*, if one says that a is equivalent to b if $a-b$ is a rational number, this is an equivalence relation for the real numbers. The equivalence class which contains a number a is then the class which contains all numbers which can be obtained by adding rational numbers to a.

equivalence of propositions. An equivalence is a proposition formed from two given propositions by connecting them by "*if, and only if.*" An equivalence is true if both propositions are true, or if both are false. The proposition "For all triangles x, x is equilateral if, and only if, x is equilateral" is true, since any particular triangle is either both equilateral and equiangular, or it is neither equilateral nor equiangular. The equivalence formed from propositions p and q is usually denoted by $p \leftrightarrow q$, or $p \equiv q$. The equivalence $p \leftrightarrow q$ is the same as the statements "p is a *necessary and sufficient* condition for q," or "p if, and only if, q"; it is equivalent to the *conjunction* of the implications $p \rightarrow q$ and $q \rightarrow p$. Propositions p and q are said to be equivalent if $p \leftrightarrow q$ is true. An equivalence is also called a biconditional statement (or proposition).

equivalence relation. A relation between elements of a given set which is a *reflexive, symmetric,* and *transitive* relation and which is such that any two elements of the set are either equivalent or not equivalent. Ordinary equality and congruence relations are examples of equivalence relations. *Syn.* Equals relations.

E-QUIV'A-LENT, *adj.* **cash equivalent of an annuity.** Same as PRESENT VALUE. See VALUE.

equivalent equations. Equations in one variable that have exactly the same roots.

equivalent figures. Equal figures (see EQUAL). Some writers always use *equivalent* where others use *equal*, and *equal* for *congruent*.

equivalent matrices. Two square matrices A and B for which there exist nonsingular square matrices P and Q such that $A=PBQ$. Two square matrices are equivalent if, and only if, one can be derived from the other by a finite number of operations of the types: (a) interchange of two rows, or of two columns; (b) addition to a row of a multiple of another row, or addition to a column of a multiple of another column; (c) multiplication of a row or of a column by a nonzero constant. Every matrix is equivalent to some diagonal matrix. This transformation PBQ of the

matrix B is called an **equivalent transformation.** If $P=Q^{-1}$, it is called a **collineatory** (or **similarity**) **transformation**; if P is the transpose of Q, it is a **congruent transformation**; if P is the Hermitian conjugate of Q, it is a **conjunctive transformation**; if $P=Q^{-1}$ and Q is orthogonal, it is an **orthogonal transformation**; if $P=Q^{-1}$ and Q is unitary, it is a **unitary transformation.** Also, see under TRANSFORMATION.

equivalent sets. Sets that can be put into one-to-one correspondence. See CORRESPONDENCE.

topologically equivalent spaces. See TOPOLOGICAL—topological transformation.

ERATOSTHENES. sieve of Eratosthenes. The process of computing all the primes not greater than a number N by writing down all the numbers from 2 to N, removing those which are multiples of 2, those which are multiples of 3, and continuing until all multiples of primes not greater than $\sqrt{N}$ have been removed. Only prime numbers will remain.

ERG, *n.* A unit of work; the work done by a force of one dyne operating over a distance of one centimeter.

ER-GOD'IC, *adj.* **ergodic theory.** The study of measure-preserving transformations. In particular, the study of theorems concerning the limits of probability and weighted means. *E.g.*, the following is a theorem of this type: If T is a measure-preserving one-to-one transformation of a bounded open region of n-dimensional space onto itself, then there is a set M of *measure zero* such that if x is a point not in M and U is a neighborhood of x, then the points $T(x)$, $T^2(x)$, $T^3(x)$, $\cdots$ are in U with a definite positive limiting frequency; *i.e.*, $\lim_{n\to\infty}\left[\sum_1^n \phi_k(x)\right]/n$ exists and is positive, where $\phi_k(x)$ is $+1$ or 0 according as $T^k(x)$ belongs to U or does not belong to U [$T^k(x)$ is the result of applying the transformation T to x successively k times]. The **ergodic theorem of Birkhoff** states that if T is a measure-preserving point transformation of the interval (0, 1) onto itself, and if the function f is Lebesgue integrable over (0, 1), then there is a function f^* which is Lebesgue integrable over (0, 1) and is such that we have

$$f^*(x)=\lim\frac{f(x)+f(Tx)+\cdots+f(T^nx)}{n+1}$$

almost everywhere on (0, 1). The **mean ergodic theorem** (a weaker result than that of Birkhoff's ergodic theorem) states that, under the same hypotheses as in Birkhoff's theorem, the same conclusion is true with *point-wise convergence almost everywhere* replaced by *convergence in the mean of order two.*

ER'ROR, *n.* (1) The difference between a given value and the true value of a quantity, taken as positive or negative according to whether the former is the larger or the smaller. (2) (*Statistics.*) Variation in measurements due to uncontrollable factors. If the uncontrollable factors are large in number, independent, approximately equal, and additive in their effect on the variation around some constant or expected value, the deviations will be normally distributed around the constant or expected value. Measurements are presumably affected by such a set of factors—hence the name **error curve** for the normal distribution. (3) (*Statistics.*) Variations in observed values of a variable due to sampling the population are often called **sampling errors.** In the analysis of variance, *error* is the random sampling fluctuations after controlling a set of factors believed to affect the parameters of the distribution. (4) (*Statistics.*) In tests of hypotheses, **error of the first type** (as defined by J. Neyman and E. Pearson) is the error of rejecting a true hypothesis. **Error of the second type** (as defined by Neyman and Pearson) is the erroneous acceptance of a false hypothesis. The probability of the second type of error is a function of the alternative true hypotheses. This function is called the **power function** of the test of the hypothesis.

error function. Any of the functions

$$Erf(x)=\int_0^x e^{-t^2}\,dt=\tfrac{1}{2}\gamma(\tfrac{1}{2},x^2),$$

$$Erfc(x)=\int_x^\infty e^{-t^2}\,dt=\tfrac{1}{2}\Gamma(\tfrac{1}{2},x^2),$$

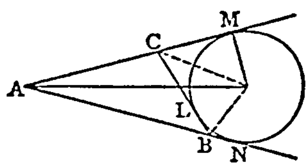
$$Erfi(x)=\int_0^x e^{t^2}\,dt=-i\cdot Erf(ix)$$

per cent error. The quotient of the error by the correct result, multiplied by 100%.

relative error. The quotient of the error by the correct result.

standard error of estimate. See CORRELATION—normal correlation.

ES-CRIBED′, *adj.* **escribed circle of a triangle.** A circle tangent to one side of the triangle and to the extensions of the other sides. *Syn.* Excircle. In the figure, the circle is an excircle of the triangle ABC, being tangent to BC at L and to AB and AC, extended, at N and M, respectively. The bisector of angle BAC passes through the center of the circle.



ES-SEN′TIAL, *adj.* **essential constant.** See CONSTANT—essential constant.

essential mapping. See INESSENTIAL.

ES-SEN′TIAL-LY, *adv.* **essentially bounded function.** See BOUNDED.

ES′TI-MATE, *n.*, *v.* As a *noun* (in *Statistics*): (1) Numerical values assigned to parameters of a distribution function, on the basis of evidence from samples, are statistical estimates. Such an estimate is called a statistic. (2) Statement as to the values of certain parameters or properties of functions on the basis of evidence.

estimate a desired quantity. To pass judgment based upon very general considerations, as contrasted to finding the quantity by exact mathematical procedure. One might *estimate* the square root of any number to the nearest integer, but one would *compute* it systematically, by some rule for extracting roots, if accuracy to three or four decimal places was required.

minimum variance unbiased estimate. The *unbiased statistic* t_n, linearly derived from a random sample of n observations, is a minimum variance estimate of T if $E(t_n - T)^2$ is smaller than for any other unbiased estimate t_n' from the same sized sample.

unbiased estimate. A statistic t_n is an unbiased estimate of the parameter T if $E(t_n) = T$ for all n, where $E(t_n)$ is the expectation of t_n.

EU-CLID′E-AN, *adj.* **Euclidean algorithm.** See ALGORITHM.

Euclidean geometry. See GEOMETRY—Euclidean geometry.

Euclidean space. (1) Ordinary three-dimensional space. (2) A space consisting of all sets (points) of n numbers $(x_1, x_2, \cdots x_n)$, where the distance $\rho(x, y)$ between $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ is defined as $\rho(x, y) = \left[\sum_{i=1}^{n} |x_i - y_i|^2\right]^{1/2}$. This is an n-dimensional Euclidean space; it is real or complex according as the coordinates $x_1, \cdots, x_n$ of $x = (x_1, \cdots, x_n)$ are real or complex numbers. An infinite dimensional Euclidean space is the (real) Hilbert space of all infinite sequences of real numbers $(x_1, x_2, \cdots)$ for which $\sum_{=1}^{\infty} x_i^2$ is finite, with $\rho(x, y)$ defined as $\left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2}$. *Syn.* Cartesian space.

locally Euclidean space. A topological space T for which there is an integer n such that each point of T has a neighborhood which is homeomorphic to an open set in n-dimensional Euclidean space. The space T is then said to be of dimension n. It has been proved that any locally Euclidean topological group is isomorphic to a Lie group (the fifth problem of Hilbert). See MANIFOLD.

EUCLID'S algorithm, axioms, postulates. See ALGORITHM, AXIOM, POSTULATE.

EULER. equation of Euler. (*Differential Geometry.*) When the lines of curvature of a surface S are parametric, the equation for the normal curvature $1/R$ for a given direction at a point of S becomes

$$\frac{1}{R} = \frac{\cos^2 \theta}{\rho_1} + \frac{\sin^2 \theta}{\rho_2},$$

where θ is the angle between the directions whose normal curvatures are $1/\rho_1$ and $1/\rho_2$. The above equation is called the equation

of Euler. See CURVATURE—normal curvature of a surface, and CURVATURE—principal curvatures of a surface at a point.

Euler characteristic. For a **curve,** the Euler characteristic is the difference between the number of vertices and the number of segments when the curve is divided into segments by points (vertices) such that each segment (together with its end points) is topologically equivalent to a closed straight line segment (can be continuously deformed into a closed interval). The Euler characteristic of a **surface** is equal to the number of vertices *minus* the number of edges *plus* the number of faces if the surface is divided into faces by means of vertices and edges in such a way that each face is topologically equivalent to a plane polygon. For both curves and surfaces, the Euler characteristic is independent of the method of subdivision. A surface has Euler characteristic 2 if and only if it is topologically equivalent to a sphere; Euler characteristic 1 if and only if it is topologically equivalent to the projective plane or to a disc (a circle and its interior); Euler characteristic zero if and only if it is topologically equivalent to a cylinder, torus, Möbius strip, or Klein bottle. See GENUS—genus of a surface, and SURFACE. For an **n-dimensional simplicial complex** K, the Euler characteristic is the number $\chi=\sum_{r=0}^{n}(-1)^r s(r)$, where $s(r)$ is the number of r-simplexes of K; χ is also equal to $\sum_{r=0}^{n}(-1)^r B_m^r$, where B_m^r is the *r-dimensional Betti number modulo m* (m a prime), and to $\sum_{0}^{n}(-1)^r B^r$, where B^r is the *r-dimensional Betti number.* Sometimes called the *Euler-Poincaré characteristic.*

Euler-Maclaurin sum formula. A formula for approximating a definite integral, say $\int_a^b f(x)\,dx$, where $f(x)$ has continuous derivatives of all orders up to the highest order used for all points of $[a, b]$, and $b-a=m$ is an integer. The formula is

$$\int_a^b f(x)\,dx=\frac{1}{2}[f(a)+f(b)]+\sum_{r=1}^{m}f(a+r)$$
$$-\sum_{r=1}^{n=1}\frac{B_r}{(2r)!}[f^{[2r-1]}(b)-f^{[2r-1]}(a)]$$
$$-f^{[2n]}(\theta m)\frac{mB_n}{(2n)!},$$

where θ is some number satisfying $0\leqq\theta\leqq 1$ and B_n is a Bernoulli number. See BERNOULLI—Bernoulli's numbers (1).

Euler's angles. The three angles usually chosen to fix the directions of a new set of rectangular space coordinate axes with reference to an old set. They are the angle between the old and the new z-axis, the angle between the new x-axis and the intersection of the new xy-plane with the old xy-plane, and the angle between this intersection and the old x-axis. This intersection is called the **nodal line** of the transformation. Euler's angles are often defined in other ways. Another common usage is to take the angles between the old and new z-axes, between the old y-axis and the normal to the plane of the two z-axes, and between this normal and the new y-axis.

Euler's constant (or **Mascheroni's constant**). The constant defined as

$$\lim_{n\to\infty}(1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+1/n-\log n)$$
$$=0.5772157\cdots.$$

It has been calculated to 260 decimal places. It is not known whether Euler's constant is an irrational number.

Euler's criterion for residues. See RESIDUE.

Euler's equation. (*Calculus of Variations.*) The differential equation

$$\frac{\partial f(x,y,y')}{\partial y}-\frac{d}{dx}\left(\frac{\partial f(x,y,y')}{\partial y'}\right)=0,$$

where

$$y'=\frac{dy}{dx}.$$

A necessary condition that $y(x)$ minimize the integral $\int_a^b f(x,y,y')\,dx$ is that $y(x)$ satisfy Euler's equation. This condition, and the more general necessary condition

$$\frac{\partial f}{\partial y}+\sum_{r=1}^{n}(-1)^r\frac{d^r}{dx^r}\left\{\frac{\partial f}{\partial y^{(r)}}\right\}=0,$$

where $y^{(r)}=\frac{d^r y}{dx^r}$, for $y(x)$ to minimize the

integral $\int_a^b f(x, y, y', \cdots, y^{(n)})\,dx$, were first discovered by Euler in 1744. For the double integral $\iint_s f(x, y, z, z_x, z_y)\,dx\,dy$, Euler's equation becomes

$$\frac{\partial f}{\partial z}-\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z_y}\right)=0,$$

where

$$z_x=\frac{\partial z(x, y)}{\partial x}, \quad z_y=\frac{\partial z(x, y)}{\partial y}.$$

Also called the Euler-Lagrange equation. See CALCULUS—calculus of variations.

Euler's formula. The formula $e^{ix}=\cos x+i\sin x$. It can be verified by expanding each of the functions e^{ix}, $\cos x$, and $\sin x$ in a Maclaurin's series and substituting the three series in the formula. Interesting special cases are those in which $x=\pi$ and 2π, for which $e^{\pi i}=-1$ and $e^{2\pi i}=1$, respectively.

Euler's ϕ-function of an integer. The number of integers not greater than the given integer and relatively prime to it. If the number is $n=a^p b^q c^r\cdots$, where $a, b, c, \cdots$ are distinct primes, then *Euler's ϕ-function* of n, written $\phi(n)$, is equal to

$$n(1-1/a)(1-1/b)(1-1/c)\cdots.$$

The values of $\phi(n)$, for $n=1, 2, 3$ and 4, are 1, 1, 2, and 2, respectively, while $\phi(12)=12(1-\frac{1}{2})(1-\frac{1}{3})=4$. *Syn.* Indicator, totient, ϕ-function.

Euler's theorem on homogeneous functions. A homogeneous function of degree n in the variables $x_1, x_2, x_3, \cdots, x_n$, multiplied by n, is equal to x_1 times the partial derivative of the function with respect to x_1, plus x_2 times the partial derivative of the function with respect to x_2, etc. *E.g.*, if $f(x, y, z)=x^2+xy+z^2$, then

$$2(x^2+xy+z^2)=x(2x+y)+y(x)+z(2z).$$

Euler's theorem for polyhedrons. For any simple polyhedron, $V-E+F=2$, where V is the number of vertices, E the number of edges, and F the number of faces. See above, Euler characteristic.

Euler's transformation of series. A transformation for oscillating series which increases the rate of convergence of convergent series and sometimes defines sums for divergent series. Consider the series $a_0-a_1+a_2-a_3+\cdots$. Euler's transformation carries this into

$$\frac{a_0}{2}+\frac{a_0-a_1}{2^2}+\frac{a_0-2a_1+a_2}{2^3}+\cdots=\sum\frac{\Delta^n a_0}{2^n},$$

where $\Delta^n a_0$ is the nth difference of the sequence $a_0, a_1, a_2, \cdots$ (*i.e.*, $\Delta^n a_0=a_0-\binom{n}{1}a_1+\binom{n}{2}a_2-\cdots+(-1)^n a_n$, where $\binom{n}{r}$ is the rth binomial coefficient of order n. *E.g.*, this transformation carries the series $1-\frac{1}{2}+\frac{1}{3}-\cdots$ into $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\cdots$ and the series $1-1+1-1+\cdots$ into $\frac{1}{2}+0+0+0+\cdots$.

E-VAL'U-ATE, *v.* To find the value of. *E.g.*, to evaluate $8+3-4$ means to reduce it to 7; to evaluate x^2+2x+2 for $x=3$ means to replace x by 3 and collect the results (giving 17); to evaluate an integral means to carry out the integration and, if it is a definite integral, substitute the limits of integration. See also, DETERMINANT—evaluation of a determinant.

E-VAL'U-A'TION, *n.* The act of evaluating.

E'VEN, *adj.* **even number.** A number that is divisible by 2. All even numbers can be written in the form $2n$, where n is an integer.

even function. See FUNCTION—even function.

even permutation. See PERMUTATION (2).

even-spaced map. See CYLINDRICAL—cylindrical map.

E-VENT', *n.* **compound event.** See COMPOUND.

dependent events. Two events such that the occurrence, or nonoccurrence, of one of them affects the occurrence, or nonoccurrence, of the other. *E.g.*, drawing a ball from a bag of different colored balls, the ball not being replaced, affects the probability of drawing any specified color in a second drawing.

independent events. Two or more events such that the occurrence or nonoccurrence of any one in a given trial does not affect the occurrence or nonoccurrence of any

of the other events. *E.g.*, the probability of drawing a certain ball from one bag is not affected by the drawing of a certain ball from another bag, nor from the same bag if the ball first drawn is replaced before the second drawing.

mutually exclusive events. Two or more events such that the occurrence of any one precludes the occurrence of all the others; if a coin is tossed, the coming up of heads and the coming up of tails on a given throw are mutually exclusive events.

simple event. An event whose probability can be obtained from consideration of a single occurrence; an event which is not a compound event. The tossing of a coin for heads (or tails) is a simple event.

EV′O-LUTE, *n.* **evolute of a curve.** See INVOLUTE—involute of a curve.

evolute of a surface. The two surfaces of center relative to the given surface *S*. See SURFACE—surfaces of center relative to a given surface. If we choose the normals to *S* as normals to the lines of curvature of *S*, we obtain the surfaces of center as loci of the evolutes of the lines of curvature of *S*. See above, evolute of a space curve. The evolute of *S* is also the evolute of any surface parallel to *S*. See PARALLEL—parallel surfaces, and INVOLUTE—involute of a surface.

mean evolute of a surface. The envelope of the planes orthogonal to the normals of a surface *S* and cutting the normals midway between the centers of principal curvature of *S*.

EV′O-LU′TION, *n.* The extraction of a root of a quantity; *e.g.*, finding a square root of 25. *Evolution* is the inverse of finding a power of a number, or **involution.**

EX-ACT′, *adj.* **exact differential equation.** See DIFFERENTIAL.

exact division. Division in which the remainder is zero. It is then said that the divisor is an **exact divisor.** The quotient is required to be of some specified type. *E.g.*, 7 is not exactly divisible by 2 if the quotient must be an integer, but 7 is exactly divisible by 2 if the quotient can be a rational number ($3\frac{1}{2}$).

exact interest. See INTEREST.

EX-CEN′TER, *n.* **excenter of a triangle.** The center of an *escribed* circle; the intersection of the bisectors of two exterior angles of the triangle.

EX-CESS′, *n.* **excess of nines.** The remainder left when any positive integer is divided by nine; the remainder when the greatest possible number of nines have been subtracted from it. It is equal to the remainder determined by dividing the sum of the digits by 9. It is customary, but not necessary, to restrict the process to positive integers. The excess of nines in 237 is 3, since $237=26\times 9+3$ (or $2+3+7=9+3$). See CASTING—casting out nines.

spherical excess. See SPHERICAL—spherical excess.

EX-CHANGE′, *n.* Payment of obligations other than by direct use of money; by use of checks, drafts, money orders, exchange of accounts, etc.

foreign exchange. Exchange carried on with other countries (between countries). The **rate of foreign exchange** is the value of the foreign money in terms of the money of one's own country (or vice versa).

EX-CIR′CLE, *n.* **excircle of a triangle.** See ESCRIBED—escribed circle of a triangle.

EX-CLU′SIVE, *adj.* **mutually exclusive events.** See EVENT.

EX′ER-CISE, *n.* A problem which is designed primarily for drill on the use of formulas, theorems, or mathematical concepts. *Syn.* Problem.

EX′IS-TEN′TIAL, *adj.* **existential quantifier.** See QUANTIFIER.

EX-PAN′SION, *n.* (1) The form a quantity takes when written as a sum of terms, or as a continued product, or in general in any type of expanded (extended) form. (2) The act, or process, of obtaining the expanded form of a quantity. (3) Increase in size.

binomial expansion. The expansion given by the binomial theorem. See BINOMIAL—binomial theorem.

coefficient of linear expansion, thermal expansion, and volume (or cubical) expansion. See COEFFICIENT.

expansion of a determinant. See DETERMINANT—expansion of a determinant.

expansion (of a function) in a series. Writing a series which converges to the function for certain values of the variables (or which "represents" the function in some other sense). The series itself is also spoken of as the *expansion* of the function.

EX-PEC-TA'TION, *n.* **expectation of life.** The average number of years that members of a given group may be expected to live after attaining a certain age, according to a mortality table. Also called **complete expectation of life** as distinguished from **curtate expectation of life,** which is the average number of *entire years* that members of a given group may be expected to live. **Joint expectation of life** is the average number of years that two (or more) persons at a given age may both (all) be expected to live, according to a mortality table.

mathematical expectation. (1) The amount of money that an individual may expect to receive if a given sum is offered him on the condition that a given event (or events) occurs. It is equal to the product of the sum offered and the probability of the occurrence of the event; if a man is to receive one dollar for tails-up throw of a coin, his expectation is $\$1 \times \frac{1}{2}$, or 50 cents. (2) (*Statistics.*) Let $f(x)$ be the relative frequency function (probability density function) of the variable x. Then $E(x) = \bar{x} = \int_a^b xf(x)\,dx$ is the expectation of the variable x over the range a to b, or more usually, $-\infty$ to ∞. The quantity

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 f(x)\,dx$$

is the variance of x, or the expectation of the squared deviations of x around its mathematical expectation. The mathematical expectation of the product xy of two *independent* variables x and y is equal to the product of their expectations. In general, let $\phi(x)$ be a function of the variable x whose frequency function is $f(x)$. Then $E(\phi(x)) = \int_a^b \phi(x)f(x)\,dx$ is the expectation of $\phi(x)$ over the range a to b, usually $-\infty$ to ∞. The arithmetic mean of a variable is the expectation of the variable. The mathematical expectation of a sum of several variables is equal to the sum of their expectations.

EX-PENS'ES, *n.* **overhead expenses.** Administrative expenses, such as salaries of officers and employees, cost of supplies, losses by credit, and rent and plant depreciation. (It sometimes includes some of the selling expenses.)

selling expenses. Expenses such as insurance, taxes, advertising, and salesmen's wages.

EX-PLIC'IT, *adj.* **explicit function.** See IMPLICIT—implicit function.

EX-PO'NENT, *n.* A number placed at the right of and above a symbol. The value assigned to the symbol with this exponent is called a **power** of the symbol, although *power* is sometimes used in the same sense as *exponent.* If the exponent is a positive integer, it indicates that the symbol is to be taken as a factor as many times as there are units in this integer. *E.g.*, $3^2 = 3 \times 3 = 9$ (the second power of 3 is 9); $x^3 = x \times x \times x$. If x is a nonzero number, the value of x^0 is defined to be 1 (if $x \neq 0$, x^0 can be thought of as the result of subtracting exponents when dividing a quantity by itself, $x^2/x^2 = x^0 = 1$). A negative exponent indicates that in addition to the operations indicated by the numerical value of the exponent, the quantity is to be reciprocated. Whether the reciprocating is done before or after the other exponential operations have been carried out is immaterial. *E.g.*, $3^{-2} = (3^2)^{-1} = (9)^{-1} = \frac{1}{9}$, or $3^{-2} = (3^{-1})^2 = (\frac{1}{3})^2 = \frac{1}{9}$. The following laws of exponents are valid when m and n are any integers (positive, negative, or zero):

(1) $a^n a^m = a^{n+m}$; (2) $a^m/a^n = a^{m-n}$;

(3) $(a^m)^n = a^{mn}$; (4) $(ab)^n = a^n b^n$;

(5) $(a/b)^n = a^n/b^n$.

If the exponent on a symbol x is a fraction p/q, then $x^{p/q}$ is defined as $(x^{1/q})^p$, where $x^{1/q}$ is the positive qth root of x if x is positive, and the (negative) qth root if x is negative and q is odd. It follows that

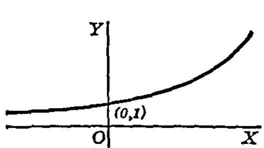
$x^{p/q}=(x^p)^{1/q}$ and that the above five laws are valid if m and n are either fractions or integers (*i.e.* **rational numbers**), provided a and b are positive numbers. If the exponent is **irrational**, the power is defined to be the quantity approximated by using rational exponents which approximate the irrational exponent; *e.g.*, 3 with exponent $\sqrt{2}$ denotes the limit of the sequence $3^{1.4}$, $3^{1.41}$, $3^{1.414}$, $\cdots$. *Tech.* If the sequence $a_1, a_2, \cdots, a_n, \cdots$ of rational numbers approaches an irrational limit a, then c^a denotes the limit of the sequence $c^{a_1}, c^{a_2}, c^{a_3}, \cdots, c^{a_n}, \cdots$. The above laws of exponents are valid for m and n any real numbers (rational or irrational) if a and b are positive. If x is a complex number, then x^m is defined to be $e^{m(\log x)}$, where this is computed by substituting $m(\log x)$ for t in the Taylor's series

$$e^t=1+t+t^2/2!+t^3/3!+t^4/4!+\cdots$$

(see EXPONENTIAL—exponential series, and LOGARITHM—logarithm of a complex number). With this definition, x^m is multiple valued and the laws of exponents are valid only in the sense that, if one member of one of the equations is computed, then this is one of the values of the other member. *E.g.*, $(2/-3)^{1/2}=(-\frac{2}{3})^{1/2}=i\sqrt{\frac{2}{3}}$ is not equal to $2^{1/2}/(-3)^{1/2}=\sqrt{2}/(i\sqrt{3})=-i\sqrt{\frac{2}{3}}$ if $(-3)^{1/2}$ is taken as $i\sqrt{3}$, but these are equal if one takes $(-3)^{1/2}$ to be $-i\sqrt{3}$. See DE MOIVRE'S THEOREM.

EX'PO-NEN'TIAL, *adj.* **derivative of an exponential.** See DIFFERENTIATION FORMULAS in the appendix.

exponential curve. The plane locus of $y=a^x$ (or, what is the same, $x=\log_a y$). It can be obtained geometrically by revolving the logarithmic curve, $y=\log_a x$, about



the line $y=x$, *i.e.*, reflecting it in this line. The curve is asymptotic to the negative x-axis, when $a>1$ as in the figure, and has its y intercept unity.

exponential equation. See EQUATION—exponential equation.

exponential function. See FUNCTION—exponential function.

exponential series. The series

$$1+x+x^2/2!+x^3/3!+\cdots+x^n/n!+\cdots.$$

This is the Maclaurin expansion of e^x. It converges to e^x for every value of x.

exponential values of sin x and cos x.

$$\sin x=\frac{e^{ix}-e^{-ix}}{2i}$$

and

$$\cos x=\frac{e^{ix}+e^{-ix}}{2},$$

where $i^2=-1$. These can be proved by use of EULER'S FORMULA.

EX-PRES'SION, *n.* A very general term used to designate any symbolic mathematical form, such, for instance, as a polynomial.

EX-SE'CANT, *n.* See TRIGONOMETRIC—trigonometric functions.

EX-TEND'ED, *adj.* **extended mean value theorem.** (1) Same as TAYLOR'S THEOREM. (2) Same as the SECOND MEAN VALUE THEOREM. See MEAN—mean value theorems for derivatives.

extended real-number system. The real-number system together with the symbols $+\infty$ (or ∞) and $-\infty$. The following definitions are used: $-\infty<a<+\infty$ if a is a real number, $a+\infty=\infty+a=\infty$ if $a\neq-\infty$, $a+(-\infty)=(-\infty)+a=-\infty$ if $a\neq+\infty$, $a\cdot\infty=\infty\cdot a=\infty$ and $a(-\infty)=(-\infty)a=-\infty$ if $0<a\leqq+\infty$, $a\cdot\infty=\infty\cdot a=-\infty$ and $a(-\infty)=(-\infty)a=+\infty$ if $-\infty\leqq a<0$, $a/\infty=a/(-\infty)=0$ if a is a real number.

EX-TE'RI-OR, *adj., n.* **alternate exterior angles.** See ALTERNATE.

exterior angle of a polygon. The angle between any side produced and the adjacent side (not produced).

exterior angle of a triangle. The angle between one side produced and the adjacent side (not produced).

exterior content. See CONTENT—content of a set of points.

exterior-interior angles. See ANGLE—angles made by a transversal.

exterior measure. See MEASURE—exterior measure.

exterior of a set. The exterior of a set E is the set of all points which have a neighborhood having no points in common with E. Same as the *interior* of the complement of E. Each such point is called an exterior point of E.

EX-TER'NAL, *adj.* **external ratio.** See POINT—point of division.

EX-TRACT', *v.* **extract a root of a number.** To find a root of the number; usually refers to finding the positive real root, or the real negative root if it be an odd root of a negative number. *E.g.*, one extracts the square root of 2 when it is found to be $1.4142\cdots$, or the cube root of -8 when it is found to be -2. *Syn.* Find, compute, or estimate a root.

EX-TRA'NE-OUS, *adj.* **extraneous root.** A number obtained in the process of solving an equation, which is not a root of the equation given to be solved. It is generally introduced either by squaring the original equation, or clearing it of fractions. *E.g.*, (1) the equation $(x^2-3x+2)/(x-2)=0$ has only one root, 1; but if one multiplies through by $x-2$ the resulting equation has the root 2 also; (2) the equation $1-\sqrt{x-1}=x$ has only one root, 1; but if one transposes and squares, thus getting rid of the radical, the resulting equation is $x^2-3x+2=0$, which has the two roots 1 and 2. The root, 2, is an extraneous root of the original equation, since its substitution in that equation gives $1-1=2$.

EX'TRA-PO-LA'TION, *n.* Estimating (approximating) the value of a function (quantity) for a value of the argument which is either greater than, or less than, all the values of the argument which are being used in the estimating (approximating). Using log 2 and log 3 one might find an approximate value of log 3.1 by extrapolation, using the formula

$$\log 3.1 = \log 3 + \tfrac{1}{10}(\log 3 - \log 2).$$

See INTERPOLATION.

EX-TREME', *adj.* **extreme terms, or extremes.** The first and last terms in a proportion; the antecedent in the first ratio, and the consequent in the second.

extreme or extremum of a function. A *maximum* or *minimum* value of the function. See MAXIMUM.

F

F, *n.* **F distribution.** See DISTRIBUTION.

F_σ set. See BOREL—Borel set.

FACE, *n.* **face amount of an insurance policy.** The amount that the company is contracted to pay when the exigency stated in the policy occurs.

face of a polyhedral angle. One of the planes that form the polyhedral angle.

face of a polyhedron. See POLYHEDRON.

face value. See PAR.

lateral face of a pyramid. See PYRAMID.

FAC'TOR, *adj., n., v.* As a *verb*, to resolve into factors. One factors 6 when he writes it in the form 2×3.

accumulation factor. See ACCUMULATION.

converse of the factor theorem. If $(x-a)$ is a factor of the polynomial $f(x)$, then a is a root of the equation $f(x)=0$.

factor of an algebraic polynomial. One of two or more polynomials whose product is the given polynomial. Usually, in elementary algebra, a polynomial with rational coefficients is considered factorable if it has two or more nonconstant polynomial factors whose coefficients are rational (sometimes it is required that the coefficients be integers). *Tech.* One of a set of polynomials whose product gives the polynomial to be factored and whose coefficients lie in a given *field* (*domain*). Unless a field is specified, the field of the coefficients of the given polynomial is understood. *Factor* is sometimes used of any quantity whatever that divides a given quantity. *E.g.*, (x^2-y^2) has the factors $(x-y)$ and $(x+y)$ in the ordinary (elementary) sense; (x^2-2y^2) has the factors $(x-\sqrt{2}y)$ and $(x+\sqrt{2}y)$ in the field of real numbers; (x^2+y^2) has the factors $(x-iy)$ and $(x+iy)$ in the complex field. See FACTORING—type forms for factoring.

factor analysis. (*Statistics.*) Let x_i be joint measurements of n variables ($i=1, \cdots, n$), and let

$$x_i = a_{i1}u_1 + \cdots + a_{im}u_m + e_i,$$

where $n>m$, the $u_1, \cdots, u_m$ and e_i are all uncorrelated random variables, and the rank of the matrix of the a_{ij} is m. This constitutes a transformation of the variables x_i into m uncorrelated variables u_j plus n error terms e_i, the u_j being called **factors** of the variables x_i. Problems incident to this method involve the estimation of the a_{ij} on the basis of observations of the variables x_i, and the existence of meaningful interpretations that may be assigned to the u_j factors. *E.g.*, nr scores on n different psychological tests may be obtained from a set of r persons. These scores may be related to a set of *factors, e.g.*, verbal facility, arithmetic ability, and form recognition, which are the psychological interpretations given to the factors u_j.

factor of an integer. An integer whose product with some integer is the given integer. *E.g.*, the factors of 12 are ± 12, ± 2, ± 3, ± 4, ± 6, ± 1. When speaking of the *factors of a number*, it is not usually intended that unity and the number itself should be included.

factor of proportionality. The constant value of the ratio of two proportional quantities. This relation is usually written in the form $y=kx$, where k is the *factor of proportionality*. *E.g.*, the distance passed over is proportional to the time, when the velocity is constant; *i.e.*, $s=kt$, where k is the *factor of proportionality*. *Syn.* Constant of proportionality.

factor of a term. Any exact divisor of the term, it being required that the quotient be of a specified type. *E.g.*, $x+1$ is a factor of $3x(x+1)$, but not of $3x$, if the quotient must be a polynomial.

factor modulo p. If $r(x)\equiv 0 \pmod p$ in the congruence $f(x)\equiv g(x)\cdot d(x)+r(x) \pmod p$, then $d(x)$ is said to be a *factor modulo p* of $f(x)$.

factor space (group, ring, vector space, etc.). See QUOTIENT—quotient space.

factor theorem. An algebraic polynomial in x is exactly divisible by $(x-a)$ if it reduces to zero when a is substituted for x. See REMAINDER—remainder theorem, and above, converse of the factor theorem.

integrating factor. (*In differential equations.*) A factor which, when multiplied into a differential equation, with right-hand member zero, makes the left-hand member an exact differential, or makes it an exact derivative. *E.g.*, if the differential equation

$$\frac{dy}{x} + \frac{y}{x^2}\,dx = 0$$

is multiplied by x^2, there results

$$x\,dy + y\,dx = 0$$

which has the solution $xy=c$. The differential equation

$$xy'' + (3-x^3)y' - 5x^2y + 4x = 0$$

has the integrating factor x^2; when multiplied by x^2 the equation becomes

$$\frac{d}{dx}(x^3y' - x^5y + x^4) = 0.$$

See ADJOINT—adjoint of a differential equation.

monomial factor. See MONOMIAL.

FAC'TOR-A-BLE, *adj.* *In arithmetic*, containing factors other than unity and itself (referring to integers). *In algebra*, containing factors other than constants and itself (referring to polynomials); x^2-y^2 is factorable in the domain of real numbers, while x^2+y^2 is not.

FAC-TO'RI-AL, *adj.*, *n.* **factorial of a positive integer.** The product of all the positive integers less than or equal to the integer. *Factorial n* is denoted by either one of the symbols $n!$ or $\underline{|n}$. *E.g.*, $1!=1$, $2!=1\cdot 2$, $3!=1\cdot 2\cdot 3$, and in general, $n! = 1\cdot 2\cdot 3\cdots n$. This definition of factorial leaves the case when n is zero meaningless. In order to make certain formulas valid in all cases, **factorial zero** is arbitrarily defined to be unity. Despite the fact that this is the value of factorial 1, there is considerable to be gained by using this definition; *e.g.*, this makes the general binomial coefficient, $n!/[r!(n-r)!]$, valid for the first and last terms, which are the terms for which $r=0$ and $r=n$, respectively.

factorial notation. The notation $n!$ or $\underline{|n}$ used in writing the factorials of a positive integer or zero. See above, factorial of a positive integer.

factorial series. See SERIES—factorial series.

FAC'TOR-ING, *p.* **type forms for factoring:**

(1) $x^2+xy=x(x+y)$;

(2) $x^2-y^2=(x+y)(x-y)$;

(3) $x^2+2xy+y^2=(x+y)^2$;

(4) $x^2-2xy+y^2=(x-y)^2$;

(5) $x^2+(a+b)x+ab=(x+a)(x+b)$;

(6) $acx^2+(bc+ad)x+bd$
$\qquad =(ax+b)(cx+d)$;

(7) $x^3\pm 3x^2y+3xy^2\pm y^3=(x\pm y)^3$;

(8) $x^3\pm y^3=(x\pm y)(x^2\mp xy+y^2)$.

(In (7) and (8) either the upper, or the lower, signs are to be used throughout.)

FAC-TO-RI-ZA'TION, *n.* The process of factoring.

factorization of a transformation. Finding two or more transformations which, when made successively, have the same effect as the given transformation. For an example, see AFFINE—affine transformation.

unique factorization theorem. The theorem states that any integer N can be expressed as a product of one and only one set of primes.

FAH'REN-HEIT, *adj.* **Fahrenheit thermometer.** A thermometer so graduated that water freezes at 32° and boils at 212°. See CONVERSION—conversion from centigrade to Fahrenheit.

FALSE, *adj.* **method of false position.** (1) Same as REGULA FALSI. (2) A method for approximating the roots of an algebraic equation. Consists of making a fairly close estimate, say r, then substituting $(r+h)$ in the equation, dropping the terms in h of higher degree than the first (since they are relatively small), and solving the resulting linear equation for h. This process is then repeated, using the new approximation $(r+h)$ in place of r. *E.g.*, the equation $x^3-2x^2-x+1=0$ has a root near 2 (between 2 and 3). Hence we substitute $(2+h)$ for x. This gives (when the terms in h^2 and h^3 have been dropped) the equation $3h-1=0$; whence $h=\frac{1}{3}$. The next estimate will then be $2+\frac{1}{3}$ or $\frac{7}{3}$.

FALTUNG, *n.* German for CONVOLUTION.

FAM'I-LY, *n.* **family of curves.** A set of curves whose equations can be obtained from a given equation by varying n essential constants which occur in the given equation is an n-parameter family of curves; *e.g.*, a set of curves whose equations are nonsingular solutions (special cases of the general solution) of a differential equation of order n. A set of concentric circles constitutes a one-parameter family of curves, the radius being the arbitrary parameter. The set of circles in the plane having a given radius is a two-parameter family of curves, the two coordinates of the center being the parameters. All the circles in a plane constitute a three-parameter family, and all the conics in a plane a five-parameter family. The set of all lines tangent to a given circle is a one-parameter family of lines, while the set of all lines in the plane is a two-parameter family.

family of surfaces. A set of surfaces whose equations can be obtained from a given equation by varying n essential constants which occur in the given equation is an n-parameter family of surfaces. The set of all spheres with a given center is a one-parameter family, while the set of all spheres is a four-parameter family.

FAREY. Farey sequence. The Farey sequence of order n is the increasing sequence of all fractions p/q for which $0\leqq p/q\leqq 1$, $q\leqq n$, and p and q are nonnegative integers with no common divisors other than 1. *E.g.*, the Farey sequence of order 5 is

$$\tfrac{0}{1},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{1}{2},\ \tfrac{3}{5},\ \tfrac{2}{3},\ \tfrac{3}{4},\ \tfrac{4}{5},\ \tfrac{1}{1}.$$

If a/b, c/d, e/f are three consecutive terms of a Farey sequence, then $bc-ad=1$ and $c/d=(a+e)/(b+f)$.

FÉJER. Féjer's theorem. Let $f(x)$ be a function of the real variable x, defined arbitrarily when $-\pi<x\leq\pi$, and defined by $f(x+2\pi)=f(x)$ for all other values of x. By Fejér's theorem is meant one of the following: (1) If $\int_{-\pi}^{\pi} f(x)\,dx$ exists, and, if this is an improper integral, $\int_{-\pi}^{\pi} |f(x)|\,dx$ exists, then the *Fourier series* associated with $f(x)$ is *summable* (C1) at all points x for which the limits from the right and left, $f(x+0)$ and $f(x-0)$, exist, and its sum

(C1) is $\frac{1}{2}\{f(x+0)+f(x-0)\}$. (2) If, in addition, $f(x)$ is continuous everywhere in an interval (a, b), then the *first Cesáro sums* converge uniformly to $f(x)$ in any interval (α, β) with $a<\alpha<\beta<b$. Both of these theorems were published by Fejér in 1904. See CESÁRO'S SUMMATION FORMULA.

FERMAT. Fermat numbers. Numbers of the type $F_n=2^{(2^n)}+1$, $n=1, 2, 3, \cdots$. Fermat thought that these numbers might all be primes. Actually, F_5 is not a prime:

$$F_5=(641)(6{,}700{,}417)=4{,}294{,}967{,}297.$$

Fermat's last theorem. The equation $x^n+y^n=z^n$, where n is an integer greater than 2, has no solution in positive integers. This theorem has never been proved, although it has been proved that it cannot have a solution, in the so-called first case in which neither x, y, nor z has a factor in common with n, unless n is greater than 253,547,889 and the least of x, y, and z is greater than $\frac{111}{77}n\ (2n^2+1)^n$; and that it cannot have a solution in the second case, namely when either x, y, or z has a factor in common with n, unless n is greater than 600.

Fermat's principle. The principle that a ray of light requires less time along its actual path than it would along any other path having the same end points. This principle was used by John Bernoulli in solving the brachistochrone problem. See BRACHISTOCHRONE.

Fermat's spiral. See PARABOLIC—parabolic spiral.

Fermat's theorem. (*Number Theory*). If p is a prime and a is prime to p, then a^{p-1} divided by p leaves a remainder of unity. *Tech.* $a^{p-1}\equiv 1 \pmod p$; *e.g.*, $2^4\equiv 1 \pmod 5$, where $p=5$ and $a=2$. See CONGRUENCE.

FERRARI'S solution of the quartic. The solution of the quartic equation

$$x^4+px^3+qx^2+rx+s=0$$

by showing that the roots of this equation are also the roots of the two equations

$$x^2+\tfrac{1}{2}px+k=\pm(ax+b),$$

where k is obtained from the following cubic equation (the RESOLVENT CUBIC):

$$k^3-\tfrac{1}{2}qk^2+\tfrac{1}{4}(pr-4s)k+\tfrac{1}{8}(4qs-p^2s-r^2)=0,$$

where

$$a=(2k+\tfrac{1}{4}p^2-q)^{1/2} \text{ and } b=(kp-r)/(2a).$$

FIBONACCI SEQUENCE. The sequence of numbers 1, 1, 2, 3, 5, 8, 13, 21, $\cdots$, each of which is the sum of the two previous numbers (these numbers are also called **Fibonacci numbers**). The ratio of one Fibonacci number to the preceding one is a *convergent* of the continued fraction

$$1+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\cdots,$$

which satisfies the equation $x=1+1/x$ and is equal to $\frac{1}{2}(\sqrt{5}+1)$. Thus the sequence $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \cdots$ has the limit $\frac{1}{2}(\sqrt{5}+1)$. See FAREY SEQUENCE.

FIELD, *n.* A set of elements a, b, $\cdots$ for which two operations, called *addition* and *multiplication*, are defined and have the properties: (i) the elements form an *Abelian group* with addition as the group operation; (ii) the elements, with the identity (0) of the additive group omitted, form an Abelian group with multiplication as the group operation; (iii) $a(b+c)=ab+ac$ for all a, b, and c in the set. *Syn.* Domain. See below, number field, for examples of fields. Also see DOMAIN—integral domain.

field of force. See FORCE.

field of study. A group of subjects that deal with closely related material, such as the field of analysis, the field of pure mathematics, or the field of applied mathematics.

field plan. (*Statistics.*) The spatial arrangement of experimental trials when the repetitions of different factors must be located at different points in space, *e.g.*, a *Latin square*, or a *randomized block experiment* in agricultural experiments.

number field. Any set of real or complex numbers such that the sum, difference, product, and quotient (except by 0) of any two members of the set is in the set. *Syn.* Number domain. The set of all rational numbers, and of all numbers of the form $a+b\sqrt{2}$ with a and b rational, are number fields. A number field is necessarily a *field*. A number field F can be enlarged by **adjoining** a number z to it, the new field consisting of all numbers which can be derived from z and the numbers of F by the operations of addition, subtraction, multiplication, and division.

ordered field. A field that contains a set of "positive" elements satisfying the conditions: (1) the sum and product of two

positive elements is positive; (2) for a given element x, one and only one of the following possibilities is valid: (a) x is positive, (b) $x=0$, (c) $-x$ is positive. An ordered field is called **complete** if every nonempty subset has a least upper bound if it has an upper bound. The real numbers form a complete ordered field.

tensor field. See TENSOR.

FIG'URE, *n.* (1) A character or symbol denoting a number, as 1, 5, 12; sometimes used in the same sense as digit. (2) A drawing, diagram or cut used to aid in presenting subject matter in text books and scientific papers.

geometric figure. See GEOMETRIC.

plane figure. See PLANE.

FIL'TER, *n.* A family F of nonempty subsets of a set X, for which the intersection of any two members of F is a member of F and for which any subset B of X which contains a member of F is itself a member of F. An **ultrafilter** is a filter which is not a proper subset of any filter (if F is an ultrafilter and A is a subset of X, then either A or the complement of A is a member of F). If X is a topological space, a filter F is said to converge to a point x if and only if each neighborhood of x is a member of F. Filters and *Moore-Smith convergence* lead to essentially equivalent theories.

FI'NITE, *adj.* **finite character.** See CHARACTER.

finite differences. See DIFFERENCE—finite differences.

finite discontinuity. See DISCONTINUITY.

finite quantity. (1) Any quantity which is *bounded*; *e.g.*, a function is *finite* on an interval if it is *bounded* on the interval. (2) A real or complex number may be said to be finite to distinguish it from the ideal number $+\infty$, $-\infty$, ∞. See COMPLEX—complex plane, and EXTENDED—extended real-number system.

finite set (group or assemblage). A set which contains a finite (limited) number of members; a set which has, for some integer n, just n members. *Tech.* A set which cannot be put into one-to-one correspondence with a part of itself. All the integers between 0 and 100 constitute a finite set. See INFINITE—infinite set.

locally finite family of sets. A family of subsets of a topological space T is *locally finite* if each point of T has a neighborhood which intersects only a finite number of these subsets.

FISHER. **Fisher's z.** The correlation-coefficient transformation

$$z=\tfrac{1}{2}\log_e\frac{1+r}{1-r},$$

where r is the correlation coefficient in a random sample from a normal bivariate population with a correlation of ρ. The sampling distribution of z is approximately normal, if the sample sizes are not small, with mean approximately z_ρ and variance $n-3$, where n is the size of the sample.

Fisher's z distribution. $z=\frac{1}{2}\log s_1^2/s_2^2$, where s_i^2 are two independent estimates, from random samples, of the variance of a normal population. See DISTRIBUTION—F distribution, and VARIANCE—variance ratio transformation.

FIT'TING, *n.* **curve fitting.** See EMPIRICAL—empirical curve, and METHOD—method of least squares.

FIXED, *adj.* **Brouwer's fixed-point theorem.** See BROUWER.

fixed assets. See ASSETS.

fixed investment. See INVESTMENT.

fixed point. A point which is not moved by a given transformation or mapping. *E.g.*, $x=3$ is a fixed point of the transformation $T(x)=4x-9$.

fixed value of a letter or quantity. A value that does not change during a given discussion or series of discussions; not arbitrary. In an expression containing several letters, some may be *fixed* and others subjected to being assigned certain values or to taking on certain values by virtue of their place in the expression; *e.g.*, if in $y=mx+b$, b is *fixed*, m arbitrary and x and y variables, the equation represents the pencil of lines through the point whose coordinates are $(0, b)$. When m has also been assigned a particular value, x and y are then thought of as taking on all pairs of values which are coordinates of points on a particular line.

Poincaré-Birkhoff fixed-point theorem. See POINCARÉ.

FLAT, *adj.* **flat angle.** Same as *straight angle*, that is, an angle of 180°.

flat price of a bond. See PRICE.

FLEC'NODE, *n.* A *node* which is also a point of inflection on one of the branches of the curve that touch each other at the node.

FLEX, *n.* Same as INFLECTION.

FLEX'ION, *n.* A name sometimes used for the rate of change of the slope of a curve; the second derivative of a function.

FLOAT'ING, *p.* **floating decimal point.** A term applied in machine computation when the decimal point is not fixed at a certain machine position throughout the computation but is placed by the machine as each operation is performed.

FLOW, *adj.* **flow chart.** See CHART.

FLUC'TU-A'TION, *n.* Same as VARIATION.

FLU'ENT, *n.* Newton's name for a varying, or "*flowing*," function.

FLU'ID, *adj., n.* **fluid pressure.** See PRESSURE—fluid pressure.

mechanics of fluids. See MECHANICS.

FLUX'ION, *n.* Newton's name for the rate of change or derivative of a "*fluent*." It is denoted in Newton's writing by a letter with a dot over it.

FO'CAL, *adj.* **focal chords of a conic.** Chords passing through a focus of the conic.

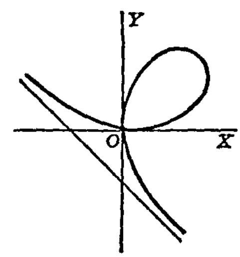
focal point. (*Calculus of Variations*) For an integral $I=\int_{x_1}^{x_2} f(x, y, y')\,dx$ and a curve C, the *focal point* of C on the *transversal* T is the point of contact of T with the envelope of transversals of C. In order for an arc $[(x_1, y_1), (x_2, y_2)]$ of T, with (x_2, y_2) on C, to minimize I, the focal point of C on T must not lie between (x_1, y_1) and (x_2, y_2) on T. See TRANSVERSALITY—transversality condition.

focal property of conics. See ELLIPSE—focal property of ellipse, HYPERBOLA—focal property of hyperbola, and PARABOLA—focal property of parabola.

focal radius of a conic. A line segment joining a focus to any point on the conic.

FO'CUS, *n.* [*pl.* **foci**] **focus of a parabola, hyperbola, or ellipse.** See PARABOLA, HYPERBOLA, and ELLIPSE.

FO'LI-UM, *n.* **folium of Descartes.** A plane cubic curve consisting of a single loop, a node, and two branches asymptotic to the same line. Its rectangular Cartesian equation is $x^3+y^3=3axy$, where the curve passes through the origin and is asymptotic to the line $x+y+a=0$.



FOOT, *n.* (1) A unit of linear measure, equal to 12 inches. See DENOMINATE NUMBERS in the appendix. (2) The point of intersection of a line with another line or a plane. In particular, the *foot of a perpendicular to a line* is the point of intersection of the line and the perpendicular; the *foot of a perpendicular to a plane* is the point in which the perpendicular cuts the plane.

foot-pound. A unit of work; the work done when a body weighing one pound is lifted one foot. More precisely, the work done when an average force of one pound produces a displacement of one foot in the direction of the force. See HORSEPOWER.

FORCE, *n.* That which pushes, pulls, compresses, distends, or distorts in any way; that which changes the state of rest or state of motion of a body. *Tech.* The time rate of change of momentum of a body. If the mass of the body does not change with time, the force **F** is proportional to the product of the mass m by its vector acceleration **a**. See NEWTON—Newton's laws of motion (2).

centrifugal force. See CENTRIFUGAL.

centripetal force. See CENTRIPETAL.

conservative force. See CONSERVATIVE.

electromotive force. See ELECTROMOTIVE.

field of force. A region of space endowed with the property that a physical object of the proper sort will experience a force acting on it if placed at any point of the region. *E.g.*, if a stationary electric charge would experience a force if placed at any point of a certain region, we should speak of the region as bearing an *electrostatic field.* Similarly, if it is a point-mass or isolated magnetic pole that is acted on, then we have a *gravitational field* or a *magnetic field*, as the case may be.

force function. A function $u(x, y, z)$ such that $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial z$ are equal, respectively, to the components X, Y, Z of a force vector $Xi+Yj+Zk=F$. Such a function is also called a POTENTIAL.

force of interest. See INTEREST.

force of mortality. See MORTALITY.

force vector. A vector equal in numerical length to a given force and having its direction parallel to the line of action of the force. See PARALLELOGRAM—parallelogram of forces.

moment of a force. See MOMENT.

parallelogram of forces. See PARALLELOGRAM—parallelogram of forces.

projection of a force. See PROJECTION—orthogonal projection.

tube of force. A tube whose boundary surface is made up of lines of force. In general, if C is a closed curve, no part of which is a line of force, and if through each point of C there passes a line of force, then the collection of these lines will make up the boundary of a *tube of force.*

unit of force. The force which will give unit acceleration to a unit mass. The force which, acting on a mass of one gram for one second, increases its velocity by one centimeter per second is called a force of one *dyne.* The force which, acting on a mass of one pound for one second, will increase the velocity of the mass one foot per second is called a force of one *poundal.*

FORCED, *p.* **forced oscillations and vibrations.** See OSCILLATION.

FOR'EIGN, *adj.* **foreign exchange.** See EXCHANGE.

FORM, *n.* (1) A mathematical expression of a certain type; see STANDARD—standard form of an equation. (2) A homogeneous polynomial expression in two or more variables. In particular, a **bilinear form** is a polynomial of the second degree which is homogeneous of the first degree in variables $x_1, x_2, \cdots, x_n$ and in variables $y_1, y_2, \cdots, y_n$; a polynomial of the form

$$P(x, y)=\sum_{i,j=1}^{n} a_{ij}x_i y_j.$$

If $x_1, \cdots, x_n;\ y_1, \cdots, y_n;\ \cdots;\ z_1, \cdots, z_n$ are m sets of variables, then an expression of the form $\sum a_{ij\ldots k}x_i y_j \cdots z_k$ is called a **multilinear form** of order m (the form is **linear, bilinear, trilinear,** etc., according as $m=1, 2, 3,$ etc.). A **quadratic form** is a homogeneous polynomial of the second degree; a polynomial of the form $\sum_{i,j=1}^{n} a_{ij}x_i x_j$. If it is positive for all real values (not all zero) of the variables $\{x_i\}$ it is called a **positive definite quadratic form**; if it is positive or zero it is **positive semidefinite.** See DISCRIMINANT—discriminant of a quadratic form, and TRANSFORMATION—congruent transformation.

standard form. See STANDARD—standard form of an equation.

FOR'MU-LA, *n.* A general answer, rule, or principle stated in mathematical language.

empirical formula. See EMPIRICAL.

formulas of integration. See INTEGRATION, and the appendix.

formulas (identities) of trigonometry. See various headings under SPHERICAL, and TRIGONOMETRY.

FOUR, *adj., n.* **four-color problem.** The determination of whether any plane map can be colored with four colors so that no two countries having a common boundary line will have the same color. It is known that 5 colors suffice, and 3 colors have been proved insufficient. It is assumed that each country is connected, *i.e.*, that it is possible to go between any two points of a given country without leaving that country.

four-step rule (or method). A rule for finding the derivative of a function: (1) add an increment Δx to x in the function,

giving $f(x+\Delta x)$; (2) subtract the function, giving $f(x+\Delta x)-f(x)$; (3) divide by Δx, obtaining

$$\frac{f(x+\Delta x)-f(x)}{\Delta x},$$

and simplify (*e.g.*, by expanding the numerator and canceling out Δx); (4) find the limit as Δx approaches 0 (sometimes by putting $\Delta x=0$). This is the mechanization of the definition of the derivative. It is used on simple examples to fix in one's mind the concept of a derivative, and then to derive general formulas for taking the derivative. If $f(x)=x^2$, the process gives:

(1) $f(x+\Delta x)=(x+\Delta x)^2$;

(2) $f(x+\Delta x)-f(x)=(x+\Delta x)^2-x^2$;

(3) $$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{(x+\Delta x)^2-x^2}{\Delta x}=2x+\Delta x;$$

(4) $$\lim_{\Delta x\to 0}(2x+\Delta x)=2x=\frac{dx^2}{dx}.$$

FOURIER. Fourier series. A series of the form

$$\tfrac{1}{2}a_0+(a_1\cos x+b_1\sin x)+(a_2\cos 2x+b_2\sin 2x)+\cdots=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

where there exists $f(x)$ such that for all n

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx,$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

and

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx.$$

The marked characteristic of a Fourier Series is that it can be used to represent functions that ordinarily are represented by different expressions in different parts of the interval, the functions being subject only to certain very general restrictions (see below, Fourier's theorem). Since the sine and cosine each have a period of 2π, the Fourier Series has a period of 2π. *E.g.*, if $f(x)$ is defined by the relations $f(x)=1$ when $-\pi\leqq x\leqq 0$, $f(x)=2$ when $0<x\leqq\pi$, then

$$\pi a_0=\int_{-\pi}^{\pi}f(x)\,dx=\int_{-\pi}^{0}dx+\int_{0}^{\pi}2\,dx=3\pi,$$

from which $a_0=3$. Similarly, $a_n=0$ for all n, $b_n=0$ for n even and $b_n=2/(n\pi)$ for n odd. Whence

$$f(x)=\tfrac{3}{2}+(2/\pi)\sin x+[2/(3\pi)]\sin 3x+[2/(5\pi)]\sin 5x+\cdots.$$

Series on ranges other than $(-\pi,\pi)$, derived from the above type of Fourier's Series, are also called FOURIER SERIES. See ORTHOGONAL—orthogonal functions.

Fourier transform. A function $f(x)$ is the *Fourier transform of* $g(x)$ if

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(t)e^{itx}\,dt$$

(the factor $1/\sqrt{2\pi}$ is sometimes omitted). Under suitable conditions on $g(x)$ (*e.g.*, those given below for Fourier's integral theorem), it then follows that

$$g(x)\approx\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-itx}\,dt,$$

where the value of the right member is

$$\lim_{h\to 0}\tfrac{1}{2}[g(x+h)+g(x-h)]$$

if $g(x)$ is of *bounded variation* in the neighborhood of x. Such functions $f(x)$ and $g(x)$ are sometimes said to be a *pair of Fourier transforms.* A function $f(x)$ is the *Fourier cosine transform* of $g(x)$, or the *Fourier sine transform* of $g(x)$, according as

$$f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}g(t)\cos xt\,dt,$$

or

$$f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}g(t)\sin xt\,dt.$$

These transformations of $g(x)$ are inverses of themselves.

Fourier's half-range series. A *Fourier Series* of the form

$$\tfrac{1}{2}a_0+a_1\cos x+a_2\cos 2x+\cdots=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx,$$

or

$$b_1\sin x+b_2\sin 2x+\cdots=\sum_{n=1}^{\infty}b_n\sin nx.$$

These are also called the **cosine** and **sine** series, respectively. Since the cosine is an *even* function, the cosine series can represent a function, $f(x)$, on the whole interval $-\pi<x<\pi$ only if $f(x)$ is an even function,

i.e., $f(-x)=f(x)$. Likewise, since the sine is an *odd* function, the sine series can represent a function, $f(x)$, on the whole interval $-\pi<x<\pi$ only if $f(x)$ is an odd function, *i.e.*, $f(-x)=-f(x)$.

Fourier's integral theorem. If $f(x)$ has at most a finite number of infinite discontinuities and is of *bounded variation* on any closed finite interval not containing such a point of discontinuity, and if $\int_{-\infty}^{\infty} |f(x)|dx$ exists, then $f(x)$ can be represented as

$$f(x)\approx\frac{1}{2\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} f(s)\cos t(x-s)\,ds,$$

where the right member of this expression has the value

$$\lim_{h\to 0} \tfrac{1}{2}[f(x+h)+f(x-h)]$$

if $f(x)$ is bounded in the neighborhood of x.

Fourier's theorem. A theorem concerning the possibility of expanding a function in a Fourier Series. The theorem, as proved by Dirichlet, is as follows: If $f(x)$ is defined and bounded on the range $(-\pi, \pi)$ and has only a finite number of maxima and minima and a finite number of discontinuities on this range and if

$$\pi a_n=\int_{-\pi}^{\pi} f(x)\cos nx\,dx,$$

and

$$\pi b_n=\int_{-\pi}^{\pi} f(x)\sin nx\,dx,$$

the series

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

converges to $f(x)$ for all x in the interval $(-\pi<x<\pi)$ for which $f(x)$ is continuous and to $\frac{1}{2}[f(x+0)+f(x-0)]$ at points for which $f(x)$ is discontinuous, $f(a+0)$ denoting the limit of $f(x)$ as x approaches a from the right, and $f(a-0)$ the limit of $f(x)$ as x approaches a from the left. These conditions on $f(x)$ are known as Dirichlet's conditions. They are more restrictive than is necessary. See FEJÉR—Fejér's theorem.

FRAC'TION, *n.* An indicated quotient of two quantities. The dividend is called the numerator and the divisor the denominator (the numerator of $\frac{3}{4}$ is 3 and the denominator is 4). A simple fraction (or common fraction or vulgar fraction) is a fraction whose numerator and denominator are both integers, as contrasted to a complex fraction which has a fraction for the numerator or denominator or both. If the numerator of a simple fraction is unity, it is called a **unit fraction.** A complex fraction can be changed to a simple fraction by inverting the denominator and multiplying, or by multiplying numerator and denominator by the least common multiple of all denominators in the complex fraction; *e.g.*,

$$\tfrac{1}{3}/(\tfrac{1}{2}+\tfrac{1}{4})=\tfrac{1}{3}/\tfrac{3}{4}=\tfrac{1}{3}\times\tfrac{4}{3}=\tfrac{4}{9},$$
$$\tfrac{1}{3}/(\tfrac{1}{2}+\tfrac{1}{4})=(12\times\tfrac{1}{3})/[12(\tfrac{1}{2}+\tfrac{1}{4})]=\tfrac{4}{9}.$$

Two simple fractions are said to be **similar** if they have the same denominators. A fraction whose numerator and denominator are real numbers is called a **rational fraction** if the numerator and denominator are rational numbers; it is called a **proper fraction** if its numerator is smaller than its denominator, and an **improper fraction** if the numerator is larger (or equal to) the denominator ($\frac{2}{3}$ is a proper fraction and $\frac{4}{3}$ is an improper fraction). A fraction whose numerator and denominator are both polynomials is also called a **rational fraction** (or *rational function*); it is a **proper fraction** if the numerator is of lower degree than the denominator, and an **improper fraction** otherwise; $x/(x^2+1)$ is a proper fraction and $x^2/(x+1)$ is an improper fraction, while $(x-y)/xy$ is a proper fraction in x and y together, but not in either x or y alone.

addition, subtraction, multiplication, and division of fractions. See SUM—sum of real numbers, SUBTRACTION, PRODUCT—product of real numbers, and DIVISION.

clearing of fractions. Multiplying both members of an equation by a common denominator of the fractions. See EXTRANEOUS—extraneous root.

continued fraction. A number plus a fraction whose denominator is a number plus a fraction, etc., such as:

$$a_1+\cfrac{b_2}{a_2+\cfrac{b_3}{a_3+\cfrac{b_4}{a_4+\cfrac{b_5}{a_5,\ \text{etc.}}}}}$$

A continued fraction may have either a finite or an infinite number of terms. In the former case it is said to be a terminating

continued fraction; in the latter, **nonterminating.** If a certain sequence of the a's and b's occurs periodically, the continued fraction is said to be **recurring** or **periodic.** The terminating continued fractions

$$a_1,\ a_1+\frac{b_2}{a_2},\ a_1+\cfrac{b_2}{a_2+\cfrac{b_3}{a_3}},\ \text{etc.}$$

are called **convergents** of the continued fraction. The quotients b_2/a_2, b_3/a_3, etc., are called **partial quotients.**

decimal fraction. See DECIMAL.

decomposition of a fraction. Writing the fraction as a sum of partial fractions.

partial fraction. See PARTIAL—partial fractions.

FRAC'TION-AL, *adj.* **fractional equation.** (1) An equation containing fractions of any sort, such as $\frac{1}{2}x+2x=1$. (2) An equation containing fractions in the variable, such as $x^2+2x+1/x^2=0$.

fractional exponent. See EXPONENT.

FRAME, *n.* **astronomical reference frame.** A frame of reference in which the sun is fixed and does not rotate relative to fixed stars. This frame is used in celestial mechanics.

frame of reference. Any set of lines or curves *in a plane*, by means of which the position of any point *in the plane* may be uniquely described; any set of planes or surfaces by means of which the position of a point *in space* may be uniquely described.

FREDHOLM. **Fredholm's determinant.** (*Integral Equations.*) Fredholm's determinant for the kernel $K(x, t)$ is the power series in λ defined as

$$D(\lambda)=1-\lambda\int_a^b K(t,t)\,dt+\frac{\lambda^2}{2!}\int_a^b\int_a^b\begin{vmatrix} K(t_1,t_1)K(t_1,t_2)\\ K(t_2,t_1)K(t_2,t_2)\end{vmatrix}dt_1\,dt_2-\frac{\lambda^3}{3!}\int_a^b\int_a^b\int_a^b\begin{vmatrix} K(t_1,t_1)\ldots K(t_1,t_3)\\ \cdots\cdots\cdots\\ K(t_3,t_1)\cdots K(t_3,t_3)\end{vmatrix}\times dt_1\,dt_2\,dt_3+\cdots.$$

Fredholm's integral equations. *Fredholm's integral equation of the first kind* is the equation $f(x)=\int_a^b K(x,t)y(t)\,dt$, and *Fredholm's integral equation of the second kind* is $y(x)=f(x)+\lambda\int_a^b K(x,t)y(t)\,dt$, in which $f(x)$ and $K(x, t)$ are two given functions and $y(x)$ is the unknown function. The function $K(x, t)$ is called the **kernel** or **nucleus** of the equation. Fredholm's equation of the second kind is said to be **homogeneous** if $f(x)\equiv 0$. Also called the **integral equations of the first and second kind**, respectively. The above is sometimes modified by letting $\lambda=1$.

Fredholm minors. The *first Fredholm minor* $D(x, y; \lambda)$ for the *kernel* $K(x, y)$ is

$$D(x,y,\lambda)=\lambda K(x,y)-\lambda^2\int_a^b\begin{vmatrix} K(x,y)K(x,t)\\ K(t,y)K(t,t)\end{vmatrix}dt+\frac{\lambda^3}{2!}\int_a^b\int_a^b\begin{vmatrix} K(x,y)K(x,t_1)K(x,t_2)\\ K(t_1,y)K(t_1,t_1)K(t_1,t_2)\\ K(t_2,y)K(t_2,t_1)K(t_2,t_2)\end{vmatrix}\times dt_1\,dt_2+\cdots.$$

The higher Fredholm minors are defined in a similar way. This definition is sometimes modified by letting $\lambda=1$.

Fredholm's solution of Fredholm's integral equation of the second kind. If $f(x)$ is a continuous function of x for $a\leqq x\leqq b$ and $K(x, t)$ is a continuous function of x and t for $a\leqq x\leqq b$, $a\leqq t\leqq b$, and if the *Fredholm determinant* $D(\lambda)$ of the *kernel* $K(x, t)$ is not zero, then the *Fredholm integral equation*

$$y(x)=f(x)+\lambda\int_a^b K(x,t)y(t)\,dt$$

has a unique continuous solution $y(x)$ given by

$$y(x)=f(x)+\frac{1}{D(\lambda)}\int_a^b D(x,t;\lambda)f(t)\,dt,$$

where $D(x, t; \lambda)$ is the *first Fredholm minor* for the kernel $K(x, t)$ and $D(\lambda)$ is the *Fredholm determinant* for $K(x, t)$. See LIOUVILLE-NEUMANN SERIES; HILBERT—Hilbert-Schmidt theory of integral equations with symmetric kernels; and VOLTERRA—Volterra's reciprocal functions.

FREE, *adj.* **free group.** A group is said to be *free* if the group has a set of generators such that no product of generators and inverses of generators is equal to the identity unless it can be written as a product of expressions of type $a\cdot a^{-1}$. *E.g.*, if a free

group has generators a and b, then expressions of type ab, aba, $a^{-1}babbab^{-1}$, etc., are all distinct members of the group. An **Abelian group** is said to be *free* if no product of generators and inverses of generators is equal to the identity unless it can be reduced to a product of expressions of type $a \cdot a^{-1}$ by use of the commutative law. If an Abelian group has a finite number of generators, then it is free if and only if no element is of finite period (the group is then a direct product of infinite cyclic groups). **An element of a group** is said to be *free* if it is not of finite period.

FREE'DOM, *n.* **degrees of freedom.** (*Statistics.*) The number of free (unrestricted and independent in the sense of random sampling) variables entering into a statistic. If a sampling distribution of n variables is independently variable in $n-p$ of the variables, there are $n-p$ degrees of freedom; p is known as the number of **linear restraints** in the distribution of the n variables. *Tech.* Let $x_1, \cdots, x_n$ be n independent randomly distributed normal variables each with zero mean and variance σ^2; further, let y_j be an orthogonally transformed variable of the $x_1, \cdots, x_n$, where the y_j are also independent and normally distributed with zero mean, and variance σ^2, $j=1, \cdots, p \leq n$. Also let

$$(x)=\sum_{i=1}^{n}(x_i^2-y_i^2- \cdots -y_p^2)$$

be a quadratic form. By the orthogonal transformation, we get $(x)=y_{p+1}^2+ \cdots + y_n^2$, which is independent of $y_1, \cdots, y_p$. The frequency function of (x) will contain a parameter $n-p$ which is the degree of freedom of the variable (x); $n-p$ is the rank of the quadratic form. *E.g.*, the sum of squares around the mean of n independent items has $n-1$ degrees of freedom if the mean is computed from the n items, for the sum of squares is restricted by the condition that

$$\sum_{i=1}^{n}(x_i-\bar{x})=0$$

FRENCH, *adj.* **French horsepower.** See HORSEPOWER.

FRENET-SERRET FORMULAS. The central formulas in the theory of space curves. If α, β, γ denote the unit vectors along the tangent, principal normal, and binormal, respectively, of a space curve C, and ρ and τ its radii of curvature and torsion, then the formulas are $\frac{d\alpha}{ds}=\frac{\beta}{\rho}$, $\frac{d\beta}{ds}=-\frac{\alpha}{\rho}-\frac{\gamma}{\tau}$, $\frac{d\gamma}{ds}=\frac{\beta}{\tau}$, where s denotes the arc-length.

FRE'QUEN-CY, *adj.*, *n.* **absolute frequency.** The number of times a value or characteristic occurs or is observed. If a variable x takes on values $x_1, x_2, \cdots, x_n$, the number of times it takes on a given value, x_i, is the *absolute frequency* of that value.

binomial frequency distribution. See BINOMIAL—binomial distribution.

cumulative frequency. See CUMULATIVE.

frequency curve. (1) A graphic picture of a frequency distribution, or of a set of frequencies of the various values of a variable. The ordinate of the curve is proportional to the frequency for the various values of the variable which are noted on the abscissa. Customarily the area under the curve depicts the total frequency, while the ratio of the area over an interval to the total area is the *relative frequency* for the interval. See HISTOGRAM, and OGIVE. (2) The various possible values of a variable and the frequencies of each value.

frequency distribution. (*Statistics.*) Subclassification of the possible values of a variable, classified into broader categories. Thus the variables, which may be continuous, *e.g.*, from 0 to 100, may be grouped arbitrarily into intervals ten units wide from 0 up to 10, 10 and up to 20, etc. The intervals on the x-axis are called class intervals. If these are marked at equal intervals in ascending order on the axis of abscissas (x-axis), and the number in each class is indicated by a horizontal line segment drawn above the x-axis at a height equal to the number in the class (the number making a grade within the particular interval), the diagram is called a histogram. If the upper ends of the ordinates at the middles of the intervals are connected by line segments, the resulting figure is called a frequency polygon.

frequency function. (1) For a *discontinuous variable*, the function of x which gives the *absolute frequencies* of the values x_i is the **absolute frequency function;** $N=\sum_{i=1}^{r} f(x_i)$ is the total frequency, where $f(x_i)$ is the absolute frequency function of the discontinuous variate. (2) For a *continuous variable*, the frequency function is defined as the first derivative of the *distribution* function of the continuous variable. If the distribution function is defined so as to have a total frequency of 1, the derived frequency function is also known as the **probability density function.** Thus: $\int_{-\infty}^{b} f(x)\,dx=F(b)$, where $f(x)$ is the frequency or probability density function and $F(x)$ is the distribution function.

frequency of a periodic function. See PERIODIC—periodic function of a real variable.

frequency polygon. (*Statistics.*) A frequency function graph obtained by joining the peaks of the frequency ordinates at the midpoints of the several successive intervals of the variable. See FREQUENCY—frequency distribution.

normal frequency curve. The graph of $y=[N/(\sigma\sqrt{2\pi})]e^{-(x-A^2)/(2\sigma^2)}$, where N is the total number of observations, A is the mean of the observations, σ the *standard deviation*, and e the base of natural logarithms. *Syn.* Normal distribution curve; probability curve. The normal frequency curve is a bell-shaped curve, symmetrical about the mean, with points of inflection at $\pm\sigma$ from the mean. Approximately 68% of the distribution is included in the interval $A\pm\sigma$, while 99.7% is included in the interval $A\pm 3\sigma$. The normal curve is very widely used because of the important property of being a very good approximation of several distribution functions, *e.g.*, the distribution of statistics from random samples from normal and a wide class of nonnormal distributions. It is the limiting form of the *binomial sampling distribution* as n increases. A linear function of N independent normally distributed variables is itself normally distributed. See CORRELATION—normal correlation.

relative frequency. The relative frequency of the occurrence of an event is the ratio of the number of times that an event happens to the number of trials in which the event can happen or fail to happen. If m_i is the *absolute frequency* of the value x_i of the variable x, and if the total number of occurrences of the variable over all its possible values is n, then m_i/n is the *relative frequency* of the value x_i of the variable x.

relative frequency function. See PROBABILITY—probability density function.

statistical frequency. The frequency with which a variable assumes the set of values included in a given class interval.

FRESNAL INTEGRALS. (1) The integrals $\int_0^x \sin t^2\,dt$ and $\int_0^x \cos t^2\,dt$, called the *Fresnal sine* and the *Fresnal cosine integrals.* These are equal, respectively, to

$$\frac{x^3}{3}-\frac{x^7}{7\cdot 3!}+\frac{x^{11}}{11\cdot 5!}-\cdots$$

and

$$x-\frac{x^5}{5\cdot 2!}+\frac{x^9}{9\cdot 4!}-\cdots,$$

which converge for all values of x. (2) The integrals

$$\int_x^\infty \frac{\cos t}{t^{1/2}}\,dt \quad\text{and}\quad \int_x^\infty \frac{\sin t}{t^{1/2}}\,dt,$$

which are equal, respectively, to

$$(U\cos x-V\sin x)$$

and

$$(U\sin x+V\cos x),$$

where

$$U=\frac{1}{x}\left(\frac{1}{x}-\frac{3!}{x^3}+\frac{5!}{x^5}-\cdots\right)$$

and

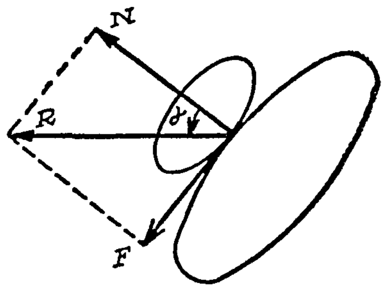
$$V=\frac{1}{x}\left(1-\frac{2!}{x^2}+\frac{4!}{x^4}-\cdots\right).$$

FRIC'TION, *n.* **angle of friction.** See below, force of friction.

coefficient of friction. See below, force of friction.

force of friction. If two bodies are in contact and one body A is at rest, or in motion without acceleration, relative to the other, then the external forces acting on A are balanced by a normal reaction force **N** perpendicular to the plane of contact and a force of friction **F** in the plane of contact.

When A is on the verge of moving, the acute angle α is called the **angle of friction,** and $\tan\alpha=\frac{F}{N}=\mu$ is the **coefficient of static friction.** When A is moving without acceleration relative to the other body, μ is the **coefficient of kinetic (sliding) friction.**



FRON-TIER′, *n.* **frontier of a set.** See INTERIOR—interior of a set.

FRUS′TUM, *n.* **frustum of any solid.** The part of the solid between two parallel planes cutting the solid. See PYRAMID, and CONE.

FUBINI. Fubini's theorem. Let m_1 and m_2 be *measures* defined on spaces X and Y and $m_1\times m_2$ be the product measure defined on the *Cartesian product* $X\times Y$. Fubini's theorem states that if $h(x, y)$ is integrable on $X\times Y$, then the subset of Y for which $h(x, y)$ is not integrable on X is of *measure zero*, the subset of X for which $h(x, y)$ is not integrable on Y is of *measure zero*, and

$$\int h\, d(m_1\times m_2)=\int f\, dm_1=\int g\, dm_2,$$

where

$$f(x)=\int h(x, y)\, dm_2$$

and

$$g(y)=\int h(x, y)\, dm_1.$$

Such results as the following are sometimes referred to as parts of Fubini's theorem: "If S is a measurable subset of $X\times Y$, then the set of points y in Y for which S_y is not measurable is of measure zero (S_y is the set of all points x of X for which (x, y) is in S)." See SIERPINSKI—Sierpinski set.

FUL′CRUM, *n.* The point at which a lever is supported. See LEVER.

FUNC′TION, *n.* An association of a certain object (or objects) from one set (the range) with each object from another set (the **domain**). *E.g.*, a function might be defined as having as its value a person's age when the person is specified—it would then be said that a person's age is a function of the person, and that the domain of this function is the set of all human beings and the range is the set of all integers which are ages of persons presently living. The expression $y=3x^2+7$ defines y as a function of x when it is specified that the domain is (for example) the set of real numbers; y is then a function of x, a value of y is associated with each real-number value of x by multiplying the square of x by 3 and adding 7 (the range of this function is the set of all real numbers not less than 3), and x is said to be the **independent variable** of the function (y is called a **dependent variable**). If this function $y=3x^2+7$ is denoted by $y=f(x)$, then the value of y when $x=2$ is $f(2)=3\cdot 2^2+7=19$. A function is said to be **single valued** if a unique value of the function is determined by a choice of the independent variable (or variables); it is **multiple valued** if more than one value of the function can correspond to a choice for the values of the independent variables (the function defined by $y=f(x)$ and $y^2=3x+1$ assigns two values to y for each $x>-\frac{1}{3}$). **A function of one variable** is a function which has only one independent variable. Both the above examples are functions of one variable; also, the area of a circle is a function of the radius; the sine of an angle is a function of the angle; the logarithm of a number is a function of the number. The symbols used for such a function are f, F, ϕ, etc., the function values corresponding to x being denoted by $f(x)$, $F(x)$, $\phi(x)$, etc.; these are customarily called the f function, the F function, the ϕ function, etc., of x. Such symbols are used when making statements that are true for several different functions, in other words, statements that are not concerned with a specific form of the function; *e.g.*, (1) $[f(x)-F(x)][f(x)+F(x)]=[f(x)]^2-[F(x)]^2$ for any specific functions whatever; (2) if f and F are each continuous on (a, b), then $f(x)+F(x)=G(x)$ defines a function G which is continuous on (a, b). A symbol like $f(x)$ is entirely general unless otherwise specified, as in

$f(x)=0$ in algebra where f is understood to be a polynomial. The notations f, F, ϕ, etc., are also used as abbreviations for specific functions under consideration. A function F can be regarded as a set of ordered pairs (x, y). Each ordered pair is said to be an element of the function and the domain of the function is the collection of all objects which occur as the first member of some element and the range is the collection of all objects which occur as the second member of some element. The function is single valued if there do not exist two different elements which have the same first member. Usually one calls a function in the above sense a **relation**; a single-valued function in the above sense is then called a function. A **function of several variables** is a function which takes on a value or values corresponding to every set of values of several variables (called the *independent variables*). The binomial $2x+xy$ is a function of x and y. Function values of functions f, F, etc., of two variables, are denoted by $f(x_1, x_2)$, $F(x, y)$, etc. A value of a function of n variables is written in the form $f(x_1, x_2, \cdots, x_n)$, etc. A function such as $z=2x+xy$, where x and y can be any real numbers, may be regarded either as a function of two variables x and y, or as a function of points (x, y). In the latter case, the *domain* is said to be the entire plane. In this way, any function of several variables can be regarded as a function of one variable; *e.g.*, a function $z=f(x_1, \cdots, x_n)$, where $x_1, \cdots, x_n$ can have any real number values, can be regarded as a function whose *domain* consists of the objects which are sequences $(x_1, \cdots, x_n)$ of n real numbers. See the various headings below.

algebraic function. A function which can be generated by algebraic operations alone. *Tech.* A function $f(x)$ such that for some polynomial $P(x, y)$ it is true that $P(x, f(x)) \equiv 0$. Any polynomial is algebraic, but (for example) $\log x$ is not.

analytic function. See various headings under ANALYTIC.

automorphic function. See AUTOMORPHIC.

Bessel function. See BESSEL.

beta function. See BETA.

characteristic function. See CHARACTERISTIC.

complementary function. See DIFFERENTIAL—linear differential equations.

composite function. See COMPOSITE.

continuous function. See CONTINUOUS.

decreasing function. See DECREASING.

dependent functions. See DEPENDENT.

entire function. See ENTIRE.

Euler's ϕ-function. See EULER.

even function. A function whose value does not change when the sign of the independent variable is changed; *i.e.*, a function such that $f(-x)=f(x)$; x^2 and $\cos x$ are even functions, for $(-x)^2=x^2$ and $\cos(-x)=\cos x$.

exponential function. (1) The function e^x, where e is the base of Napierian logarithms. (2) The function a^x, where a is a constant. (3) A function in which the variable or variables appear in exponents and possibly also as a base, such as 2^{x+1} or x^x. For a complex number z, the function e^z may be defined either by $e^z=e^x(\cos y+i \sin y)$, where $z=x+iy$, or by $e^z=1+z+z^2/2!+z^3/3!+\cdots$.

function of class C^n. A function which is continuous and has continuous derivatives of all orders up to and including the nth. The functions of class C^0 are the continuous functions.

function of class L_p. A function f is of class L_p on an interval (or measurable set) Ω if it is (Lebesgue) *measurable* and $|f(x)|^p$ is *summable* over Ω. The space $L_p (p \geqq 1)$ of all functions of class L_p is a complete normed *vector space* (*i.e.*, a *Banach space*) if addition and multiplication by scalars are taken as ordinary addition and multiplication and

$$\|f\| = \left[\int_\Omega |f|^p d\Omega\right]^{1/p}$$

is defined as the "length" or norm of f. Minkowski's inequality is then equivalent to $\|f+g\| \leqq \|f\|+\|g\|$, and Hölder's inequality is equivalent to

$$\int_\Omega |fg| d\Omega \leqq \|f\| \cdot \|g\|$$

if f is of class L_p and g of class L_q and $p+q=pq$ $(p>1, q>1)$.

function-element of an analytic function of a complex variable. See ANALYTIC—analytic continuation of an analytic function of a complex variable.

function theory. See THEORY.

gamma function. See GAMMA—gamma function.

Hamiltonian function. (*Physics.*) The sum of the *kinetic energy* and the *potential energy.*

harmonic function. See HARMONIC.

holomorphic function. See ANALYTIC—analytic function of a complex variable.

hyperbolic functions. See HYPERBOLIC.

implicit function. See IMPLICIT—implicit function.

increasing function. See INCREASING.

integral function. Same as ENTIRE FUNCTION.

inverse function. See INVERSE.

logarithmic function. A function of the form $\log f(x)$.

measurable function. See MEASURABLE.

meromorphic function. See MEROMORPHIC.

monogenic function. See MONOGENIC.

monotonic functions. Functions which are either monotonic increasing or monotonic decreasing. See MONOTONIC.

odd function. A function whose sign changes, but whose absolute value does not change, when the sign of the dependent variable is changed; *i.e.*, a function such that $f(-x)=-f(x)$; x^3 and $\sin x$ are odd functions, for

$$(-x)^3=-x^3 \quad \text{and} \quad \sin(-x)=-\sin x.$$

orthogonal functions. See ORTHOGONAL.

periodic function. See PERIODIC.

rational integral function of one variable. Same as POLYNOMIAL IN ONE VARIABLE. See POLYNOMIAL.

regular function. See ANALYTIC—analytic function of a complex variable.

step function. See STEP—step function.

stream function. Let $f(x, y)$ denote the flux, in an incompressible fluid, across some curve AP where A is a fixed point and P is variable. The function $f(x, y)$ is dependent only on the position of P, since the flux across two curves joining A and P must be the same; otherwise fluid would be added or taken away from the space between the two curves. When P moves in such a way as to keep $f(x, y)$ constant, P traces curves across which there is no flux. These curves are called stream lines, and if the equation of such a stream line is $F(x, y)=0$ then $F(x, y)$ is called a *stream function.*

subadditive and subharmonic functions. See ADDITIVE—additive set function, and SUBHARMONIC.

summable function. See SUMMABLE—summable function.

transcendental function. See TRANSCENDENTAL.

trigonometric function. See TRIGONOMETRIC—trigonometric functions.

unbounded function. See UNBOUNDED.

vector function. A function whose values are vectors. The expression

$$F=f_1 i+f_2 j+f_3 k,$$

where f_1, f_2, and f_3 are scalar functions, defines a vector (valued) function.

FUNC'TION-AL, *adj., n.* As an *adjective:* Of, relating to, or affecting a function. As a *noun:* A correspondence between a class C_1 of functions and another class C_2 of functions, not necessarily distinct from C_1, in such a manner that to each y of C_1 corresponds one or more f of C_2. The "degenerate" case in which C_2 is a class of numbers is also included as a possible case. Many authors in the modern theories of abstract spaces use the term only in the case in which C_2 is a class of numbers, but in these generalizations the elements of the class C_1 need not be functions. *E.g.*, $\int_a^b \alpha(x)y(x)\,dx$, $\max_{a\leqq x\leqq b}|y(x)|$, $dy(x)/dx$, and $\alpha(x)y(x)+\int_a^b \beta(x, s)y(s)\,ds$ are *functionals* of y. In each of these examples, the class C_1 is a suitably restricted class of real functions y of a real variable x, while C_2 is the class of real numbers in the first two examples and a suitable restricted class of real functions of a real variable in the last two examples. Two simple examples of a functional in which both C_1 and C_2 are classes of real functions of two real variables are

$$\frac{\partial^2 y(s, r)}{\partial s^2}+\frac{\partial^2 y(s, r)}{\partial r^2} \quad \text{and} \quad \int_r^s y(s, t)y(t, r)\,dt.$$

A functional f defined on a vector space is a linear functional if $f(x+y)=f(x)+f(y)$ and $f(ax)=af(x)$ for any vectors x and y and scalar a. If f has real or complex number values, then f is continuous for each x if and only if there is a number M such that $|f(x)|\leqq M\cdot\|x\|$ for each x. The least such number M is called the norm of f. See CONJUGATE—conjugate space.

differential of a function. If $f(y)$ is a *functional* from a class C_1 of functions to a class C_2 of functions, then a differential of $f(y)$ at y_0 with increment $\delta y(x)$ is a continuous, additive functional $\delta f(y_0, \delta y)$ from C_1 to C_2 such that $f(y_0+\delta y)-f(y_0)=\delta f(y_0, \delta y)+$ "higher order terms in $\delta y(x)$" for all $\delta y(x)$ in some neighborhood of the "zero function" in C_1. In order that this definition be applicable, the classes C_1 and C_2 must be such that the notions of addition, subtraction, zero function, neighborhood, and continuity are meaningful, and the meaning of "higher order terms in $\delta y(x)$" must be specified. *E.g.*, if C_1 and C_2 are both the *Banach space* of all real continuous functions $y(x)$ of a real variable x in $a \leqq x \leqq b$ with $\|f\| = \max\limits_{a \leqq x \leqq b} |f(x)|$ and $\rho(f, g) = \|f-g\|$ is the *distance* between f and g, then $\delta f(y_0, \delta y)$ is a *Fréchet differential* of $f(y)$ at y_0 if it is a continuous additive functional from C_1 to C_2 and

$$f(y_0+\delta y)-f(y_0)=\delta f(y_0, \delta y)+\|\delta y\|\epsilon(y_0, \delta y),$$

where $\|\epsilon(y_0, \delta y)\|$ tends to zero with $\|\delta y(x)\|$ uniformly for all functions $\delta y(x)$ continuous in $a \leqq x \leqq b$. If $\alpha(x)$ and $\beta(x, s)$ are fixed continuous functions, then the *Fréchet differential* of $f(y)=\alpha(x)y(x)+\int_a^b \beta(x, s)y^2(s)\, ds$ exists for each y_0 in C_1 and is given by $\delta f(y_0, \delta y)=\alpha(x)\delta y(x)+2\int_a^b \beta(x, s)y_0(s)\, \delta y(s)\, ds$.

functional determinant. See JACOBIAN.

functional notation. A notation used to denote the general concept *function* and often as an abbreviation of some specific function. The notation consists of a letter placed before parentheses containing a number of letters representing the independent variable or variables of the function, followed by the interval or intervals over which the variable or variables range; for example, one writes $f(x)$ on (a, b), or $f(x)$, $a<x<b$, and $f(x, y)$, $a \leqq x \leqq b$, $c \leqq y \leqq d$. The interval is commonly omitted when obtainable from the context or the nature of the function. A function is frequently denoted by a single symbol, such as f, the number (or object) in the range of f which is associated with an x in the domain being denoted by $f(x)$. See FUNCTION.

FUND, *n.* Money (sometimes other assets immediately convertible into money) which is held ready for immediate demands.

endowment fund. A fund permanently appropriated for some objective, such as carrying on a school or church.

reserve fund. (1) *In insurance:* See RESERVE. Used to take care of additional cost of policy in later years. (2) *In business:* A sum held ready to meet emergencies or take advantage of opportunities to purchase at low prices.

sinking fund. A fund accumulated by periodic investments for some specific purpose such as retiring bonds, replacing equipment, providing pensions, etc. (The amount of the sinking fund is the amount of the annuity formed by the payments.)

FUN'DA-MEN'TAL, *adj.* **four fundamental operations of arithmetic.** Addition, subtraction, multiplication, and division.

fundamental assumption. See ASSUMPTION.

fundamental coefficients and quadratic forms of a surface. See under SURFACE.

fundamental group. Let S be a set which has the property that any two of its points can be connected by a *path*, a *path* being the image (for a continuous mapping) of a (directed) interval. The fundamental group of S is the *quotient group* of the group of all paths, with initial and terminal point at a designated base point P, and the subgroup of all paths which are homotopic to the path consisting of the single point P (using the quotient group is equivalent to defining two paths to be *equal* whenever they are homotopic to each other). The *product* of paths f and g is the path obtained by attaching g to the end of f; the *inverse* of f is the path obtained by reversing the direction assigned to f. Two fundamental groups with different base points are isomorphic. If the fundamental group contains only the identity, then S is *simply connected.* The fundamental group of a circle is an infinite cyclic group. For a torus, it is the commutative group generated by two elements a and b. A closed orientable surface of genus p has a fundamental group generated by $2p$ elements a_i, b_i which satisfy the relation

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \cdots a_pb_pa_p^{-1}b_p^{-1}=1$$

and is not commutative unless $p=1$ (the surface is then a torus). A closed non-orientable surface has a fundamental group generated by q elements a_i with the relation

$$a_1a_1a_2a_2\cdots a_qa_q=1,$$

which for $q=1$ is a group of order two generated by a single element (the surface is then the projective plane). The commutators of the fundamental group generate a group which is isomorphic with the 1-dimensional *homology group* (based on the integers).

fundamental identities of trigonometry. See TRIGONOMETRIC—trigonometric functions.

fundamental lemma of the calculus of variations. If $\alpha(x)$ is continuous for $a\leqq x\leqq b$ and $\int_a^b \alpha(x)\phi(x)\,dx=0$ for all $\phi(x)$ which have continuous first derivatives in $a\leqq x\leqq b$ and have $\phi(a)=\phi(b)=0$, then $\alpha(x)\equiv 0$ throughout the interval $a\leqq x\leqq b$.

fundamental numbers and functions in integral equations. Same as the EIGENVALUES and EIGENFUNCTIONS.

fundamental period of a periodic function of a complex variable. Same as PRIMITIVE PERIOD. See PERIODIC—periodic function of a complex variable, and various headings under PERIOD.

fundamental sequence. Same as CAUCHY SEQUENCE. See SEQUENCE.

fundamental theorem of algebra. Every polynomial equation of degree $n\geqq 1$ with complex coefficients has at least one root, which is a complex number (real or imaginary). See GAUSS—Gauss' proof of the fundamental theorem of algebra; WINDING—winding number.

fundamental theorem of the integral calculus. A theorem giving a method of summing elements of area, volume, etc. (finding the area, volume, etc.) by the use of antiderivatives. If

$$\int_a^b f(x)\,dx$$

is defined as $F(b)-F(a)$, where $F(x)$ is a function such that

$$\frac{dF(x)}{dx}=f(x),$$

then the fundamental theorem of the integral calculus states: If $f(x)$ is continuous and single valued, then

$$\lim_{n\to\infty}[f(x_1)\Delta_1x+f(x_2)\Delta_2x+f(x_3)\Delta_3x+\cdots$$

$$+f(x_n)\Delta_nx]=\lim_{n\to\infty}\sum_1^n f(x_i)\Delta_ix=\int_a^b f(x)\,dx,$$

where $\Delta_1x, \Delta_2x, \Delta_3x, \cdots \Delta_nx$ are n nonoverlapping subintervals of the interval (a, b), whose sum is $(b-a)$, the maximum length of the subintervals approaching zero as n becomes infinite, and x_i is some value of x in the interval Δ_ix. If

$$\int_a^b f(x)\,dx$$

is defined as the above limit, then the *fundamental theorem of calculus* is stated thus: If $\int_a^b f(x)\,dx$ exists and $f(x)$ is continuous at the interior point x of (a, b), then the derivative of $\int_a^x f(t)\,dt$ is equal to $f(x)$. The fundamental theorem of the integral calculus is sometimes taken as the statement that the above summation has a limit under the given hypothesis (or if $f(x)$ is bounded and continuous *almost everywhere*). See DARBOUX'S THEOREM, INTEGRAL, and ELEMENT—element of integration.

FUTURE, *adj.* **future value of a sum of money.** See AMOUNT.

G

G_δ set. See BOREL—Borel set.

GAME, *n.* A set of rules, for individuals or groups of individuals involved in a competitive situation, giving their permissible actions, the amount of information each receives as the competition progresses, the probabilities associated with the chance events that might occur during the competition, the circumstances under which the competition ends, and the amount each individual pays or receives as a consequence. An n-person game is a game in which there are exactly n players or interests (a two-person game is a game in which there ar

exactly two players or conflicting interests). The **theory of games** is the mathematical theory, founded largely by the mathematician John von Neumann, of optimal behavior in situations involving conflict of interest. See BOX, DUEL, HER, MAZUR, MINIMAX—minimax theorem, MORRA, NIM, PAYOFF, PLAYER, STRATEGY, and the headings below.

coin-matching game. A two-person zero-sum game in which each of the two players tosses a coin of like value. If the two coins show like faces—either both "heads" or both "tails"—the first player wins, while if they show unlike faces the second player wins.

"Colonel Blotto" game. In the theory of games, the problem of dividing attacking and defending forces between fortresses under the assumption that at each fortress each side loses a number of men equal to the number of men in the smaller force involved at the fortress, and that the fortress is then occupied by the side having survivors; occupation of a fortress is considered as being equivalent to having a certain number of survivors, and the payoff is then measured in terms of the total number of survivors at the fortresses.

circular symmetric game. A finite two-person zero-sum game whose matrix is a *circulant* (*i.e.*, the elements of each row are the elements of the previous row slid one place to the right, with the last element put first).

completely mixed game. A game having a unique solution that is also a simple solution; equivalently, a game such that every possible strategy has positive probability in the solution. See below, solution of a game.

continuous game. An infinite game in which each player has a closed and bounded continuum of pure strategies, usually taken (without loss of generality) to be identified with the numbers of the closed interval $[0, 1]$. See below, finite and infinite games.

concave and convex games. A **concave game** is a two-person zero-sum game for which the payoff function $M(x, y)$ is a *concave function* of the strategy x of the maximizing player (the dual of the convex game with payoff function $-M(y, x)$). A **convex game** is a two-person zero-sum game for which the payoff function $M(x, y)$ is a *convex function* of the strategy y of the minimizing player (the dual of the concave game with payoff function $-M(y, x)$). A **concave-convex game** is a two-person zero-sum game for which the payoff function $M(x, y)$ is a *concave function* of the strategy x of the maximizing player and is a *convex function* of the strategy y of the minimizing player.

cooperative game. A game in which the formation of coalitions is possible and permissible. A game is **noncooperative** if the formation of coalitions is either not possible or not permissible. See COALITION.

extensive form of a game. The general description of a game in terms of its moves, information patterns, etc. See below, normal form of a game.

finite and infinite games. A game is **finite** if each player has only a finite number of possible pure strategies; it is **infinite** if at least one player has an infinite number of possible pure strategies (*e.g.*, a pure strategy might ideally consist of choosing an instant from a given interval of time at which to fire a gun). See STRATEGY.

game of survival. A two-person zero-sum game that continues until one player loses all.

game with perfect information. A game such that at each move each player knows the outcome of all previous moves of the game. Such a game of necessity has a saddle point, and accordingly each player has an optimal pure strategy. A game with imperfect information is a game in which at least one move is made by a player not knowing the outcome of all previous moves.

normal form of a game. A description of a game in terms of its strategies and associated payoff matrix or function. See above, extensive form of a game.

polynomial game. A continuous game having payoff function of the form $M(x, y) = \sum_{i,j=0}^{m,n} a_{ij}x^iy^j$, where the strategies x, y range over the closed interval $[0, 1]$. See below, separable game.

positional game. A game with simultaneous moves by the players, in which each player knows at all times the outcome of all previous moves. See above, game with perfect information.

saddle point of a game. See SADDLE.

separable game. A continuous game having payoff function of the form $M(x, y) = \sum_{i,j=0}^{m,n} a_{ij} f_i(x) g_j(y)$, where the strategies x, y range over the closed interval $[0, 1]$, the a_{ij} are constants, and the $f_i(x)$ and $g_j(y)$ are continuous functions. A *polynomial game* is a particular instance of a separable game.

solution of a two-person zero-sum game. A pair of optimal mixed (or pure) strategies, one for each of the players of the game. A **simple solution** is a solution X, Y such that each pure strategy for the minimizing player, when used against the maximizing player's optimal mixed strategy X, and each pure strategy for the maximizing player, when used against the minimizing player's optimal mixed strategy Y, yields an expected value of the payoff exactly equal to the value of the game. A game might have a solution without having a simple solution. The *coin-matching game* is an example of a game having a simple solution. A set of **basic solutions** of a game is a finite set S of solutions of the game, such that every solution can be written as a convex linear combination of the members of S, but such that there is no proper subset of S in terms of which each solution can be so written. See PLAYER, and STRATEGY.

symmetric game. A finite two-person zero-sum game with a (square) skew-symmetric payoff matrix, *i.e.*, a payoff matrix for which $a_{ij} = -a_{ji}$ for all i and j. More generally, a two-person zero-sum game with payoff function $M(x, y)$ satisfying $M(x, y) = -M(y, x)$ for all x and y. The *value* of such a game is zero, and both players have the same *optimal strategies*.

value of a game. The number v associated with any two-person zero-sum game for which the minimax theorem holds. See MINIMAX—minimax theorem, and SADDLE—saddle point of a game.

zero-sum game. A game in which the sum of the winnings of the various players is always zero. Thus games like poker, in which we consider only the financial payoff, are zero-sum games unless there is a house charge for playing. A **non-zero-sum game** is a game in which, for at least one play, the sum of the winnings of the various players is not zero.

GAM'MA, *n.* The third letter of the Greek alphabet, written, lower case, γ; capital, Γ.

gamma function. Gamma of x is written $\Gamma(x)$ and is defined to be

$$\int_0^\infty t^{x-1} e^{-t}\, dt$$

(for $x > 0$, or the real part of x greater than zero if x is complex). It can easily be shown that $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = 1$, and from these results that $\Gamma(n) = (n-1)!$, when n is any positive integer. Also,

$$\Gamma(z) = \left[z e^{\gamma z} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right) e^{-z/n} \right\} \right]^{-1}$$

where γ is *Euler's constant*, which is an analytic function of the complex number z except for $z = 0, -1, -2, \cdots$. The **incomplete gamma functions,** $\gamma(a, x)$ and $\Gamma(a, x)$, are defined by

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t}\, dt,$$

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t}\, dt.$$

It follows that $\Gamma(a) = \gamma(a, x) + \Gamma(a, x)$ and that

$$\gamma(a+1, x) = a\gamma(a, x) - x^a e^{-x},$$

$$\Gamma(a+1, x) = a\Gamma(a, x) + x^a e^{-x},$$

$$\gamma(a, x) = \sum_0^\infty \frac{(-1)^n x^{a+n}}{n!(a+n)}.$$

GATE, *n.* In a computing machine, a switch that allows the passage of a signal if and only if one or more other signals are present; thus a gate is the machine equivalent of the logical "and". See CONJUNCTION, and BUFFER.

inverse gate. Same as BUFFER.

GAUSS. Gauss' differential equation. See HYPERGEOMETRIC—hypergeometric differential equation.

Gauss' equation. (*Differential Geometry.*) An equation expressing the total curvature $K = \dfrac{DD'' - D'^2}{EG - F^2}$ in terms of the fundamental coefficients of the first order, E, F, G, and their partial derivatives of the first and second orders:

$$K = \frac{1}{2H}\left\{\frac{\partial}{\partial u}\left[\frac{F}{EH}\frac{\partial E}{\partial v} - \frac{1}{H}\frac{\partial G}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{2}{H}\frac{\partial F}{\partial u} - \frac{1}{H}\frac{\partial E}{\partial v} - \frac{F}{EH}\frac{\partial E}{\partial u}\right]\right\};$$

where $H=\sqrt{EG-F^2}$, or in terms of Christoffel symbols,

$$K=\frac{1}{H}\left\{\frac{\partial}{\partial u}\left(\frac{H}{G}\begin{bmatrix}2 & 2\\ & 1\end{bmatrix}\right)-\frac{\partial}{\partial v}\left(\frac{H}{G}\begin{bmatrix}1 & 2\\ & 1\end{bmatrix}\right)\right\}$$
$$=\frac{1}{H}\left[\frac{\partial}{\partial v}\left(\frac{H}{E}\begin{bmatrix}1 & 1\\ & 2\end{bmatrix}\right)-\frac{\partial}{\partial u}\left(\frac{H}{E}\begin{bmatrix}1 & 2\\ & 2\end{bmatrix}\right)\right].$$

In tensor notation:

$$x^i{}_{,\alpha\beta}=d_{\alpha\beta}X^i.$$

For isothermic parameters,

$$E=G=\lambda(u,v),\quad F=0,$$

the formula reduces to

$$K=-\frac{1}{2\lambda}\left[\frac{\partial^2\log\lambda}{\partial u^2}+\frac{\partial^2\log\lambda}{\partial v^2}\right];$$

and for geodesic parameters, $E=1$, $F=0$, $G=[\mu(u,v)]^2$, $\mu\geqq 0$, the formula reduces to

$$K=-\frac{1}{\mu}\frac{\partial^2\mu}{\partial u^2}.$$

See GAUSS—theorem of Gauss, and CODAZZI EQUATIONS.

Gauss' formulas (or Delambre's analogies). Formulas stating the relations between the sine (or cosine) of half of the sum (or difference) of two angles of a spherical triangle and the other angle and the three sides. If the angles of the triangle are A, B, and C and the sides opposite these angles are a, b, and c, respectively, then Gauss' formulas are:

$$\cos\tfrac{1}{2}c\sin\tfrac{1}{2}(A+B)=\cos\tfrac{1}{2}C\cos\tfrac{1}{2}(a-b),$$
$$\cos\tfrac{1}{2}c\cos\tfrac{1}{2}(A+B)=\sin\tfrac{1}{2}C\cos\tfrac{1}{2}(a+b),$$
$$\sin\tfrac{1}{2}c\sin\tfrac{1}{2}(A-B)=\cos\tfrac{1}{2}C\sin\tfrac{1}{2}(a-b),$$
$$\sin\tfrac{1}{2}c\cos\tfrac{1}{2}(A-B)=\sin\tfrac{1}{2}C\sin\tfrac{1}{2}(a+b).$$

Gauss' fundamental theorem of electrostatics. The surface integral of the exterior normal component of the electric intensity over any closed surface all of whose points are free of charge is equal to 4π times the total charge enclosed by the surface. In the corresponding theorem for gravitational matter, the constant is -4π.

Gauss' mean value theorem. Let u be a regular harmonic function in a region R. Let P be a point in R, S a sphere with center at P and lying entirely (boundary and interior points) within R, and A the area of S; then $u(P)=(1/A)\iint_S u\,dS$. For R a plane region and C a circle with perimeter c, $u(P)=\frac{1}{c}\int_c u\,ds$.

Gauss plane. Same as COMPLEX PLANE.

Gauss' proof of the fundamental theorem of algebra. The first known proof of this theorem. A geometrical proof consisting essentially of substituting a complex number, $a+bi$, for the unknown of the equation, separating the real and imaginary parts of the result, and then showing that the two resulting functions of a and b are zero for some pair of values of a and b.

Gaussian integer. See INTEGER.

theorem of Gauss. The famous theorem that the total curvature of a surface is a function of the fundamental coefficients of the first order of the surface and their partial derivatives of the first and second orders. See above, Gauss' equation.

GELFOND-SCHNEIDER THEOREM. If α and β are algebraic numbers with $\alpha\neq 0$ and $\alpha\neq 1$, and if β is not a rational number, then any value of α^β is transcendental (*i.e.*, is a real or complex number which is not a root of a polynomial equation whose coefficients are integers). This theorem was proved independently by Gelfond (1934) and Schneider (1935).

GEN'ER-AL, *adj.* Not specific or specialized; covering all known special cases. Examples are the *general polynomial equation* (see EQUATION—polynomial equation) and *general term* (see TERM—general term).

general solution of a differential equation. See DIFFERENTIAL—solution of a differential equation.

general term. See TERM—general term.

GEN'ER-AL-IZED, *adj.* **generalized mean value theorem.** (1) Same as TAYLOR'S THEOREM. (2) Same as SECOND MEAN VALUE THEOREM. See MEAN—mean value theorems for derivatives.

generalized ratio test. See RATIO—ratio test.

GEN'ER-AT'ING, *p.* **generating function.** A function F that, through its representation by means of an infinite series of some sort, gives rise to a certain sequence of constants or functions as coefficients in the series. *E.g.*, the function $(1-2ux+u^2)^{-1/2}$ is a generating function of the *Legendre*

polynomials $P_n(x)$ by means of the expansion

$$(1-2ux+u^2)^{-1/2}=\sum_0^\infty P_n(x)u^n.$$

GEN'ER-A'TOR, *n.* Same as GENERATRIX (the feminine form of *generator*).

rectilinear generators. See RULED—ruled surface.

generator of a surface of translation. See SURFACE—surface of translation.

generators of a group. A set of generators of a group G is a subset S of G such that each member of G can be represented (using the group operations) in terms of members of S, repetitions of members of S being allowed. The set S of generators is **independent** if no member of S is in the group generated by the other members of S.

GEN'ER-A'TRIX, *n.* **generatrix of a ruled surface.** A straight line which forms the surface by moving according to some law. The elements of a cone are different positions of its generatrix. See RULED—ruled surface.

GE'NUS, *n.* **genus of a surface.** A *closed orientable surface* is topologically equivalent to a sphere with an even number $2p$ of holes (made by removing discs) which have been connected in pairs by p *handles* (shaped like the surface of half of a doughnut). A *closed nonorientable surface* is topologically equivalent to a sphere which has had a certain number q of discs replaced by *cross-caps.* The numbers p and q are said to be the *genus* of the surface. In either of the above cases, the surface not being closed means that some discs have been removed and the hole left open. A torus is a sphere with one handle; a Möbius strip is a sphere with one cross-cap and one "hole"; a Klein bottle is a sphere with two cross-caps; a cylinder is a sphere with two "holes". In general, the *Euler characteristic* of a surface is equal to $2-2p-q-r$, where p is the number of handles (which can be zero for a nonorientable surface), q is the number of cross-caps (zero for an orientable surface), and r is the number of holes (or boundary curves).

GE'O-DES'IC, *adj., n.* A curve C on a surface S such that at each point of C the principal normal of C coincides with the normal to S; a curve whose geodesic curvature vanishes identically. See below, geodesic curvature of a curve on a surface. It follows that if a straight line lies on a surface, then the line is a geodesic for the surface. A geodesic is a curve that yields a stationary value of the length integral

$$s=\int_{t_0}^{t_1}\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\,dt.$$

In terms of the arc-length parameter s, the Euler-Lagrange equations for this calculus of variations problem are the system of second-order differential equations

$$\frac{d^2x^i(s)}{ds^2}+\Gamma^i_{\alpha\beta}(x^1(s),\cdots,x^n(s))\frac{dx^\alpha(s)}{ds}\frac{dx^\beta(s)}{ds}=0,$$

where the $\Gamma^i_{\alpha\beta}$ are the Christoffel symbols of the second kind based on the metric tensor $g_{ij}(x^1,\cdots,x^n)$. See RIEMANN—Riemannian spaces.

center of geodesic curvature. The center of geodesic curvature of a curve C on a surface S at a point P of C is the *center of curvature*, relative to P, of the curve C', where C' is the orthogonal projection of C on the plane tangent to S at P. See below, geodesic curvature of a curve on a surface.

geodesic circle on a surface. If equal lengths are laid off from a point P of a surface S along the geodesics through P on S, the locus of the end points is an orthogonal trajectory of the geodesics. The locus of end points is called a geodesic circle with center at P and radius r. See below, geodesic polar coordinates.

geodesic coordinates in Riemannian space. Coordinates y^i such that the Christoffel symbols

$$\Gamma^i_{\alpha\beta}(y^1,\cdots,y^n)$$

are all zero when evaluated at the point in question which we take to be the origin $y^1=y^2=\cdots=y^n=0$. Thus the coordinate system is locally Cartesian. If the x^i are general coordinates, then the coordinate transformation

$$x^i=q^i+y^i-\frac{1}{2!}[\Gamma^i_{\alpha\beta}(x^1,\cdots,x^n)]_{x^i=q^i}\,y^\alpha y^\beta$$

defines implicitly geodesic coordinates y^i.

See below, geodesic parameters (coordinates) for a surface, and geodesic polar coordinates for a surface.

geodesic curvature of a curve on a surface (at a point). For a directed curve C: $u=u(s)$, $v=v(s)$, on a surface S: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, let π be the plane tangent to S at a point P of C, and let C' be the orthogonal projection of C on π. Let the positive direction of the normal to the cylinder K projecting C on C' be determined so that the positive directions of the tangent to C, the normal to K, and the normal to S, at P, have the same mutual orientation as the positive x, y, and z axes; and let ψ be the angle between the positive directions of the principal normal to C and the normal to K at P. Then the geodesic curvature $1/\rho_g$ of the curve C on the surface S at the point P is defined by $\frac{1}{\rho_g}=\frac{\cos\psi}{\rho}$, where ρ is the curvature of C at P. Thus the geodesic curvature of C is numerically equal to the curvature of C' and is positive or negative according as the positive directions of the principal normal to C and the normal to K at P lie on the same or opposite sides of the normal to S. If the positive direction on C is reversed, the geodesic curvature changes sign. See below, radius of geodesic curvature.

geodesic ellipse on a surface. Let P_1 and P_2 be distinct points on a surface S (or let C_1 and C_2 be curves on S such that C_1 and C_2 are not geodesic parallels of each other on S). Let u and v measure geodesic distances on S from P_1 and P_2 (or from C_1 and C_2), respectively. Then the curves $u'=$const. and $v'=$const., where $u'=\frac{1}{2}(u+v)$, $v'=\frac{1}{2}(u-v)$, are called **geodesic ellipses** and **hyperbolas**, respectively, on S relative to P_1 and P_2 (or to C_1 and C_2). These names are given because, for instance, the sum of the geodesic distances from P_1 and P_2 (or from C_1 and C_2) to a variable point of a fixed geodesic ellipse has a constant value.

geodesic hyperbola on a surface. See above, geodesic ellipse on a surface.

geodesic parallels on a surface. Given a smooth curve C_0 on a surface S, there exists a unique family of geodesics on S intersecting C_0 orthogonally; if segments of equal length s be measured along the geodesics from C_0, then the locus of their end points is an orthogonal trajectory C_s of the geodesics. The curves C_s are called **geodesic parallels** on S. See below, geodesic parameters for a surface.

geodesic parameters (coordinates) for a surface. Parameters u, v for a surface S such that the curves $u=$const. are the members of a family of geodesic parallels, while the curves $v=\text{const.}=v_0$ are members of the corresponding orthogonal family of geodesics, of length u_2-u_1 between the points (u_1, v_0) and (u_2, v_0). See above, geodesic parallels on a surface. A necessary and sufficient condition that u, v be geodesic parameters is that the first fundamental form of S reduce to $ds^2=du^2+G\,dv^2$. See below, geodesic polar coordinates for a surface.

geodesic polar coordinates for a surface. These are geodesic parameters u, v for a surface S, except that the curves $u=\text{const.}=u_0$, instead of being geodesic parallels, are concentric geodesic circles, of **radius** u_0, and center, or **pole,** P corresponding to $u=0$; the curves $v=v_0$ are the geodesic radii; and, for each v_0, v_0 is the angle at P between the tangents to $v=0$ and $v=v_0$. Necessary and sufficient conditions that u, v be geodesic polar coordinates are that the first fundamental quadratic form of S reduce to $ds^2=du^2+\mu^2\,dv^2$, $\mu\geqq 0$, and that at $u=0$ we have $\mu=0$ and $\partial\mu/\partial u=1$. All points on $u=0$ are singular points corresponding to P. See above, geodesic parameters for a surface.

geodesic radius. The radius of a geodesic circle on a surface; *i.e.*, the geodesic distance on the surface from the center of the circle to its boundary.

geodesic representation of one surface on another. A representation such that each geodesic on one surface corresponds to a geodesic on the other.

geodesic torsion of a curve on a surface at a point. The geodesic torsion of the surface at the point in the direction of the curve. See below, geodesic torsion of a surface at a point in a given direction.

geodesic torsion of a surface at a point in a given direction. The torsion of the geodesic on the surface through the point and in the given direction. See TORSION—torsion of a curve, and above, geodesic torsion of a curve on a surface at a point.

geodesic triangle on a surface. A triangle formed by three geodesics, intersecting by pairs on the surface. See CURVATURE—integral curvature of a geodesic triangle.

radius of geodesic curvature. The reciprocal of the geodesic curvature. See above, geodesic curvature of a curve on a surface.

umbilical geodesic. See UMBILICAL.

GE'O-GRAPH'IC, *adj.* Pertaining to the surface of the earth.

geographic coordinates. Same as SPHERICAL COORDINATES in the sense of coordinates on a sphere. Spherical coordinates use the longitude and colatitude of a point on a sphere of radius r.

geographic equator. See EQUATOR.

GE'O-MET'RIC, or GE'O-MET'RI-CAL, *adj.* Pertaining to geometry; according to rules or principles of geometry; done by geometry.

geometric average. The geometric average of n positive numbers is the positive nth root of their product. The geometric average of two numbers is the middle term in a geometric progression of three terms including the two given numbers. There are always two such means, but in common usage G.M. is understood to denote the positive root of the product, unless otherwise indicated. The *geometric means* of 2 and 8 are $\pm\sqrt{16}$ or ± 4. See AVERAGE. *Syn.* Geometric mean.

geometric figure. Any combination of points, lines, planes, circles, etc.

geometric locus. Any system of points, curves, or surfaces defined by certain general conditions or equations, such as the *locus of points* equidistant from a given point, or the *locus* of the *equation* $y=x$.

geometric magnitude. Any magnitude having a geometric interpretation; length, area, volume, angle, etc.

geometric mean. See above, geometric average.

geometric progression. See PROGRESSION.

geometric series. See SERIES.

geometric solid. Any portion of space which is occupied conceptually by a physical solid; *e.g.*, a cube or a sphere.

geometric solution. The solution of a problem by strictly geometric methods, as contrasted to algebraic or analytic solutions.

geometric surface. Same as SURFACE.

method of geometric exhaustion. A method used by the Greeks to find such areas as that of the ellipse, a segment of a parabola, etc. Consists of finding an increasing (or decreasing) sequence of areas (expressed in terms of familiar areas of plane geometry), whose terms are always less than (greater than) the desired area and increasing (decreasing); then (in modern terminology) showing that it approaches the desired area as a limit. The idea of exhaustion enters when it is argued that the desired area cannot be different from a certain value, since, if it were, it would either be less than some term of the increasing sequence or greater than some term of the decreasing sequence.

GE-OM'E-TRY, *n.* The science that treats of the shape and size of things. *Tech.* The study of invariant properties of given elements under specified groups of transformations.

analytic geometry. The geometry in which position is represented analytically (by coordinates) and algebraic methods of reasoning are used for the most part.

Euclidean geometry. The study of the ordinary 2- and 3-dimensional spaces studied by Euclid, or the study of Euclidean spaces of any number of dimensions. See EUCLIDEAN—Euclidean space.

metric differential geometry. The study, by means of differential calculus, of properties of general elements of curves and surfaces which are invariant under rigid motions.

plane analytic geometry. Analytic geometry in the plane (in two dimensions), devoted primarily to the graphing of equations in two variables and finding the equations of loci in the plane.

plane (elementary) geometry. The branch of geometry that treats of the properties and relations of plane figures (such as angles, triangles, polygons, circles) which can be drawn with ruler and compasses.

projective geometry. See PROJECTIVE.

solid analytic geometry. Analytic geometry in three dimensions; devoted primarily to the graphing of equations (in three

variables) and finding the equations of loci in space.

solid (elementary) geometry. The branch of geometry which studies figures in space (three dimensions) whose plane sections are the figures studied in plane elementary geometry, such as angles between planes, cubes, spheres, polyhedrons.

synthetic geometry. See SYNTHETIC—synthetic geometry.

GIBBS. Gibbs' phenomenon. Quite generally, for a sequence of transformations $T_n(x), n=1, 2, \cdots$, of a function $f(x)$, if the interval

$$[\liminf_{x\to x_0,\, n\to\infty} T_n(x), \limsup_{x\to x_0,\, n\to\infty} T_n(x)]$$

contains points outside the interval

$$[\liminf_{x\to x_0} f(x), \limsup_{x\to x_0} f(x)],$$

then the sequence is said to exhibit a Gibbs phenomenon at $x=x_0$. Most particularly, the phrase is applied to methods of summability of the Fourier series for a function having a single jump at x_0.

GIBRAT. Gibrat's distribution. If the logarithm of the variable x is *normally distributed*, x is distributed according to Gibrat's distribution:

$$G(x)=\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(\log x)^2}.$$

GIRTH, *n.* The length of the perimeter of a cross section of a surface when that length is the same for all right cross sections in planes parallel to the plane of that cross section.

GNO'MIC, *adj.* **gnomic map.** See AZIMUTHAL—azimuthal map.

GOLDBACH CONJECTURE. The conjecture (unproved) that any even number (except 2) can be represented as the sum of two prime numbers.

GOMPERTZ. Gompertz curve. A curve whose equation is of the form $\log y = \log k + (\log a)b^x$, or $y=ka^{b^x}$, where $0<a<1$, $0<b<1$. The value of y at $x=0$ is ka, and, as $x\to\infty$, $y\to k$. The increments in y as x increases are such that the difference of increments of $\log y$ are proportional to the corresponding differences in $\log y$. This is one of the types of curves known as **growth curves.**

Gompertz's law. The force of mortality (risk of dying) increases geometrically; is equal to a constant multiple of a power of a constant, the exponent being the age for which the force of mortality is being determined. See MAKEHAM'S LAW.

GRAD, *n.* One-hundredth part of a right-angle in the centesimal system of measuring angles, also called a GRADE OR DEGREE.

GRADE, *n.* (1) The slope of a path or curve. (2) The inclination of a path or curve, the angle it makes with the horizontal. (3) The sine of the inclination of the path, vertical rise divided by the length of the path. (4) An inclined path. (5) A class of things relatively equal. (6) A division or class in an elementary school. (7) A rating, given students on their work in a given course. (8) One-hundredth part of a right angle. See GRAD.

GRA'DI-ENT, *n.* (*Physics.*) The rate at which a variable quantity, such as temperature or pressure, changes in value; in these instances, called *thermometric gradient*, and *barometric gradient*, respectively.

gradient of a function. (*Vector analysis.*) The vector whose components along the axes are the partial derivatives of the function with respect to the variables. Written: $\text{grad} f = \nabla f = if_x + jf_y + kf_z$, where f_x, f_y and f_z are the partial derivatives of f, a function of x, y, and z. $\text{Grad} f(x, y, z)$ is a vector whose component in any direction is the derivative of f in that direction. Its direction is that in which the derivative of f has its maximum, and its absolute value is equal to that maximum. $\text{Grad} f$, evaluated at a point P: (x_1, y_1, z_1), is normal to the surface $f(x, y, z)=c$ at P, where c is the constant $f(x_1, y_1, z_1)$. See VARIATION—variation of a function on a surface.

method of conjugate gradients. See CONJUGATE.

GRAD'U-AT'ED, *adj.* Divided into intervals, by rulings or other marks, such as the graduations on a ruler, a thermometer or a protractor.

GRAEFFE. Graeffe's method for approximating the roots of an algebraic equation with numerical coefficients. The method consists of replacing the equation by an equation whose roots are the 2^kth power of the roots of the original equation. If the roots $r_1, r_2, r_3 \cdots$ are real and such that $|r_1| > |r_2| > |r_3| > \cdots$, then k can be made large enough that the ratio of r_1^{2k} to the coefficient of the next to the highest degree term is numerically as near unity as one desires and also the ratio of $r_1^{2k} r_2^{2k}$ to the coefficient of the third highest degree term is numerically as near unity as desired, etc. From these relations, $|r_1|, |r_2|, \cdots$ can be determined. If the roots are complex or equal, variations of this method can be used to obtain them.

GRAM, *n.* A unit of weight in the metric system; one-thousandth of a standard kilogram of platinum preserved in Paris. It was intended to be the weight of one cubic centimeter of water at 4°C. (the temperature at which its density is a maximum), and this is very nearly true. See DENOMINATE NUMBERS in the appendix.

GRAM-CHARLIER SERIES. (1) Type A: a series used in a certain system of deriving frequency functions based on a Fourier integral theorem. In particular, the frequency function

$$f(x)= \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)}\left[1+\frac{1}{3!} u_3 H_3+\frac{u_4-3}{4!} H_4+\cdots\right],$$

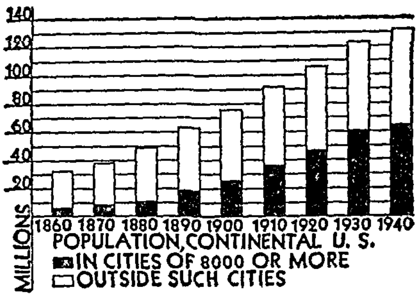
where x is in standard deviation units, u_i is the ith moment, and H_i are *Hermite polynomials*. Successive terms in the series do not necessarily diminish monotonically. Thus a satisfactory approximation may not be obtained by the first few terms. This is essentially a system of representing a given function by means of a series of derivatives of the normal distribution curve. (2) Type B: A *Poisson distribution*, instead of the normal, is used as a base for the series.

GRAM'I-AN, *n.* (1) For n vectors $u_1, u_2, \cdots, u_n$ in n dimensions, the determinant with $u_i \cdot u_j$ as the element in the ith row and jth column, where $u_i \cdot u_j$ is the *scalar product* of u_i and u_j (the *Hermitian scalar product* if u_i and u_j have complex components). This determinant being zero is a necessary and sufficient condition for the linear dependence of $u_1, \cdots, u_n$. (2) For n functions $\phi_1, \phi_2, \cdots, \phi_n$, the determinant with $\int_\Omega \phi_i\phi_j \, d\Omega$ as the element in the ith row and jth column. This determinant is zero if and only if the functions ϕ_i are linearly dependent in the interval or region of integration Ω if suitable restrictions are satisfied by the functions. *E.g.*, (a) that each ϕ_i be continuous; or (b) that each ϕ_i be (Lebesgue) measurable and $|\phi_i|$ be (Lebesgue) integrable (linearly dependent here meaning that there exist $a_1, \cdots, a_n$, not all zero, such that $\sum_{i=1}^{n} a_i\phi_i = 0$ *almost everywhere* in Ω). Under condition (b), the Gram determinants (1) and (2) become equivalent when the vectors and functions are regarded as elements of *Hilbert space*. *Syn.* Gram determinant.

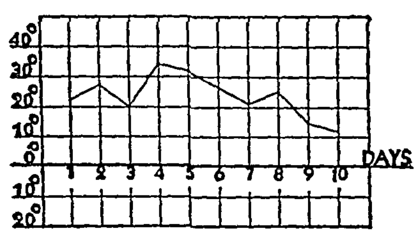
GRAPH, *n.* (1) A drawing which shows the relation between certain sets of numbers (see below, bar graph, broken line graph, circular graph). (2) A representation of some quantity by a geometric object, such as the representation of a complex number by a point in the plane (see COMPLEX—complex number). (3) A drawing which depicts a functional relation. *E.g.*, the graph of an equation in two variables is (*in the plane*) the curve which contains those points, and only those points, whose coordinates satisfy the given equation. *In space*, it is the cylinder which contains those points, and only those points, whose coordinates satisfy the given equation (*i.e.*, whose right section is the graph in a plane of the given equation). The graph of an equation in three variables is a surface which contains those points and only those points whose coordinates satisfy the equation. The graph of a first degree linear equation in Cartesian coordinates is a straight line in the plane or a plane in three dimensions. The graph of a set of simultaneous equations is either: (1) The graphs of all the equations, showing their intersections, or: (2) The intersection of the graphs of the equations.

bar graph. A graph consisting of parallel

bars (see figure) whose lengths are proportional to certain quantities given in a set of data. Used to convey a better idea of the meaning of the data than is derived directly from numbers.



broken line graph. A graph formed by segments of straight lines which join the points representing certain data. The days during a certain period of time might be indicated by successive, equally spaced points on the x-axis and ordinates drawn at each point proportional in length to the highest temperature on those days. If the upper ends of these ordinates be connected by line segments, a broken line graph results.



circular graph. A compact scheme for geometrically comparing parts of a whole to the whole. The whole is represented by the area of a circle, while the parts are represented by the areas of sectors of the circle.

GRAPH, *v.* To draw the graph of. See above, bar graph, broken line graph, etc.

GRAPH'IC-AL, or GRAPH'IC, *adj.* Pertaining to graphs, or drawings-to-scale; working by drawings-to-scale rather than with algebraic tools.

graphical solution of an equation, $f(x)=0$. Estimating the real roots from the graph of the equation $y=f(x)$. The real roots are the values of the variable for which the function is zero; hence they are the abscissas of the points at which the graph crosses the x-axis. See ROOT—root of an equation.

graphical solution of inequalities. Finding the region in the plane or space where the inequality, or inequalities, hold true. *E.g.*, (1) $x>2$ has for its solution all points in the region to the right of the line whose equation in rectangular coordinates is $x=2$; (2) the inequality $x^2+y^2+z^2<1$ has for its solution all points within the sphere $x^2+y^2+z^2=1$.

GRAPH'ING, *n.* Drawing the graph of an equation or the graph representing a set of data. See CURVE—curve tracing, and below, statistical graphing.

graphing by composition. A method of graphing which consists of writing the given function as the sum of several functions whose graphs are easier to draw, plotting each of these functions, then adding the corresponding ordinates. The graph of $y=e^x-\sin x$ can be readily drawn by drawing the graphs of each of the equations $y=e^x$ and $y=-\sin x$, then adding the ordinates of these two curves, which correspond to the same values of x. *Syn.* Graphing by composition of ordinates.

statistical graphing. Representing a set of statistics diagrammatically. *E.g.*, (1) a curve or broken line may be drawn through points whose ordinates represent the statistical values obtained at time intervals which are indicated on the axis of abscissas. (2) Adjoining bars may be drawn, from a common line, representing (in length) certain statistical values. This is called a **bar graph.** Many other schemes for representing statistics diagrammatically are employed. The fundamental idea in all of them is to enable the reader to study the statistics better than he could were they presented as a mere collection of numbers. See GRAPH—bar graph, broken line graph, and FREQUENCY—normal frequency curve.

GRAV'I-TA'TION, *n.* **law of universal gravitation.** The law of attraction, formulated by Newton, in accordance with which two particles of masses m_1 and m_2 interact so that the force of attraction is proportional to the product of the masses and

varies inversely as the square of the distance between the particles. In symbols, $F=k\frac{m_1m_2}{r^2}$, where r is the distance between the particles and k is the universal **constant of gravitation** whose value, determined by experiments, in the c.g.s. system of units, is 6.675×10^{-8} cm.^{3} per gram sec^2.

GRAV'I-TY, *n.* **acceleration of gravity.** See ACCELERATION—acceleration of a falling body.

center of gravity. See CENTER—center of mass.

GREAT, *adj.* **great circle.** See CIRCLE—great circle.

GREAT'ER, *adj.* One **cardinal number** is greater than a second when the set of units represented by the second is a part of that represented by the first, but not conversely; one cardinal number is greater than a second if the units of the first can be paired one-to-one with a subset of the units of the second, but not conversely. *E.g.*, 5 is greater than 3, since any set of 5 objects contains a set of 3 objects, but no set of 3 objects contains a set of 5 objects. One **real number** is greater than a second when the number must be added to the second to make them equal is positive; one real number is greater than a second when it is to the right of the second in the number scale: $\cdots -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots$. Thus 3 is greater than 2 (written $3>2$); and $-2>-3$, because 1 must be added to -3 to make -2. For **ordinal numbers** α and β which have ordinal types corresponding to well-ordered sets, α is greater than β if $\alpha\neq\beta$ and any set of ordinal type β can be put into a one-to-one, order-preserving correspondence with an initial segment of any set of ordinal type α. For any numbers A and B, the statements "A is less than B" and "B is greater than A" are equivalent.

GREAT'EST, *adj.* **greatest common divisor.** See DIVISOR.

greatest common measure. Same as GREATEST COMMON DIVISOR.

greatest lower bound. See BOUND.

GREEK ALPHABET. See the appendix.

GREEN. Green's formulas:

$$(1)\ \iiint u\,\nabla^2u\,dV+\iiint\nabla u\cdot\nabla u\,dV=\iint u\,\partial u/\partial n\,dS;$$

$$(2)\ \iiint u\,\nabla^2v\,dV+\iiint\nabla u\cdot\nabla v\,dV=\iint u\,\partial v/\partial n\,dS;$$

$$(3)\ \iiint u\,\nabla^2v\,dV-\iiint v\,\nabla^2u\,dV=\iint(u\,\partial v/\partial n-v\,\partial u/\partial n)\,dS.$$

The second of these may be obtained from Green's theorem:

$$\iiint\nabla\cdot\phi\,dV=\iint\phi\cdot\nu\,dS$$

by taking ϕ to be $u\nabla v$ so that $\nabla\cdot\phi=u\nabla^2v+\nabla u\cdot\nabla v$. The first is the special case of the second with $v=u$, and the third may be obtained from the second by permuting u and v and subtracting. The volume integrals are taken over a volume which meets the requirements of Green's theorem, while the surface integrals are taken over the entire boundary of the volume. The symbol $\partial u/\partial n$ denotes the directional derivative of u in the direction of the exterior normal to the surface, *i.e.*, $\partial u/\partial n=\nabla u\cdot\nu$ if ν is the unit exterior normal.

Green's function. For a region R with boundary surface S, and for a point Q interior to R, the Green's function $G(P, Q)$ is a function of the form $G(P, Q)=1/(4\pi r)+V(P)$, where r is the distance PQ, $V(P)$ is harmonic, and G vanishes on S. The solution $U(Q)$ of the Dirichlet problem can be represented in the form

$$U(Q)=-\iint_S f(P)\frac{\partial G(P,Q)}{\partial n}\,d\sigma_P.$$

Green's functions, Neumann's functions, and Robin's functions are sometimes called *Green's functions of the first, second, and third kinds*, respectively. See BOUNDARY—first boundary value problem of potential theory (the Dirichlet problem).

Green's theorem. (1) *In the plane*: Let R be a finite region in the plane and C its boundary. Then the *line integral* of $P\,dx+Q\,dy$ around C in a direction such as to keep the interior of R always on the left is

equal to the integral over R of $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$, provided P, Q, and these partial derivatives are continuous and single-valued throughout R and C. This is the special case of Stokes' theorem when the surface lies in the x-y plane. (2) *In space:* Let V be a region of space and S be its boundary. Then the integral of $P\,dy\,dz+Q\,dx\,dz+R\,dx\,dy$ over S is equal to the integral over V of $\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)$, provided P, Q, R, and these partial derivatives are single-valued and continuous throughout V and S. In vector notation, with $\mathbf{F}=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$, this is

$$\iint_S \mathbf{F}\cdot\boldsymbol{\nu}\,dS=\iiint_V \nabla\cdot\mathbf{F}\,dV,$$

where $\boldsymbol{\nu}$ is the unit vector normal to dS and pointing out of V and $\nabla\cdot\mathbf{F}$ is the *divergence* of $\mathbf{F}$. See above, Green's formulas. It is necessary to restrict the region R in (1) and V in (2). A sufficient restriction on R is that it can be divided into a finite number of regions such that the boundary of each region can be divided into two curves $y=f_1(x)$ and $y=f_2(x)$, and also into two curves $x=g_1(y)$ and $x=g_2(y)$, where f_1, f_2, g_1 and g_2 are continuous and single-valued. The analogous condition for V is also sufficient. *Syn.* Divergence theorem, Gauss' theorem, Ostrogradski's theorem.

GREGORY-NEWTON FORMULA. A formula for interpolation. If $x_0, x_1, x_2, x_3 \cdots$ are successive values of the argument, and $y_0, y_1, y_2, y_3, \cdots$ the corresponding values of the function, the formula is:

$$y=y_0+k\Delta_0+\frac{k(k-1)}{2!}\Delta_0{}^2+\frac{k(k-1)(k-2)}{3!}\Delta_0{}^3+\cdots,$$

where $k=(x-x_0)/(x_1-x_0)$, x (the value of the argument for which y is being computed) lies between x_0 and x_1,

$$\Delta_0=y_1-y_0,$$
$$\Delta_0{}^2=y_2-2y_1+y_0,$$
$$\Delta_0{}^3=y_3-3y_2+3y_1-y_0,$$

etc., the coefficients in $\Delta_0{}^n$ being the binomial coefficients of order n. If all the terms in the formula except the first two are dropped, the result is the ordinary interpolation formula used with logarithmic and trigonometric tables and in approximating roots of an equation, namely

$$y=y_0+[(x-x_0)/(x_1-x_0)](y_1-y_0).$$

(This is, incidentally, the two-point form of the equation of a straight line.)

GROSS, *adj., n.* Twelve dozen; 12×12.

gross capacity, price, profit, etc. The totality before certain parts have been deducted to leave the balance designated by the term *net*. *E.g.*, the *gross profit* is the sale price minus the initial or first cost. When the overhead charges have been deducted the remainder is the *net profit*.

gross premium. See PREMIUM.

gross tonnage. See TONNAGE.

GROUP, *n.* A set of elements subject to some rule of combination (usually called multiplication) such that: (1) the product of any two elements (alike or different) is unique and is in the set; (2) the set contains a unique element (called the **identity** or **unit** element) such that its product with any element, in either order, is again that same element; (3) for every element in the set there is an element (called the **inverse**) such that the product of the two, in either order, is the identity element; (4) the *associative law* holds. The number of elements in a group is called its order. The cube roots of unity form a group under ordinary multiplication. The positive and negative integers and zero form a group under ordinary addition, the identity being zero and the inverse of an element its negative. A group is **Abelian** (or **commutative**) if (in addition to the four assumptions listed above) it satisfies the commutative law. *I.e.*, $ab=ba$, where a and b are any two elements of the group. Any groups for which all elements are powers of one element is said to be **cyclic.** A cyclic group is necessarily Abelian. The cube roots of unity form a cyclic, Abelian group. A group containing only a finite number of elements is said to be a **finite group**; otherwise it is an **infinite group** (the set of all integers, with ordinary addition, is an infinite group). The number of elements in a finite group is called the **order** of the group (see PERIOD—period of an element of

a group). A group whose elements are elements of another group (and subject to the same rule of combination in both groups) is a **subgroup** of the latter. The group consisting of the cube roots of unity is a subgroup of the group consisting of the sixth roots of unity, the combination operation being ordinary multiplication. The product of any two elements within a subgroup is in the subgroup, but the product of one within the subgroup by one not in it is not in the subgroup.

alternating group. A group consisting of all even permutations on n objects. See PERMUTATION—permutation group.

composite group. See below, simple group.

direct product of groups. See PRODUCT—direct product of groups.

free group. See FREE—free group.

full linear group. The full linear group (of dimension n) is the group of all nonsingular matrices of order n with complex numbers as elements and matrix multiplication as the group operation.

fundamental group. See FUNDAMENTAL.

group character. See CHARACTER.

group without small subgroups. A topological group for which there is a neighborhood U of the identity such that the only subgroup completely contained in U is the subgroup consisting of the identity alone.

invariant subgroup. See INVARIANT—invariant subgroup.

Lie group. See LIE.

modular group. See MODULAR.

normal divisor or subgroup of a group. See INVARIANT—invariant subgroup of a group.

order of a finite group. See above, GROUP.

perfect group. See COMMUTATOR.

permutation group. See PERMUTATION—permutation group.

quotient (or factor) group. See QUOTIENT—quotient space.

real linear group. The real linear group (of dimension n) is the group of all nonsingular matrices of order n with real numbers as elements and matrix multiplication as the group operation. See above, full linear group.

representation of a group. See REPRESENTATION.

semigroup. See SEMI.

simple group. A group that has no *invariant subgroups* other than the identity alone and the whole group. A group which is not simple is called **composite.**

solvable group. A group that contains a sequence of invariant subgroups, beginning with itself and ending with the identity, such that: (1) Each invariant subgroup is contained in the preceding one; (2) the quotient of the order of any one of the invariant subgroups by the order of the following one is a prime integer.

symmetric group. A group of all permutations on n letters. See PERMUTATION—permutation group.

topological group. See TOPOLOGICAL—topological group.

GROUPING TERMS. A method of factoring consisting of rearranging terms, when necessary, inserting parentheses, and taking out a factor; *e.g.*,

$$\begin{aligned} x^3+4x^2-8-2x &= x^3+4x^2-2x-8 \\ &= x^2(x+4)-2(x+4) \\ &= (x^2-2)(x+4). \end{aligned}$$

GROWTH, *adj.* **growth curve.** (*Statistics.*) A curve designed to indicate the general pattern of growth of some variable. These are of several types. See GOMPERTZ CURVE, and LOGISTIC—logistic curve.

GU-DER-MANN'I-AN, *n.* The function u of the variable x defined by the relation $\tan u = \sinh x$; u and x also satisfy the relations $\cos u = \operatorname{sech} x$ and $\sin u = \tanh x$. The Gudermannian of x is written $\operatorname{gd} x$.

GY-RA'TION, *n.* **radius of gyration.** See RADIUS.

H

HAAR. Haar measure. See MEASURE—Haar measure.

HADAMARD. Hadamard's conjecture. The wave equation for $3, 5, \cdots$ space dimensions satisfies Huygens' principle, while that for 1 or an even number of space dimensions does not. Hadamard's conjecture is that no equation essentially different from the *wave equation* satisfies Huygens' principle. See HUYGENS' PRINCIPLE.

Hadamard's inequality. For a determinant of order n and value D, with real or complex elements a_{ij}, the inequality

$$|D|^2 \leq \prod_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right).$$

Hadamard's three-circle theorem. The theorem that if the complex function $f(z)$ is analytic in the ring $a<|z|<b$, and $m(r)$ denotes the maximum of $|f(z)|$ on a concentric circle of radius r in the ring, then $\log m(r)$ is a convex function of $\log r$. The name of the theorem, coined by Landau, reflects the fact that three radii are needed in order to express the convexity inequality. The result has been extended by Hardy to mean-value functions $m_t(r)$ of arbitrary nonnegative order t, of which the maximum-value function $m(r)$ is the limiting case as $t \to +\infty$.

HAHN-BANACH THEOREM. Let L be a linear subset contained in a Banach space B. Let f be a real-valued continuous linear functional defined on L. Then there is a real-valued continuous linear functional F defined on all of B such that $f(x)=F(x)$ if x is in L and the norm of f on L is equal to the norm of F on B. If B is a complex Banach space, then f and F can be complex-valued. See CONJUGATE—conjugate space.

HALF, *adj.* **half-angle formulas of plane trigonometry.** See TRIGONOMETRY.

half-angle and half-side formulas of spherical trigonometry. See TRIGONOMETRY.

half-line. A line terminated by a point in one direction and extending indefinitely (without limit) in the opposite direction. *Syn.* Ray.

half-plane. The part of a plane which lies on one side of a line in the plane. The line might or might not be included.

HAMEL. Hamel basis. If L is a vector space whose scalar multipliers are the elements of a field F, then it can be shown (using Zorn's Lemma) that there exists a set B of elements of L (called a **Hamel basis** for L) which has the properties that the elements of a finite subset of B are *linearly independent* and each element of L can be written as a *finite linear combination* (with coefficients in F) of elements of B. *E.g.*, there is a Hamel basis B (necessarily noncountable) for the real numbers regarded as a factor space with rational numbers as scalar multipliers; each real number x can be written as $\sum_1^n a_i b_{\alpha_i}$ in exactly one way, with the a_i's rational and the b_{α_i}'s in B.

HAMILTON. Hamilton-Cayley theorem. The theorem that every matrix satisfies its characteristic equation. See CHARACTERISTIC—characteristic equation of a matrix.

Hamiltonian. (1) In *classical particle mechanics*, a *function* of n generalized coordinates q_i and momenta p_i commonly symbolized by H and defined by

$$H=\sum_{i=1}^{n} p_i \dot{q}_i - L,$$

where p_i is the generalized momentum associated with q_i $(p_i = \partial L/\partial \dot{q}_i)$, $\dot{q}_i$ is the first time derivative of the ith **generalized coordinate,** L is a **Lagrangian function.** If the Lagrangian function does not contain the time explicitly, H is equal to the total energy of system. H satisfies the canonical equations of motion

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i \ (i=1, \cdots, n).$$

(2) In *quantum theory*, an *operator* H which gives the equation of motion for the wave function ψ in the form

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi.$$

Hamilton's principle. The principle that over short intervals of time, and in a conservative field of force, a particle moves in such a way as to minimize the action integral $\int_{t_1}^{t_2} (T-U)\,dt$, where $T=\frac{1}{2}m\Sigma \dot{q}_i \cdot \dot{q}_i$ denotes kinetic energy and $U=U(q_1, q_2, q_3)$ is the potential function satisfying $m\ddot{q}_i = -U_{q_i}$. Thus (in a conservative field of force) trajectories are extremals of the action integral.

HAN'DLE, *n.* **handle of a surface.** See GENUS—genus of a surface.

HANKEL. Hankel function. A function of one of the types

$$H_n^{(1)}(z)=\frac{i}{\sin n\pi}\left[e^{-n\pi i}J_n(z)-J_{-n}(z)\right]$$
$$=J_n(z)+iN_n(z),$$
$$H_n^{(2)}(z)=\frac{-i}{\sin n\pi}\left[e^{n\pi i}J_n(z)-J_{-n}(z)\right]$$
$$=J_n(z)-iN_n(z),$$

where J_n and N_n are Bessel and Neumann functions (limits of these expressions are used when n is a nonzero integer). The Hankel functions are solutions of Bessel's differential equation (if n is not an integer). Both $H_n^{(1)}$ and $H_n^{(2)}$ are unbounded near zero; they behave exponentially at ∞ (like e^{iz} and e^{-iz}, respectively). Also called *Bessel functions of the third kind.*

HAR-MON'IC, *adj.* **damped harmonic motion.** The motion of a body which would have simple harmonic motion except that it is subjected to a resistance proportional to its velocity. The equation of motion is

$$x=ae^{-ct}\cos(kt+\phi).$$

The exponential factor continuously reduces the amplitude. The differential equation of the motion is

$$\frac{d^2x}{dt^2}=-(c^2+k^2)x-2c\frac{dx}{dt}.$$

harmonic analysis. The study of the representation of functions by means of linear operations (*summation* or *integration*) on characteristic sets of functions; in particular, the representation by means of *Fourier series.*

harmonic average. See AVERAGE.

harmonic conjugates of two points. See CONJUGATE—harmonic conjugates with respect to two points.

harmonic division of a line. A line segment is said to be divided *harmonically* when it is divided externally and internally in the same ratio. See RATIO—harmonic ratio.

harmonic function. (1) A function $u(x, y)$ which satisfies Laplace's equation in two variables:

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0.$$

Some kind of regularity condition is usually assumed, such as that u has continuous partial derivatives of the first and second order in some given region. Two harmonic functions u and v are said to be conjugate harmonic functions if they satisfy the *Cauchy-Riemann partial differential equations*; *i.e.*, if and only if $u+iv$ is an *analytic function* (it is assumed here that u and v have continuous first-order partial derivatives). The conjugate of a harmonic function can be found by integration, using the Cauchy-Riemann equations. (2) A solution $u(x, y, z)$ of Laplace's equation in three variables:

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}=0.$$

Some kind of regularity condition is usually assumed, such as that u has continuous partial derivatives of the first and second order in some given region. (3) Sometimes a function of type $A\cdot\cos(kt+\phi)$ or $A\cdot\sin(kt+\phi)$ is called a harmonic function, or a simple harmonic function (see below, simple harmonic motion). Then a sum such as $3\cos x+\cos 2x+7\sin 2x$ is called a compound harmonic function.

harmonic mean between two numbers. Denoted by H. M. The middle term of three successive terms in an harmonic progression; the reciprocal of the arithmetic mean of their reciprocals; *e.g.*, the harmonic mean between 1 and $\frac{1}{3}$ is the reciprocal of the arithmetic mean between 1 and 3, which is $\frac{1}{2}$. Stated algebraically, the H. M. between a and b is the reciprocal of $\frac{1}{2}(1/a+1/b)$, which is $2ab/(a+b)$.

harmonic mean of n numbers. The reciprocal of the arithmetical mean of their reciprocals. The harmonic mean of $a_1, a_2, a_3, \cdots, a_n$ is $[(a_1^{-1}+a_2^{-1}+a_3^{-1}+\cdots+a_n^{-1})/n]^{-1}$.

harmonic progression. A sequence of quantities whose reciprocals form an arithmetic progression; denoted by H. P. In *music*, strings of the same material, same diameter, and the same tension, whose lengths are proportional to the terms in a harmonic progression, produce harmonic tones. The sequence $1, \frac{1}{2}, \frac{1}{3}, \cdots, 1/n, \cdots$, is an harmonic progression.

harmonic ratio. See RATIO—harmonic ratio.

harmonic series. See SERIES.

simple harmonic motion. Motion like that of the projection upon a diameter of a circle of a point moving with uniform speed around the circumference; the motion of a particle moving on a straight line under a force proportional to the particle's distance from a fixed point and directed toward that point. If the fixed point is the origin and the x-axis the line, the acceleration of the particle is $-k^2x$, where k is a constant. *I.e.*, the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -k^2x.$$

The general solution of this equation is $x = a\cos(kt+\phi)$. The particle moves back and forth (oscillates) between points at a distance a on either side of the origin. The time for a complete oscillation is $2\pi/k$. The distance a is called the **amplitude** and $2\pi/k$ the **period.** The angle $\phi + kt$ is called the **phase** and ϕ the initial phase.

spherical harmonic. A spherical harmonic of degree n is an expression of type

$$r^n\{a_nP_n(\cos\theta) + \sum_{m=1}^{n} [a_n^m \cos m\phi + b_n^m \sin m\phi]P_n^m(\cos\theta)\},$$

where P_n is a *Legendre polynomial* and P_n^m is an *associated Legendre function.* Any spherical harmonic is a homogeneous polynomial of degree n in x, y, and z and is a particular solution of Laplace's equation (in spherical coordinates); any solution of Laplace's equation which is analytic near the origin is the sum of an infinite series $\sum_0^\infty H_n$, where H_n is a spherical harmonic of degree n.

surface harmonic. A surface harmonic of degree n is an expression of type

$$a_nP_n(\cos\theta) + \sum_{m=1}^{n} [a_n^m \cos m\phi + b_n^m \sin m\phi]P_n^m(\cos\theta),$$

where P_n is a *Legendre polynomial* and P_n^m is an *associated Legendre function.* A surface harmonic of type $(\cos m\phi)P_n^m(\cos\theta)$ or $(\sin m\phi)P_n^m(\cos\theta)$ is called a **tesseral harmonic** if $m<n$, a **sectoral harmonic** if $m=n$, and is a solution of the differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 y}{\partial\theta^2} + n(n+1)y = 0.$$

A tesseral harmonic is zero on $n-m$ parallels of latitude and $2m$ meridians (on a sphere with center at the origin of spherical coordinates); a sectoral harmonic is zero along $2n$ meridians (which divide the surface of the sphere into sectors).

zonal harmonic. A function $P_n(\cos\theta)$, where P_n is the *Legendre polynomial* of degree n. The function $P_n(\cos\theta)$ is zero along n great circles on a sphere with center at the origin of a system of spherical coordinates (these circles pass through the poles and divide the sphere into n zones). See above, spherical harmonic.

HARVARD MARK I, II, III, IV. Certain automatic digital computing machines built at Harvard University.

HA'VER-SINE, *n.* See TRIGONOMETRIC—trigonometric functions of an acute angle.

HEAT, *adj., n.* **heat equation.** The parabolic second-order partial differential equation

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

where $u = u(x, y, z; t)$ denotes temperature, (x, y, z) are space coordinates, and t is the time variable; the constant k is the thermal conductivity of the body, c its specific heat, and ρ its density.

HEINE-BOREL THEOREM. If an infinite set M of intervals is such that each point of a given closed and bounded interval I is an interior point of at least one of the intervals of M, then there exists a finite number of intervals of M such that each point of I is an interior point of one of the intervals of this finite set. *Abstractly* (for metric spaces or topological spaces satisfying the second axiom of countability): If E is closed and compact and M an aggregate of open sets such that every element of E belongs to at least one of the sets of M, then there exists a finite number of sets of M such that each point of E belongs to at least one of these sets. In

this form, the theorem is frequently called the **Borel-Lebesgue theorem.**

HEL'I-COID, *n.* A surface generated by a plane curve or a twisted curve which is rotated about a fixed line as axis and also is translated in the direction of the axis in such a way that the ratio of the two rates is constant. A helicoid can be represented parametrically by equations $x = u\cos v$, $y = u\sin v$, $z = f(u) + mv$. For $m = 0$, the helicoid is a surface of revolution; and for $f(u) = \text{const.}$, the surface is a special right conoid called a right helicoid. See below, right helicoid.

right helicoid. A surface that can be represented parametrically by the equations $x = u\cos v$, $y = u\sin v$, $z = mv$. It is shaped rather like a propeller screw. If u is held fixed, $u \neq 0$, the equations define a helix (the intersection of the helicoid and the cylinder $x^2 + y^2 = u^2$). The right helicoid is the one and only real ruled minimal surface.

HE'LIX, *n.* A curve which lies on a cylinder or cone and cuts the elements under constant angle. See below, circular helix, conical helix, and cylindrical helix.

circular helix. A curve which lies on a right circular cylinder and cuts the elements of the cylinder under constant angle. Its equations, in parametric form, are $x = a\sin\theta$, $y = a\cos\theta$, $z = b\theta$, where a and b are constants and θ is the parameter. The thread of a bolt may be a circular helix. See HELIX.

conical helix. A curve which lies on a cone and cuts the elements of the cone under constant angle.

cylindrical helix. A curve which lies on a cylinder and cuts the elements of the cylinder under constant angle. A cylindrical helix is a circular helix if the cylinder is a right circular cylinder.

HELMHOLTZ. Helmholtz' differential equation. The equation

$$L\frac{dI}{dt} + RI = E.$$

The equation is satisfied by the current I in a circuit which has resistance R and inductance L, where E is the impressed or external electromotive force.

HEM'I-SPHERE, *n.* A half of a sphere bounded by a great circle.

HEP'TA-GON, *n.* A polygon having seven sides. A heptagon whose sides are all equal and whose interior angles are all equal is a **regular heptagon.**

HER, *n.* A game in which one player deals one card to his opponent and one to himself, at random from an ordinary deck of cards. Each looks only at his own card. The dealer's opponent may elect to keep his own card or to exchange cards with the dealer, except that the dealer is not compelled to relinquish a king. Thereafter, the dealer may elect to keep the card he then has or to exchange it for a new card dealt from the deck, except that if the new card is a king he must keep the card he already has. High card wins. This is a game with both personal moves and chance moves. See MOVE.

HERMITE. Hermite polynomials. The polynomials $H_n(x) = (-1)^n e^{x^2}\dfrac{d^n e^{-x^2}}{dx^n}$. The functions $e^{-\frac{1}{2}x^2}H_n(x)$ are *orthogonal functions* on the interval $(-\infty, \infty)$ with

$$\int_{-\infty}^{\infty} [e^{-\frac{1}{2}x^2}H_n(x)]^2\,dx = 2^n n!\sqrt{\pi}.$$

The Hermite polynomial H_n is a solution of *Hermite's differential equation* with the constant $\alpha = n$. For all n, $H_n'(x) = 2nH_{n-1}(x)$ and $e^{x^2-(t-x)^2} = \sum_{n=1}^{\infty} H_n(x)t^n/n!$.

Hermite's differential equation. The differential equation $y'' - 2xy' + 2\alpha y = 0$, where α is a constant. Any solution of Hermite's equation, multiplied by $e^{-\frac{1}{2}x^2}$, is a solution of $y'' + (1 - x^2 + 2\alpha)y = 0$.

Hermite's formula of interpolation. An interpolation formula for functions of period 2π. The formula, which is a trigonometric analogue of Lagrange's formula, is

$$f(x) = \frac{f(x_1)\sin(x - x_2)\cdots\sin(x - x_n)}{\sin(x_1 - x_2)\cdots\sin(x_1 - x_n)} + \cdots$$

to n terms. See LAGRANGE—Lagrange's formula of interpolation.

HERMITIAN, *adj.* **Hermitian conjugate of a matrix.** The transpose of the *complex conjugate* of the matrix. Called the **adjoint** of the matrix by some writers on quantum mechanics. *Syn.* Associate matrix. See ADJOINT—adjoint of a transformation.

Hermitian form. A bilinear form in conjugate complex variables whose matrix is Hermitian; an expression of the form

$$\sum_{i,j=1}^{n} a_{ij}x_i\bar{x}_j,$$

where $a_{ij}=\bar{a}_{ji}$. See TRANSFORMATION—conjunctive transformation.

Hermitian matrix. A matrix which is its own Hermitian conjugate; a square matrix such that a_{ij} is the complex conjugate of a_{ji} for all i and j, where a_{ij} is the element in the ith row and jth column.

Hermitian transformation. For bounded linear transformations (which include any linear transformation of a finite-dimensional space); same as SELF-ADJOINT TRANSFORMATION, or SYMMETRIC TRANSFORMATION (see SELF, and SYMMETRIC). For unbounded linear transformations, *Hermitian* usually means *self-adjoint*.

skew Hermitian matrix. A matrix which is the negative of its *Hermitian conjugate*; a square matrix such that a_{ij} is the complex conjugate of $-a_{ji}$ for all i and j, where a_{ij} is the element in the ith row and jth column.

HERON'S FORMULA. Same as HERO'S FORMULA. The latter name is usually given to it.

HERO'S FORMULA. A formula expressing the area of a triangle in terms of the sides, a, b, c. It is

$$A=\sqrt{s(s-a)(s-b)(s-c)},$$

where $s=\frac{1}{2}(a+b+c)$. Named after Heron, who was sometimes called Hero.

HES'SI-AN, *n.* For a function $f(x_1, x_2, \cdots, x_n)$ of n variables, the Hessian of f is the nth-order determinant whose element in the ith row and jth column is $\dfrac{\partial^2 f}{\partial x_i\,\partial x_j}$. The Hessian is analogous to the second derivative of a function of one variable, as a Jacobian is analogous to the first derivative. *E.g.*, the Hessian of a function $f(x, y)$ is

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2,$$

which is useful in determining maxima, minima, and saddle points (see MAXIMUM, and SADDLE—saddle point).

HEX'A-GON, *n.* A polygon of six sides.

regular hexagon. A hexagon whose sides are all equal and whose interior angles are all equal.

simple hexagon. Six points, no three of which are collinear, and the six lines determined by joining consecutive vertices.

HEX-AG'O-NAL, *adj.* **hexagonal prism.** A prism having hexagons for bases. See PRISM.

HEX'A-HE'DRON, *n.* A polyhedron having six faces.

regular hexahedron. See POLYHEDRON—regular polyhedron.

HIGH'ER, *adj.* **higher plane curve.** An algebraic curve of degree higher than the second. Sometimes includes transcendental curves.

HIGH'EST, *adj.* **highest common factor.** Same as GREATEST COMMON DIVISOR. See DIVISOR.

HILBERT. Hilbert parallelotope. See PARALLELOTOPE.

Hilbert-Schmidt theory of integral equations with symmetric kernels. A theory built on the orthogonality of *eigenfunctions* corresponding to distinct *eigenvalues*. Some characteristic results are: (1) $K(x, t)$ has at least one eigenvalue, all eigenvalues are real, and $\phi_1(x)$ and $\phi_2(x)$ are orthogonal if they are eigenfunctions corresponding to unequal eigenvalues; (2) there is an *orthonormal sequence* of eigenfunctions $\phi_i(x)$ corresponding to the eigenvalues λ_i (not necessarily all distinct) such that (*a*) if $\sum_{n=1}^{\infty}\dfrac{\phi_n(x)\phi_n(t)}{\lambda_n}$ is uniformly convergent for $a\leqq x\leqq b$, $a\leqq t\leqq b$, then it is equal to

$K(x, t)$, (b) if for the function $f(x)$ there is a continuous function $g(x)$ such that

$$f(x)=\int_a^b K(x, t)g(t)\,dt,$$

then

$$f(x)=\sum_{i=1}^{\infty} a_i\phi_i(x),$$

where

$$\lambda_i a_i=\int_a^b g(t)\phi_i(t)\,dt,$$

the series converging absolutely and uniformly; (4) if $f(x)$ is continuous and λ is not an eigenvalue, then

$$\theta(x)=f(x)+\lambda\int_a^b K(x, t)\theta(t)\,dt$$

has a unique continuous solution $\theta(x)$ given by

$$\theta(x)=f(x) +\lambda\sum_{n=1}^{\infty}\left[\frac{1}{\lambda_n-\lambda}\int_a^b f(t)\phi_n(t)\,dt\right]\phi_n(x),$$

the series being absolutely and uniformly convergent; (5) if λ is an eigenvalue, there is a solution if, and only if, $f(x)$ is orthogonal to each eigenfunction corresponding to λ and the solution is given by the above (with the terms for which $\lambda_n=\lambda$ omitted) plus any linear combination of these eigenfunctions.

Hilbert space. The space H of all sequences $x=(x_1, x_2, \cdots)$ of complex numbers, where $\sum_{i=1}^{\infty}|x_i|^2$ is finite. The sum $x+y$ is defined as $(x_1+y_1, x_2+y_2, \cdots)$, the product ax as $(ax_1, ax_2, \cdots)$, and the **inner product** or **Hermitian scalar product** as $(x, y)=\sum_{i=1}^{\infty} x_i\bar{y}_i$, where $x=(x_1, x_2, \cdots)$ and $y=(y_1, y_2, \cdots)$. *Abstractly*, Hilbert space is a *vector space* which satisfies the postulates: (1) There exists a numerically-valued function (x, y) defined for every pair of elements and having the properties: $(ax, y)=a(x, y)$ for all complex numbers a; $(x+y, z)=(x, z)+(y, z)$; $(x, y)=\overline{(y, x)}$; $(x, x)\geqq 0$ and $(x, x)=0$ only if $x=0$. (2) The space is *separable* and *complete*, the norm of an element x being defined as $(x, x)^{1/2}=\|x\|$ and a *neighborhood* of x as the sphere of all y satisfying $\|x-y\|<\epsilon$ for some fixed ϵ. (3) The space is *not* finite-dimensional, meaning that for any positive integer n there are n elements $x_1, \cdots, x_n$ such that $a_1x_1+a_2x_2+\cdots+a_nx_n=0$ is true only when $a_1, a_2, \cdots, a_n$ are all zero. Any two spaces satisfying these postulates are equivalent, *i.e.*, can be put into one-to-one correspondence in such a way that the correspondence preserves the operations of addition and multiplication by complex numbers and preserves the inner product (x, y). The above space of sequences is such a space. Another is the set of all complex-valued (Lebesgue) measurable functions f defined *almost everywhere* on an interval (a, b) for which $|f|^2$ is Lebesgue integrable. Two functions are considered identical if they are equal *almost everywhere* on (a, b); the operations of addition and multiplication by complex numbers are defined as ordinary addition and multiplication; and (f, g) is defined as $\int_a^b f(x)\overline{g(x)}\,dx$. If ordinary (Riemann) integration is used, all postulates are satisfied except that of completeness. A space satisfying all the postulates except (3) is called a **unitary space** (postulate (2) is then necessarily satisfied). It is sometimes not assumed that Hilbert space is separable, in which case a Hilbert space is equivalent to some space of the above type where the number of components x_a of an element x is not necessarily countable.

HIS'TO-GRAM, *n.* A graphic representation of a frequency function in which the several frequencies associated with the component intervals comprising the range of the variable are indicated by the areas of contiguous vertical bars. If the intervals are equal the heights serve as an exact measure. See FREQUENCY—frequency distribution.

HITCHCOCK. Hitchcock transportation problem. See TRANSPORTATION.

HOD'O-GRAPH, *n.* If the velocity vectors of a moving particle be laid off from a fixed point, the extremities of these vectors trace out a curve called the *hodograph* of the moving particle. The *hodograph* of uniform motion in a straight line is a point. The *hodograph* of uniform motion in a circle is another circle whose radius is equal to the speed of the particle. If $\bar{v}=f(t)$ is

HO'MOL'O-GOUS, *adj.* **homologous elements** (such as terms, points, lines, angles). Elements that play similar roles in distinct figures or functions. The numerators or the denominators of two equal fractions are *homologous terms.* The vertices of a polygon and those of a projection of the polygon on a plane are *homologous points* and the sides and their projections are *homologous lines.* *Syn.* Corresponding. Also see HOMOLOGY—homology group.

HO-MOL'O-GY, *adj.* **homology group.** Let K be an n-dimensional simplicial complex [or a topological simplicial complex or a complex in a suitable more general sense (see COHOMOLOGY—cohomology group)] and T^r be the set of all r-dimensional *cycles* of K, defined by use of a group G (see CHAIN—chain of simplexes). An r-dimensional cycle is **homologous to zero** if it is 0 or is the *boundary* of an $(r+1)$-dimensional chain of K, while two r-dimensional cycles are **homologous** if their difference is homologous to zero. The commutative quotient group T^r/H^r, where H^r is the group of all cycles which are homologous to zero, is called an **r-dimensional homology group** or **Betti group.** The elements of a homology group are therefore classes of mutually homologous cycles. This definition depends on the particular group G whose elements are used as coefficients in forming chains. However, if the homology groups over the groups of integers are known, the homology groups over any group G can be determined. The zero-dimensional homology group is isomorphic with the group G. If G is the group of integers, the 1-dimensional homology group of the torus has two generators of infinite order (a small circle around the torus and a large circle around the "hole"); the 1-dimensional homology group of the surface of an ordinary sphere contains only the identity (any two 1-cycles are homologous, any 1-cycle being a boundary of a 2-chain). See FUNDAMENTAL—fundamental group.

HO'MO-MOR'PHISM, *n.* A correspondence of a set D (the domain) with a set R (the range) such that each element of D determines a unique element of R and each element of R is the correspondent of at least one element of D. If R is a subset of D, the homorphism is said to be an **endomorphism.** If D and R are **topological spaces,** it is required that the correspondence be continuous (see CONTINUOUS—continuous correspondence of points). If operations such as multiplication, addition, or multiplication by scalars are defined for D and R, it is required that these correspond as described in the following. If D and R are **groups** (or semigroups) with the operation denoted by $\cdot$, and x corresponds to x^* and y to y^*, then $x\cdot y$ must correspond to $x^*\cdot y^*$. If D and R are **rings** (or integral domains or fields) and x corresponds to x^* and y to y^*, then xy must correspond to x^*y^* and $x+y$ to x^*+y^*. If D and R are **vector spaces,** multiplication and addition must correspond as for rings and scalar multiplication must correspond in the sense that if a is a scalar and x corresponds to x^*, then ax corresponds to ax^*. If the vector space is normed (*e.g.*, if it is a Banach space or Hilbert space), then the correspondence must be continuous. This is equivalent to requiring that there be a number M such that $\|x^*\| \leqq M\|x\|$ if x corresponds to x^* (a homomorphism of normed vector spaces is also a *bounded linear transformation*). See ISOMORPHISM, ISOMETRY, and IDEAL.

HO'MO-SCE-DAS'TIC, *adj.* (*Statistics.*) Having equal variance. *E.g.*, several *distributions* are homoscedastic if their variances are equal. In a *bivariate distribution,* one of the variables is homoscedastic if, for given values of the second variable, the variance of the first variable is the same regardless of the values of the second variable. In a *multivariate distribution,* one of the variables is homoscedastic if its conditional distribution function has a constant variance regardless of the particular set of values of the other variable.

HO-MO-THET'IC, *adj.* **homothetic figures.** Figures so related that lines joining corresponding points pass through a point and are divided in a constant ratio by this point.

homothetic transformation. See SIMILITUDE—transformation of similitude.

HOMOTOPIC, *adj.* See DEFORMATION—continuous deformation.

the vector equation of the path of the particle, $d\bar{v}/dt=\bar{f}'(t)$ is the equation of the *hodograph*.

HOLD'ER, *n.* (*Finance.*) The one who owns a note, not necessarily the payee named in the note. See NEGOTIABLE.

HÖLDER. Hölder's condition. See LIPSCHITZ—Lipschitz condition.

Hölder's definition of the sum of a divergent series. If the series is Σa_n, Hölder's definition gives the sum as

$$\lim_{n\to\infty} s_n' = \lim_{n\to\infty} \frac{s_1+\cdots+s_n}{n},$$

where

$$s_n = \sum_{i=1}^{n} a_i,$$

or

$$\lim_{n\to\infty} \frac{s_1'+\cdots+s_n'}{n},$$

where

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the vector equation of the path of the particle, $d\bar{v}/dt=f''(t)$ is the equation of the *hodograph.*

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HOOKE. Hooke's law. The basic law of proportionality of stress and strain published by Robert Hooke in 1678. In its simple form it states that within elastic limits of materials the elongation produced by the tensile force is proportional to the tensile force. If the elongation is denoted by e and the tensile stress by T, then $T = Ee$, where E is a constant depending on the properties of the material. The constant E is called the modulus in tension. This law is found experimentally to be valid for many substances when the forces and the deformations produced by them are not too great. See MODULUS—Young's modulus, and below, generalized Hooke's law.

generalized Hooke's law. The law in the theory of elasticity asserting that, for sufficiently small strains, each component of the stress tensor is a linear function of the other components of this tensor. The coefficients of the linear forms connecting the components of these tensors are elastic constants. It is known that the general elastic medium requires 21 such constants for its complete characterization. A homogeneous, isotropic elastic medium is characterized by two constants, Young's modulus and Poisson's ratio. See MODULUS—Young's modulus, and POISSON—Poisson's ratio.

HO-RI'ZON, *n.* **horizon of an observer on the earth.** The circle in which the earth, looked upon as a plane, appears to meet the sky; the great circle on the celestial sphere which has its pole at the observer's zenith. See HOUR—hour angle and hour circle.

HOR'I-ZON'TAL, *adj.* Parallel to the earth's surface looked upon as a plane; parallel to the plane of the horizon. *Tech.* In a plane perpendicular to the plumb line.

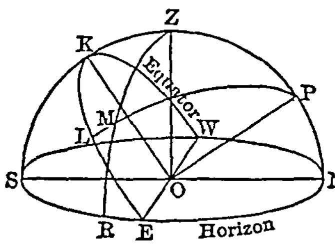
HORNER'S METHOD. A method for approximating the real roots of an algebraic equation. Its essential steps are as follows: (1) Isolate a positive root between two successive integers (if the equation has only negative real roots, *transform* it to one whose roots are the negatives of those of the given equation). (2) *Transform* the equation into an equation whose roots are decreased by the lesser of the integers between which the root lies, by the substitution $x' = x - a$. The root of the new equation will lie between zero and unity. (3) Isolate the root of the new equation between successive tenths. (4) *Transform* the last equation into an equation whose roots are decreased by the smaller of these tenths and isolate the root of this equation between hundredths. Continue this process to one decimal place more than the place to which the answer is to be correct. The root sought is then the total amount by which the roots of the original equation were reduced, namely, the lesser integer, plus the lesser tenth, plus the lesser hundredth, etc., the last decimal being rounded off to make the result accurate to the desired decimal place. Fractions may be avoided in locating the roots by transforming the equation in step (4) to one whose roots are ten times as large, and repeating the same transformation each time another digit in the root is sought. Synthetic division is generally used to expedite the work of substituting values for the variable. Roots often can be approximated quickly after the first step by solving the equation obtained by dropping the terms of higher degree than the first. See REMAINDER—remainder theorem, INTERPOLATION, and REGULA FALSI.

HORSEPOWER. A unit of power; a measure of how fast work is being done. Several values have been assigned to this unit. The one used in England and America is the *Watts horsepower.* It is defined as 550 foot-pounds per second, at sea level and 50° latitude. The Watts horsepower is 1.0139 times the French horsepower. See FOOT—foot-pound.

HOUR, *n.* One twenty-fourth of the average time required for the earth to make one complete revolution about its axis relative to the sun; *i.e.*, one twenty-fourth of a *mean solar day.* See TIME.

hour angle and hour circle. In the figure, let O be the place of the observer, $NESW$ the circle in which the plane of the observer's horizon cuts the celestial sphere, EKW the circle in which the plane of the earth's equator cuts the celestial sphere, NS the north-and-south line, and

east-and-west line. The circles $NESW$ and EKW are called respectively the astronomical **horizon** and the **celestial equator.** Z is the zenith and P the **north celestial pole.** $SZPN$ is called the **celestial meridian** or **meridian** of O. Let M be any celestial object, and draw great circles ZR and PL which pass through M and are perpendicular to the horizon and the equator, respectively. RM is the **altitude** of M, and NR its **azimuth.** LM is the **declination** of



M, and KOL is its **hour angle.** LP is called the **hour circle** of M. If M is north of the equator the declination is taken as positive; if south, as negative. The hour angle is positive if M is west of the meridian, negative if it is east of the meridian. The hour angle of a celestial object changes at the rate of 15° an hour or 360° a day (sidereal time), the hour circle passing through it appearing to rotate in the opposite direction to that of the earth, that is, to the west.

kilowatt-hour. See KILOWATT.

HULL, *n.* **convex hull of a set.** See CONVEX.

HUN′DRED, *n.* **hundred's place.** See PLACE—place value.

HUN′DREDTH, *adj.* **hundredth part of a number.** The quotient of the number and 100, or $\frac{1}{100}$ times the number.

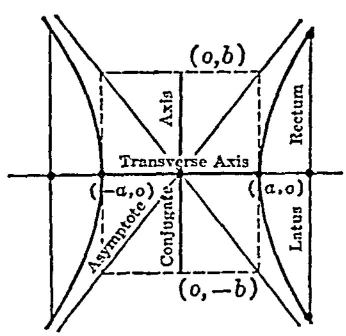
HUYGEN'S FORMULA. The length of an arc of a circle is *approximately* equal to twice the chord subtending half the arc plus one-third of the difference between twice this chord and the chord subtending the entire arc; or eight-thirds of the chord subtending half the arc minus one-third of the chord subtending the whole arc.

HUYGENS' PRINCIPLE. If, for an initial-value problem in a space of n dimensions, the domain of dependence of each point is a manifold of at most $n-1$ dimensions, then the problem is said to satisfy Huygens' principle. See DEPENDENCE—domain of dependence, HADAMARD—Hadamard's conjecture.

HY-PER′BO-LA, *n.* A curve with two branches which is a plane section of a circular conical surface; the locus of a point whose distances from two fixed points, called the **foci,** have a constant difference. The standard form of the equation in rectangular Cartesian coordinates is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

where the hyperbola is symmetric about the x- and y-axes and cuts the x-axis in the points whose coordinates are $(a, 0)$ and $(-a, 0)$, as in the figure below. The intercepts on the y-axis are imaginary. The axes of symmetry of the hyperbola are called the **axes of the hyperbola** (regardless of whether they coincide with the coordinate axes). The segment (of length $2a$) of the axis which cuts the hyperbola is called the **transverse** (real) axis and the **conjugate axis** is the line segment of length $2b$ (as illustrated). The line segments a and b are called the **semitransverse** and **semiconjugate** axes, respectively. (Transverse and conjugate axes are also used in speaking of the entire axes of symmetry.) If c is the distance from the center to a focus, then $c^2=a^2+b^2$ and the **eccentricity** of the hyperbola is c/a. Two hyperbolas are said to be **similar** if they have the same eccentricity. The extremities of the transverse axes are called the **vertices** of the hyperbola; the double ordinate at a focus, *i.e.*, the chord through a focus and perpendicular to the transverse axis, is called a **latus rectum.** See CONIC.



asymptote to a hyperbola. See ASYMPTOTE.

auxiliary circle of the hyperbola. See below, parametric equations of the hyperbola.

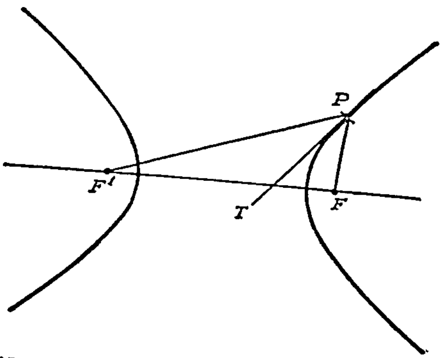
conjugate hyperbolas. Hyperbolas for which the real (transverse) and conjugate axes of one are, respectively, the conjugate and real axes of the other. Their standard equations are $x^2/a^2 - y^2/b^2 = 1$ and $x^2/a^2 - y^2/b^2 = -1$. They have the same asymptotes.

diameter of a hyperbola. See DIAMETER—diameter of a conic.

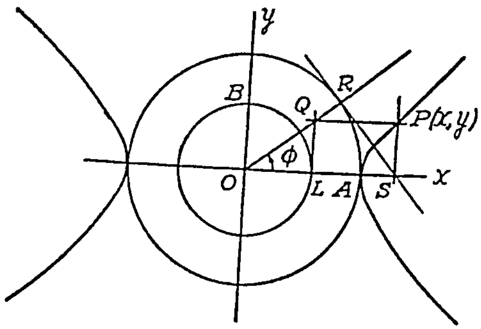
director circle of a hyperbola. See DIRECTOR.

equiangular (or equilateral) hyperbola. Same as RECTANGULAR HYPERBOLA.

focal (or reflection) property of the hyperbola. The angle formed by the focal radii drawn from any point P (see figure) on the hyperbola is bisected by the tangent line drawn to the hyperbola at P. If the hyperbola be constructed from a polished strip of metal, a ray of light emanating from one focus (F_2) is reflected along a line whose extension passes through the other focus (F_1).



parametric equations of the hyperbola. With the origin as center draw two circles with radii equal to the semiconjugate and semitransverse axes of the required hyperbola (see figure). These are called the **eccentric circles** of the hyperbola. Draw OR intersecting the eccentric circle of radius OA (the **auxiliary circle**) in R. Draw a tangent line to the auxiliary circle at R. It crosses the x-axis at S. Draw a tangent line to the other eccentric circle at L. This line intersects OR at Q. Through Q and S draw lines parallel, respectively, to the x- and y-axes. The intersection of these lines gives a point P on the hyperbola. The angle LOQ, designated by ϕ, is called the eccentric angle of the hyperbola. Letting a and b represent OA and OB, respectively, we find that the rectangular Cartesian coordinates of P are $x = a \sec\phi$, $y = b\tan\phi$. These equations are called the parametric equations of the hyperbola.



rectangular hyperbola. A hyperbola whose major and minor axes are equal. Its equation, in standard form, is $x^2 - y^2 = a^2$. The equations of the asymptotes are $y = x$ and $y = -x$. *Syn.* Equiangular hyperbola, equilateral hyperbola.

HY'PER-BOL'IC, *adj.* **hyperbolic cylinder.** See CYLINDER.

hyperbolic functions. The functions *hyperbolic sine* of z, *hyperbolic cosine* of z, etc., written $\sinh z$, $\cosh z$, etc. They are defined by the relations

$$\sinh z = \tfrac{1}{2}(e^z - e^{-z}), \quad \cosh z = \tfrac{1}{2}(e^z + e^{-z}),$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z},$$

$$\operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z}.$$

The Taylor's series for $\sinh z$ and $\cosh z$ are

$$\sinh z = z + z^3/3! + z^5/5! + \cdots,$$
$$\cosh z = 1 + z^2/2! + z^4/4! + \cdots.$$

The hyperbolic functions for z real are related to the hyperbola in a manner somewhat similar to the way the trigonometric functions are related to the circle. The hyperbolic and trigonometric functions are connected by the relations: $\sinh iz = i \sin z$, $\cosh iz = \cos z$, $\tanh iz = i \tan z$, where

$i^2 = -1$. Some of the properties of the hyperbolic functions are:

$$\sinh(-z) = -\sinh z,$$
$$\cosh(-z) = \cosh z,$$
$$\cosh^2 z - \sinh^2 z = 1,$$
$$\operatorname{sech}^2 z + \tanh^2 z = 1,$$
$$\coth^2 z - \operatorname{csch}^2 z = 1.$$

See EXPONENTIAL—exponential values of $\sin x$ and $\cos x$.

hyperbolic logarithms. Another name for *natural logarithms*. See LOGARITHM.

hyperbolic paraboloid. See PARABOLOID.

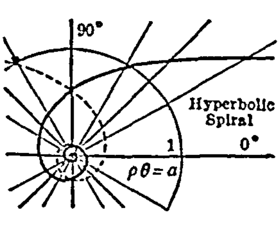
hyperbolic partial differential equation. A real second-order partial differential equation of the form

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} + F\left(x_1, \cdots, x_n, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right) = 0,$$

for which the quadratic form $\sum_{i,j=1}^{n} a_{ij} y_i y_j$ is *nonsingular* and *indefinite*; *i.e.*, by means of a real linear transformation the quadratic form can be reduced to a sum of n squares, but not all of the same sign. The term is often reserved for the case where all but one of the squares are of the same sign, although this case is also sometimes called **normal hyperbolic.** A typical example is the *wave equation*. See INDEX—index of a quadratic form.

hyperbolic point of a surface. A point of the surface at which the total curvature is negative; a point at which the Dupin indicatrix is a hyperbola.

hyperbolic (or reciprocal) spiral. A plane curve whose radius vector varies inversely with the vectorial angle. Its polar equation is $\rho\theta = a$, where a is the constant of proportionality. It is asymptotic to a straight line parallel to the polar axis and at a distance a above it.



inverse hyperbolic functions. The inverses of the hyperbolic functions; written $\sinh^{-1} z$, $\cosh^{-1} z$, etc., and read inverse hyperbolic sine of z, etc. Also called **arc-hyperbolic functions.** The explicit forms of the functions can be derived from the definitions of the hyperbolic functions; they are

$$\sinh^{-1} z = \log(z + \sqrt{z^2+1}),$$
$$\cosh^{-1} z = \log(z \pm \sqrt{z^2-1}),$$
$$\tanh^{-1} z = \tfrac{1}{2} \log \frac{1+z}{1-z},$$
$$\operatorname{ctnh}^{-1} z = \tfrac{1}{2} \log \frac{z+1}{z-1},$$
$$\operatorname{sech}^{-1} z = \log \frac{1 \pm \sqrt{1-z^2}}{z},$$
$$\operatorname{csch}^{-1} z = \log \frac{1 + \sqrt{1+z^2}}{z}.$$

See INVERSE—inverse function.

Riemann surface of hyperbolic type. See TYPE.

HY-PER'BO-LOID, *n.* Certain quadric surfaces having a (finite) center and some of its plane sections hyperbolas; a term referring to the so-called hyperboloids of one sheet and of two sheets.

asymptotic cone of a hyperboloid. See ASYMPTOTIC.

center of a hyperboloid. The point of symmetry of the hyperboloid. This is the intersection of the three principal planes of the hyperboloid.

confocal hyperboloids. See CONFOCAL—confocal quadrics.

conjugate hyperboloids. See CONJUGATE.

hyperboloid of one sheet. A hyperboloid that is cut in an ellipse or hyperbola by every plane parallel to a coordinate plane. If its equation is written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

planes parallel to the xy-plane cut the surface in ellipses, while planes parallel to the xz- or yz-plane cut it in hyperbolas (see figure). The surface is a *ruled surface*. It contains two sets of rulings (two families of straight lines), and through each point of the surface there passes one member of

each family. The equation of the two families of lines are:

$$\frac{x}{a}-\frac{z}{c}=p\left(1-\frac{y}{b}\right),$$

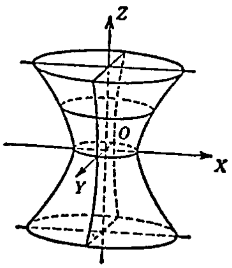
$$p\left(\frac{x}{a}+\frac{z}{c}\right)=1+\frac{y}{b},$$

and

$$\frac{x}{a}-\frac{z}{c}=p\left(1+\frac{y}{b}\right),$$

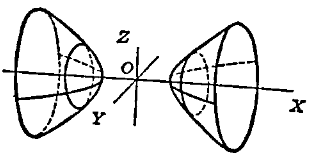
$$p\left(\frac{x}{a}+\frac{z}{c}\right)=1-\frac{y}{b},$$

where p is an arbitrary parameter. The product of the two equations in either set gives the original equation of the hyperboloid. Therefore the lines represented by these sets must lie on the hyperboloid.



Either set of rulings is called a set of **rectilinear generators,** since it may be used to generate the surface (see RULED—ruled surface). A **hyperboloid of revolution of one sheet** is a hyperboloid of one sheet whose elliptical sections parallel to the xy-plane, when in the position illustrated above, are circles. The parameters a and b are equal in this case and the surface can be generated by revolving the hyperbola, $x^2/a^2-z^2/c^2=1$, about the z-axis.

hyperboloid of two sheets. A surface whose sections by planes parallel to two of the three coordinate planes (see figure) are



hyperbolas and whose sections by planes parallel to the third plane are ellipses, except for a finite interval where there is no intersection (the intersection is imaginary). When the surface is in the position illustrated, its equation is of the form $x^2/a^2-y^2/b^2-z^2/c^2=1$. A **hyperboloid of revolution of two sheets** is a hyperboloid of two sheets whose elliptical sections are circles. The parameters b and c (in the equation of the hyperboloid of two sheets) are equal in this case, and the *hyperboloid of revolution of two sheets* can be generated by revolving the hyperbola, $x^2/a^2-y^2/c^2=1$, about the x-axis.

similar hyperboloids. See SIMILAR.

HY'PER-GE'O-MET'RIC, *adj.* **hypergeometric differential equation.** The differential equation

$$x(1-x)\frac{d^2y}{dx^2}+[c-(a+b+1)x]\frac{dy}{dx}-aby=0.$$

When $c\neq 1, 2, 3, \cdots$, the general solution (for $|x|<1$) is $y=c_1F(a, b;\ c;\ x)+c_2x^{1-c}F(a-c+1,\ b-c+1;\ 2-c;\ x)$, where $F(a, b; c; x)$ is the *hypergeometric function.* *Syn.* Gauss' differential equation.

hypergeometric function. For $|z|<1$, the hypergeometric function $F(a, b; c; z)$ is the sum of the hypergeometric series (see below). This function has an analytic continuation which is analytic in the complex plane with the line from $+1$ to $+\infty$ omitted. For $|x|>1$ and $a-b$ not an integer or zero, the hypergeometric function can be expressed in the form:

$$F(a, b; c; z) = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(b)\Gamma(a-c)}(-z)^{-a}\times F(a,\ 1-c+a;\ 1-b+a;\ z^{-1}) + \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(a)\Gamma(b-c)}(-z)^{-b}\times F(b,\ 1-c+b;\ 1-a+b;\ z^{-1}),$$

where z is not real and $\Gamma(z)$ is the *gamma function.* See JACOBI—Jacobi's polynomials, and GAUSS—Gauss' differential equation.

hypergeometric series. A series of the form

$$1+\sum_{n=1}^{\infty}\frac{\left[\begin{array}{r}a(a+1)\cdots(a+n-1)b(b+1)\cdots\\(b+n-1)z^n\end{array}\right]}{n!c(c+1)(c+2)\cdots(c+n-1)}$$

where c is not a negative integer. The series converges absolutely for $|z|<1$. A necessary and sufficient condition for it

to converge when $z=1$ is that $a+b-c$ be negative (or that its real part be negative if it is complex). Also called *Gaussian Series.* See GAUSS—Gauss' differential equation.

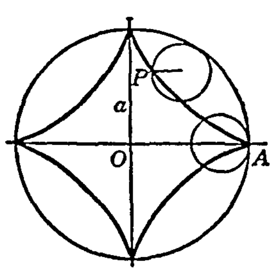
HY'PER-PLANE, *n.* A subset H of a linear space L such that H contains all x for which there are numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$, and elements $h_1, h_2, \cdots, h_n$ of H, satisfying $x=\Sigma\lambda_i h_i$ and $\Sigma\lambda_i=1$; it is also usually required that H be a *maximal* proper subset of this type (see BARYCENTRIC—barycentric coordinates). Equivalently, H is a hyperplane if there is a maximal linear subset M of L such that, for any element h of H, H consists precisely of all sums of type $x+h$ with x belonging to M. If L is a normed linear space, it is usually required that H be closed; this is equivalent to requiring the existence of a continuous linear functional f and a number c for which H is the set of all x with $f(x)=c$.

hyperplane of support. See SUPPORT—plane (and hyperplane) of support.

HY-PO-CY'CLOID, *n.* The plane locus of a point, P, fixed on a circle which rolls on the inside of a given fixed circle. If a is the radius of the fixed circle, b the radius of the rolling circle, and θ the angle subtended by the arc which has contacted the fixed circle, the parametric equations of the hypocycloid are:

$$x=(a-b)\cos\theta+b\cos[(a-b)\theta/b],$$
$$y=(a-b)\sin\theta-b\sin[(a-b)\theta/b].$$

The criteria for the number of arches is the same as for the *epicycloid.* The hypocycloid of four cusps (which is the case shown



in the figure) has the rectangular Cartesian equation $x^{2/3}+y^{2/3}=a^{2/3}$. The hypocycloid has a cusp of the first kind at every point at which it touches the fixed circle. See EPICYCLOID.

HY-POT'E-NUSE, *n.* The side opposite the right angle in a plane right triangle. See TRIANGLE.

HY-POTH'E-SIS, *n.* [*pl.* **hypotheses**]. (1) An assumed proposition used as a premise in proving something else; a condition; that from which something follows. See IMPLICATION. (2) A proposition held to be probably true because its consequences, according to known general principles, are found to be true. The word has always been applied in this sense to the theories of the planetary system. (3) (*Statistics.*) A statement specifying a population or distribution. It is framed in such a manner that it can be denied on the basis of sample evidence if the hypothesis is not true.

admissible hypothesis. (*Statistics.*) Any hypothesis that is regarded as possibly true.

composite hypothesis. (*Statistics.*) A statement which specifies a set of distributions by restricting certain or all of the parameters to a range. Any *nonsimple hypothesis* is composite.

linear hypothesis. (*Statistics.*) Under the assumption that the parameters B_i satisfy a set of linear relations involving x_{ij} $(i=1, \cdots p; j=1, \cdots N)$ of which the x_{ij} are normally and independently distributed with equal variances, it is a **linear hypothesis** that there are s equations, linearly independent of the preceding set, in p $(s\leq p)$ parameters B_i. The first set of equations is assumed *a priori* and the second set is subject to testing *via* the tests of a linear hypothesis. *E.g.*, in a linear regression problem, the linear regression is $y=f(x)$, which is *a priori* true, and the hypothesis that the values of the parameters in $f(x)$ are equal to some value (*e.g.*, zero) is the hypothesis to be tested. The *analysis of variance* is a special case of the general class of tests of linear hypotheses.

null hypothesis. A particular statistical hypothesis usually specifying the population from which a random sample is assumed to have been drawn, and which is to be nullified if the evidence from the random sample is unfavorable to the hypothesis, *i.e.*, if the random sample has a

low probability under the null hypothesis and a higher one under some admissible alternative hypothesis.

simple hypothesis. (*Statistics.*) A hypothesis that specifies the distribution exactly.

test of hypothesis. (*Statistics.*) A hypothesis specifying a distribution (in n dimensions) may be **tested** in the sense that randomly obtained observations or evidence may be evaluated to determine the probability of drawing such evidence by random sampling methods under the assumption that the hypothesis is correct. Also called a **test of significance.** The possibility of alternative hypotheses being true is implicit in the form of the question. If the evidence, or sample, proves to be improbable (to an arbitrary and discretionary degree) and if it is probable under other alternative admissible hypotheses, one may be willing to reject the tested (null) hypothesis. The relative frequency with which one is willing to erroneously reject the tested hypothesis (when in fact it was correct) is discretionary, and the relative frequency or probability of such rejection is called the **size of the critical region** of the test. The **critical region** constitutes that set of samples that leads to rejection of the tested hypothesis. The larger this size or level, the more probable it is that the tested hypothesis will be rejected, when it should be so rejected (*i.e.*, when some alternative is the correct hypothesis). The probability of rejecting the tested hypothesis when some alternative is true is called the **power of the test** with respect to that alternative. To be **discriminatory** or **unbiased** a test should have a power with respect to alternative hypotheses that is greater than the size of the critical region (or level of significance). If it is not, the test is a **biased test.** The power of a test may be increased by increasing the size of the sample, and also by choosing that critical region which maximizes the power, since not all critical regions have the same power with respect to certain alternative hypotheses. *E.g.*, the tested hypothesis may be that a population of light bulbs has an average life length of 2500 hours in a normal distribution. If some alternative is true, say 3000 hours, the obtained sample may prove to be a possible sample under either hypothesis. A set of possible samples must be prescribed in advance such that if a sample is observed to fall in that set, the tested hypothesis of 2500 hours will be rejected. Clearly this set should be one in which it is as probable as possible that a sample will fall if some alternative, *e.g.*, 3000 hours, is true. The set of samples with means of over 2800 hours may be selected as that set for which there is a probability of .05 of observing a sample if the tested hypothesis of 2500 hours is true. On the other hand, if 3000 hours is the truth about the mean, the probability of getting a randomly drawn sample that has a mean of over 2800 hours may be .80. Thus the size of the critical region, or the **level of significance**, is .05 and the **power** of that region with respect to the 3000 alternative hypothesis is .80. If it is admitted that the life length may be less than 2500 hours, the critical region should include samples with means that are less than some particular value, selected so that the critical region should include, *e.g.*, .05 of the possible samples. Then the probability of rejecting the 2500-hour hypothesis on the basis of this critical region will now increase as the truth deviates farther in *either* direction from the 2500-hour hypothesis.

HY-PO-TRO′CHOID, *n.* Same as *hypocycloid*, except that the describing point may lie within the circle or on the radius extended. If h is the distance from the center of the rolling circle to the describing point, and the other parameters are the same as for the *hypocycloid*, the parametric equations are:

$$x=(a-b)\cos\theta+h\cos[(a-b)\theta/b],$$
$$y=(a-b)\sin\theta-h\sin[(a-b)\theta/b].$$

The cases when h is less than b, or greater than b, are similar to the corresponding cases for the *trochoid* (see TROCHOID).

I

I′CO-SA-HE′DRON, *n.* A polyhedron having twenty faces.

regular icosahedron. An icosahedron whose faces are congruent equilateral triangles and whose polyhedral angles are

congruent. See POLYHEDRON—regular polyhedron.

I-DE'AL, *adj., n.* Let R be a set which is a ring with respect to operations called addition and multiplication (it may also be an integral domain, algebra, etc.). A subset I which is an additive *group* (or, equivalently, a subset I which is such that $x-y$ belongs to I whenever x and y belong to I) is called a **left ideal** if cx belongs to I whenever c belongs to R and x belongs to I (it is a **right ideal** if xc belongs to I whenever c belongs to R and x belongs to I). It is a **two-sided ideal** (or simply an **ideal**) if cx and xc belong to I whenever c belongs to R and x belongs to I. For any ideal I in a ring R, there is a homomorphism which maps R onto the *quotient ring* R/I. This homomorphism maps each member of I onto zero. Also, R/I is isomorphic to the image of R under any homomorphism for which I is the set of those elements which map onto zero. A subset I of a ring is an ideal if there is a homomorphism of the ring for which I is the set of those elements which map onto zero. An ideal is a **principal ideal** if it contains an element such that all elements of the ideal are multiples of this element. *E.g.*, the set of even integers is a principal ideal in the integral domain of all integers. The product AB of two ideals A and B is the ideal obtained by multiplying every member of A by every member of B and then forming all possible sums of these products. If D is the integral domain of all algebraic integers and a **prime ideal** in D is one which is not the ideal I of all multiples of 1 and has no factors other than itself and 1, then any ideal of D can be represented uniquely (except for order of the factors) as a product of prime ideals. See ALGEBRAIC—algebraic number, MODULE, and QUOTIENT—quotient space.

ideal point. A point at infinity; a term used to complete the terminology of certain subjects (*e.g.*, projective geometry) so that it is not necessary to state exceptions to certain theorems. Instead of saying that two straight lines in the same plane intersect except when they are parallel, it is said that two straight lines in a plane always intersect, intersecting in the ideal point being synonymous with being parallel. Thus an *ideal point* is thought of as a direction, the direction of a certain set of parallel lines. In *homogeneous coordinates*, the ideal points are the points for which $x_3=0$; the points $(x_1, x_2, 0)$, where x_1 and x_2 are not both zero. The point $(x_1, x_2, 0)$ lies on any line whose slope is equal to x_2/x_1. See COORDINATE—homogeneous coordinates in the plane, and INFINITY—point at infinity in the complex plane.

I-DEM-FAC'TOR, *n.* The dyadic $\mathbf{ii}+\mathbf{jj}+\mathbf{kk}$, called *idemfactor* because its *scalar product* with any vector in either order does not change the vector. See DYAD.

I-DEM'PO-TENT, *adj.* An idempotent quantity is one unchanged under multiplication by itself. Unity and the matrix

$$\left|\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 0\end{matrix}\right|$$

are idempotent.

I-DEN'TI-CAL, *adj.* **identical figures.** Figures that are exactly alike in form and size; two triangles with three sides of one equal to three sides of the other are identical. *Syn.* Congruent.

identical quantities. Quantities which are alike in form as well as value. Quantities which form the left and right members of an *identity* are not necessarily identical, usually being different in form although always having the same values for all values of the variables.

I-DEN'TI-TY, *adj., n.* A statement of equality, usually denoted by $\equiv$, which is true for all values of the variables (with the exception of values of the variables for which each member of the statement of equality does not have meaning) if there be any involved (see EQUATION). The two quantities which are equated are the same except possibly in form. $2+3\equiv 5$, $2\times 3\equiv 6$, and $(x+y)^2\equiv x^2+2xy+y^2$ are identities. The equality sign, $=$, is quite commonly used in place of the identity sign, $\equiv$.

identity matrix. See MATRIX.

Pythagorean identities and other trigonometric identities. See TRIGONOMETRY—identities of plane trigonometry.

ILLIAC. An automatic digital computing machine built at the University of Illinois.

IM′AGE, *n.* **image of a point.** See MAPPING.

spherical image. See SPHERICAL.

IM-AG′I-NA′RY, *adj.* **imaginary axis.** See COMPLEX—complex numbers, and ARGAND DIAGRAM.

imaginary curve (surface). A term used to provide continuity in speaking of loci of equations, the imaginary part of the curve (surface) corresponding to imaginary values of the variables which satisfy the equation. The equation $x^2+y^2+z^2=1$ has for its real locus the sphere with radius one and center at the origin, but is also satisfied by $(1, 1, i)$ and many other points whose coordinates are not all real. See CIRCLE—imaginary circle, ELLIPSE, ELLIPSOID, and INTERSECTION.

imaginary number. See COMPLEX—complex number.

imaginary part of a complex number. If the complex number z is written in the form $z=x+iy$, where x and y are real, then the *imaginary part* of z is y, written $I(z)=y$, $\operatorname{Im}(z)=y$, or $\Im(z)=y$.

imaginary roots. Roots of an equation or number which are complex numbers whose imaginary part is not zero. *E.g.*, the roots of $x^2+x+1=0$ are $-\frac{1}{2}\pm\frac{1}{2}\sqrt{3}\cdot i$. See FUNDAMENTAL—fundamental theorem of algebra, ROOT—root of a number, and COMPLEX—complex number.

IM-BED′, *v.* See SPACE—enveloping space.

IM GROSSEN. *German* for IN THE LARGE. See SMALL—in the small.

IM KLEINEN. *German* for IN THE SMALL. See SMALL—in the small.

IM-ME′DI-ATE, *adj.* **immediate annuity.** See ANNUITY.

IM′PLI-CA′TION, *n.* A proposition formed from two given propositions by connecting them in the form "*If*$\cdots$, *then*$\cdots$." The first statement is the **antecedent** (or **hypothesis**) and the second the **consequent** (or **conclusion**). An implication is true in all cases except when the antecedent is true and the consequent is false. *E.g.*, the following implications are true: "If $2\cdot3=7$, then $2\cdot3=8$"; "If $2\cdot3=7$, then $2\cdot3=6$"; "If $2\cdot3=6$, then $3\cdot4=12$." Such a proposition as "If a quadrilateral is a square, then it is a parallelogram" can be written as "For any quadrilateral x, if x is a square, then x is a parallelogram," which is true, since when x is a specific quadrilateral the expression "If x is a square, then x is a parallelogram" is a true proposition. For propositions p and q, the implication "*if* p, *then* q" is usually written as $p\rightarrow q$, or $p\supset q$, and read "p *implies* q." The implication $p\rightarrow q$ has the same meaning as the propositions "p is a *sufficient condition* for q," or "q is a *necessary condition* for p." *Syn.* Conditional statement (or proposition). See CONVERSE—converse of an implication, and EQUIVALENCE—equivalence of propositions.

IM-PLIC′IT, *adj.* **implicit differentiation.** See DIFFERENTIATION—implicit differentiation.

implicit function. A function defined by an equation of the form $f(x, y)=0$ (in general $f(x_1, x_2, \cdots, x_n)=0$). If y is thought of as the dependent variable, $f(x, y)=0$ is said to define y as an *implicit* function of x. Sometimes such equations can be solved for y, *i.e.*, written in the form $y=F(x)$. When this has been done, y is called an **explicit** function of x. In $x+y^3+2x^2y+xy=0$, y is an *implicit* function of x, while in $y=x^2+1$, y is an *explicit* function of x.

implicit function theorem. A theorem stating conditions under which an equation, or a system of equations, can be solved for certain dependent variables. For a **function of two variables,** the implicit function theorem states conditions under which an equation in two variables possesses a unique solution for one of the variables in a neighborhood of a point whose coordinates satisfy the given equation. *Tech.* If $F(x, y)$ and $D_yF(x, y)$, the partial derivative of F with respect to y, are continuous in the neighborhood of the point (x_0, y_0) and if $F(x_0, y_0)=0$ and $D_yF(x_0, y_0)\neq0$, then there is a number $\epsilon>0$ such that there exists one and only one function, $y=f(x)$, which is such that $y_0=f(x_0)$ and which is continuous and satisfies $F[x, f(x)]=0$ for $|x-x_0|<\epsilon$. *E.g.*, the function x^2+xy^2+y-1 and its partial derivative with respect to y, namely, $2xy+1$, are both continuous in the neigh-

borhood of $(1, 0)$, and $x^2+xy^2+y-1=0$ while $2xy+1 \neq 0$ when $x=1, y=0$. Hence there exists a unique solution for y, in the neighborhood of $(1, 0)$, which gives $y=0$ for $x=1$. That solution is

$$y=\frac{-1+\sqrt{1-4x(x^2-1)}}{2x}.$$

The **general implicit function theorem** states conditions under which a system of $n+p$ equations in n dependent variables and p independent variables possesses solutions for the dependent variables in a neighborhood of a point whose coordinates satisfy the given equations. Consider a system of n equations between the $n+p$ variables

$$u_1, u_2, \cdots, u_n, \quad \text{and} \quad x_1, x_2, \cdots, x_p,$$

namely

$$f_1(x_1, x_2, \cdots, x_p; \quad u_1, u_2, \cdots, u_n)=0,$$
$$f_2(x_1, x_2, \cdots, x_p; \quad u_1, u_2, \cdots, u_n)=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f_n(x_1, x_2, \cdots, x_p; \quad u_1, u_2, \cdots, u_n)=0.$$

Suppose that these equations are satisfied for the values $x_1=x_1^0, \cdots, x_p=x_p^0$, $u_1=u_1^0, \cdots, u_n=u_n^0$, that the functions f_i are continuous in the neighborhood of this set of values and possess first partial derivatives which are continuous for this set of values of the variables and, finally, that the Jacobian of these functions does not vanish for $x_i=x_i^0$, $u_k=u_k^0$ $(i=1, 2, \cdots, p;$ $k=1, 2, \cdots, n)$. Under these conditions there exists one and only one system of continuous functions

$$u_1=\phi_1(x_1, x_2, \cdots, x_p)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u_n=\phi_n(x_1, x_2, \cdots, x_p)$$

defined in some neighborhood of

$$(x_1^0, x_2^0, \cdots, x_p^0)$$

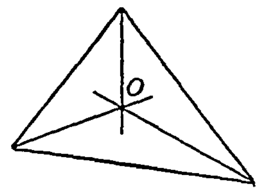
which satisfy the above equations and which reduce to

$$u_1^0, u_2^0, \cdots, u_n^0, \text{ for } x_1=x_1^0, x_2=x_2^0, \cdots, x_p=x_p^0.$$

IM-PROP'ER, *adj.* **improper fraction.** See FRACTION.

improper integral. See INTEGRAL.

IN'CEN'TER, *n.* **incenter of a triangle.** The center of the inscribed circle; the intersection of the bisectors of the interior angles of the triangle.



INCH, *n.* A unit of measure of length, or distance; one twelfth of one foot, approximately 2.54 centimeters. See DENOMINATE NUMBERS in the appendix.

IN-CIR'CLE, *n.* Same as INSCRIBED CIRCLE.

IN'CLI-NA'TION, *n.* **inclination of a line in the plane.** An angle from the positive direction of the x-axis to the line, the angle usually being taken to be greater than or equal to 0° and less than 180° (see ANGLE —angle of intersection.

inclination of a line in space with respect to a plane. The smaller angle the line makes with its orthogonal projection in the plane.

inclination of a plane with respect to a given plane. The smaller of the dihedral angles which it makes with the given plane.

IN'COME, *n.* **income rate.** (*Finance.*) Same as YIELD.

income tax. A tax on incomes (salaries or profits), levied by the federal or other governments. The tax is determined by taking a certain per cent of the remainder of the income after certain deductions (exemptions) have been made.

IN'COM-MEN'SU-RA-BLE, *adj.* **incommensurable line segments.** Two line segments that have no common measure, such as the hypotenuse and a leg of an isosceles right triangle; two lines whose lengths are represented by incommensurable numbers.

incommensurable numbers. Numbers that are not both integral multiples of the same number. The numbers $\sqrt{2}$ and 3 are incommensurable, for no number is contained in both an integral number of times.

IN'COM-PAT'I-BLE, *adj.* **incompatible equations.** Same as INCONSISTENT EQUATIONS.

IN'COM-PLETE', *adj.* **incomplete beta and gamma functions.** See BETA—beta function, GAMMA—gamma function.

incomplete induction. See INDUCTION—mathematical induction.

IN-CON-SIS'TENT, *adj.* **inconsistent equations.** Two (or more) equations that are not satisfied by any one set of values of the variables; equations that are not consistent. *E.g.*, $x+y=2$ and $x+y=3$ are inconsistent. See CONSISTENCY. *Syn.* Incompatible.

IN-CREAS'ING, *adj.* **increasing function** of a single variable. A function whose value increases as the independent variable increases; a function whose graph, in Cartesian coordinates, rises as the abscissa increases. If a function possesses a derivative, then the function is increasing in an interval if the derivative is non-negative throughout the interval, provided the derivative is not identically zero in any interval. An increasing function is often said to be **strictly increasing,** to distinguish it from a *monotonic increasing* function. *Tech.* A function f is strictly increasing on an interval (a, b) if

$$f(y) > f(x)$$

for any two numbers x and y (of this interval) for which $x<y$.

increasing premium policy. An insurance policy on which the early premiums are smaller than the later ones.

monotonic increasing. See MONOTONIC.

IN'CRE-MENT, *n.* A change in a variable; an amount added to a given value of a variable, usually thought of as a small amount (positive or negative).

increment of a function. The change in the function due to changes in the values of the independent variables (these changes in the independent variables are called *increments of the independent variables* and may be either positive or negative). If the function is $f(x)$ and the change in the independent variable, x, is Δx, then the increment in $f(x)$ is $\Delta f = f(x+\Delta x) - f(x)$. If $f(x)$ has a derivative, then

$$f(x+\Delta x) - f(x) = f'(x)\Delta x + \epsilon \cdot \Delta x,$$

where ϵ is an *infinitesimal* which approaches zero as Δx approaches zero; $f'(x)\Delta x$ is the **principal part** of Δf, or the **differential** of f. If $u=u(x, y)$ is a function of x and y, the increment of u, written Δu, is equal to $u(x+\Delta x, y+\Delta y) - u(x, y)$. By the use of the *mean value theorem* for a function of two variables this expression can be written as the sum $(D_x u \Delta x + D_y u \Delta y) + (\epsilon_1 \Delta x + \epsilon_2 \Delta y)$, where $D_x u$ and $D_y u$ denote the partial derivatives of u with respect to x and y respectively, and ϵ_1 and ϵ_2 are infinitesimals which approach zero as Δx and Δy approach zero, provided there is a neighborhood of the point (x, y) in which $D_x u$ and $D_y u$ both exist and one (at least) is continuous. The sum of the two terms in the first parenthesis is called the **principal part** of Δu, or **total differential** of u, written $du = D_x u\,dx + D_y u\,dy$. The increments of x and y, Δx and Δy, have been written dx and dy, in order to avoid confusion of notation in the cases $u=x$ and $u=y$. The increments dx and dy are also the *differentials* of the *independent* variables x and y. See DIFFERENTIAL.

IN-DEF'I-NITE, *adj.* **indefinite integral.** See INTEGRAL—indefinite integral.

IN'DE-PEND'ENCE, *n.* **statistical independence.** The probability function of x and y jointly, $f(x, y)$, is equal to $f(x)$ times $f(y)$ if, and only if, x and y are **statistically independent.** For any number of variables, $x_1, \cdots, x_n$, if the joint probability is the product of the several probability functions, then the variables are all statistically independent. Independent variables are noncorrelated, but not necessarily conversely.

stochastic independence. Same as STATISTICAL INDEPENDENCE.

IN'DE-PEND'ENT, *adj.* **independent equations.** A system of equations such that no one of them is necessarily satisfied by a set of values of the independent variables which satisfy all the others. See CONSISTENCY.

independent events. See EVENT.

independent functions. A set of functions $u_1, u_2, \cdots u_n$, where $x_1, x_2, \cdots x_n$ are the independent variables, such that there does not exist a relation $F(u_1, u_2, \cdots, u_n) \equiv 0$, where not all $\partial F/\partial u_i$ are identically zero. The functions are independent if and only if the *Jacobian*

$$\frac{D(u_1, u_2, \cdots, u_n)}{D(x_1, x_2, \cdots, x_n)}$$

does not vanish identically provided not all of $\partial F/\partial u_i$ vanish at any point under consideration and u_i have continuous first partial derivatives. The functions $2x+3y$ and $4x+6y+8$ are dependent, since $4x+6y+8=2(2x+3y)+8$. The functions

$$u_1=2x+3y+z, \quad u_2=x+y-z$$

and $u_3=x+y$, are independent. Their Jacobian does not vanish; it is

$$\begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}$$

which equals -1.

independent variable. See FUNCTION.

linearly independent quantities. Quantities which are not linearly dependent. See DEPENDENT—linearly dependent.

IN'DE-TER'MI-NATE, *adj.* **indeterminate equation.** See EQUATION—indeterminate equation.

indeterminate forms. Expressions of the type $\infty-\infty$, $0/0$, ∞/∞, $0\cdot\infty$, ∞°, 0°, 1^∞, which are undefined. These may arise from replacing different members of composite functions by their limits before combining the members properly. The correct procedure is to find the limits of the difference, quotient, etc., not the difference, quotient, etc., of the limits. See L'HOSPITAL'S RULE.

IN'DEX, *n.* [*pl.* **indices**]. A number used to point out a specific characteristic or operation.

contravariant and covariant indices. See TENSOR.

dummy (or umbral) index. See SUMMATION—summation convention.

free index. See SUMMATION—summation convention.

index of a matrix. See below, index of a quadratic form.

index of a point relative to a curve. See WINDING—winding number.

index of precision. The constant h in the normal frequency (probability) curve, $y=Ke^{-h^2(x-A)^2}$. See FREQUENCY—normal frequency curve.

index of a quadratic form. The number of positive terms when the quadratic form is reduced to a sum of squares by means of a linear transformation. Likewise, the **index of a Hermitian form** is the number of terms with positive coefficients when the form is reduced to the type $\sum_{i=1}^{n} a_i z_i \bar{z}_i$ by means of a linear transformation. See TRANSFORMATION—congruent transformation, conjunctive transformation. Analogously, the **index of a symmetric or of a Hermitian matrix** is the number of positive elements when it is transformed to diagonal form. For forms or matrices, the number of positive terms diminished by the number of negative terms is called the **signature.**

index of a radical. An integer placed above and to the left of a radical to indicate what root is sought; *e.g.*, $\sqrt[3]{64}=4$. The index is omitted when it would be 2; $\sqrt{x}$ rather than $\sqrt[2]{x}$ indicates the square root of x.

index of refraction. See REFRACTION.

index of a subgroup. The quotient of the order of the group by the order of the subgroup. See GROUP.

IN'DI-CA'TOR, *n.* **indicator diagram.** A diagram in which the ordinates of a curve represent a varying force, the abscissas the distance passed over, and the area beneath the curve the work done.

indicator of an integer. See EULER—Euler's ϕ-function.

IN'DI-CA'TRIX, *n.* **binormal indicatrix of a space curve.** The locus of the extremities of the radii of the unit sphere parallel to the positive directions of the binormals of the given space curve. See INDICATRIX—spherical indicatrix of a space curve. *Syn.* Spherical indicatrix of the binormal to a space curve.

principal normal indicatrix of a space curve. The locus of the extremities of the radii of the unit sphere parallel to the positive directions of the principal normals of

the given space curve. *Syn.* Spherical indicatrix of the principal normal to a space curve.

spherical indicatrix of the principal normal to a space curve. See above, principal normal indicatrix of a space curve.

spherical indicatrix of a ruled surface. The intersection of the *director cone* of the ruled surface with the unit sphere, when the vertex of the cone is at the origin. See DIRECTOR—director cone of a ruled surface.

spherical indicatrix of a space curve. The curve traced out on a unit sphere by the end of a radius which is always parallel to a tangent that moves along the curve. If the curve is a plane curve, its spherical indicatrix lies on a great circle of the sphere. Hence, the amount of deviation of the spherical indicatrix from a great circle gives some idea of the amount of deviation of the curve from being a plane curve, *i.e.*, of the amount of torsion of the curve. *Syn.* Spherical representation of a space curve. See above, principal normal indicatrix of a space curve, and binomial indicatrix of a space curve.

tangent indicatrix of a space curve. Same as the SPHERICAL INDICATRIX. See above, principal normal indicatrix of a space curve, and binormal indicatrix of a space curve.

IN'DI-CES, *n.* Plural of **index.** See INDEX.

IN'DI-RECT', *adj.* **indirect differentiation.** The differentiation of a function of a function by use of the formula $df(u)/dx = (df(u)/du)(du/dx)$. See CHAIN—chain rule.

indirect proof. (1) Same as REDUCTIO AD ABSURDUM PROOF. (2) Proving a proposition by first proving another theorem from which the given proposition follows.

IN-DORSE', *v.* **to indorse a note or other financial arrangement.** To accept the responsibility for carrying out the obligation of the maker of the paper provided the maker does not meet his own obligation; to assign a paper to a new payee by signing it on the back (this makes the indorser liable for the maker's obligation unless the signature is accompanied by the statement *without recourse*).

IN-DORSE'MENT, *n.* The act of indorsing; that which is written when indorsing. See INDORSE.

IN-DUC'TION, *n.* **mathematical induction.** A method of proving a law or theorem by showing that it holds in the first case and showing that, if it holds for all the cases preceding a given one, then it also holds for this case. Before the method can be applied it is necessary that the different cases of the law depend upon a parameter which takes on the values $1, 2, 3, \cdots$. The essential steps of the proof are as follows: (1) Prove the theorem for the first case. (2) Prove that if the theorem is true for the nth case (or for the first through the nth cases), then it is true for the $(n+1)$th case. (3) Conclude that it must then be true for all cases. For, if there were a case for which it is not true, there must be a *first* case for which it is not true. Because of (1), this is not the first case. But because of (2), it cannot be any other case [since the previous case could not be true without the next case (known to be false) being true; it could not be false because the next case is the first false case]. *E.g.*, (1) since the sun rose today, if it can be shown that if it rises any one day, it will rise the next day, then it follows by mathematical induction that it will always rise. (2) To prove $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$. If $n=1$, the right member becomes 1, which completes step (1). Adding $(n+1)$ to both members gives

$$1+2+3+\cdots+n+(n+1)$$
$$=\tfrac{1}{2}n(n+1)+n+1=\tfrac{1}{2}(n+1)(n+2),$$

which completes step (2). Therefore the statement is true for all values of n. Mathematical induction is called **complete induction,** in contrast to **incomplete induction** which draws a "conclusion" from the examination of a finite number of cases.

transfinite induction. See TRANSFINITE.

IN-DUC'TIVE, *adj.* **inductive methods.** Drawing conclusions from several known cases; reasoning from the particular to the general. See MATHEMATICAL—mathematical induction.

IN'E-QUAL'I-TY, *n.* A statement that one quantity is less than (or greater than) another. If the quantity a is less than the quantity b, their relation is denoted symbolically by $a<b$, and the relation a greater than b is written $a>b$. Inequalities

have many of the properties of equations. They are still true after any quantities have been added or subtracted from both members and after both members have been multiplied or divided by any positive number. However, multiplication or division by negative numbers changes the sense of the inequality. Since $3 > 2$,

$$3+1>2+1, \quad 3-1>2-1 \quad \text{and}$$
$$2\times 3>2\times 2; \quad \text{but} \quad -3<-2.$$

An inequality which is not true for all values of the variables involved is a **conditional inequality**; an inequality which is true for all values of the variables (or contains no variables) is an **unconditional inequality** (or **absolute inequality**). *E.g.*, $(x+2)>3$ is *a conditional inequality, because it is true* only for x greater than 1; while $(x+1)>x$, $3>2$, and $(x-1)^2+3>2$ are unconditional inequalities. The direction (*greater than* or *less than*) in which the inequality sign points is the sense of the inequality. This is used in such phrases as *same sense* and *opposite sense*. The inequalities $a<b$ and $c<d$, or $b>a$ and $d>c$, are said to have the *same sense*; the inequalities $a<b$ and $d>c$ are said to have *opposite senses*.

graphical solution of inequalities. See GRAPHICAL.

inequalities of Abel, Bessel, Cauchy, Hadamard, Hölder, Jensen, Minkowski, Newton, Schwartz, Tchebycheff, and Young. See the respective names.

simultaneous inequalities. See SIMULTANEOUS.

IN-ER'TI-A, *n.* **moment of inertia, principal axes of inertia, products of inertia.** See MOMENT—moment of inertia.

inertia of a body. Its resistance to change of its state of motion, or rest; the property of a body which necessitates exertion of force upon the body to give it acceleration. *Syn.* Mass.

law of inertia. A law of mechanics stating that material bodies not subjected to action by forces either remain at rest or move in a straight line with constant speed. This law was deduced by Galileo in 1638 and incorporated by Isaac Newton in the *Principia* (1867) as one of the postulates of mechanics. This law is also known as *Newton's First Law of Motion.*

IN-ER'TIAL, *adj.* **inertial coordinate system.** See COORDINATE.

IN'ES-SEN'TIAL, *adj.* A mapping of a topological space X into a topological space Y is said to be **inessential** if it is *homotopic* to a mapping whose range is a single point (see DEFORMATION—continuous deformation). A mapping is **essential** if it is not inessential. A mapping into a circle (or an n-sphere) whose range is not the entire circle (or sphere) is *inessential.* A mapping of an interval (or an n-cell) into a circle (or an n-sphere) is *inessential.* A mapping of a circle into a circle is *essential* if and only if the winding number of the image of the circle (relative to its center) is not zero.

IN'FER-ENCE, *n.* **statistical inference.** The making of statements, or the process of drawing judgments, about a population on the basis of random samples in such a manner that the probability of making correct inferences is determinable under various alternative hypotheses about the population being sampled.

IN-FE'RI-OR, *adj.* **limit inferior.** (1) See SEQUENCE—accumulation point of a sequence. (2) The **limit inferior** of a function $f(x)$ at a point x_0 is the smallest number L such that for any $\epsilon>0$ and neighborhood U of x_0 there is a point $x\neq x_0$ of U for which $f(x)<L+\epsilon$. This limit is denoted by $\liminf_{x\to x_0} f(x)$ or $\underline{\lim}_{x\to x_0} f(x)$. The limit inferior of $f(x)$ as $x\to x_0$ is equal to the limit as $\epsilon\to 0$ of the g.l.b. of $f(x)$ for $|x-x_0|<\epsilon$ and $x\neq x_0$, and may be positively or negatively infinite. (3) The **limit inferior** of a sequence of point sets U_1, $U_2, \cdots$ is the set consisting of all points belonging to all the sets from a certain place on. It is equal to the sum over p of the intersection of $U_p, U_{p+1}, \cdots$, or $\sum_{p=1}^{\infty}\prod_{n=p}^{\infty} U_n$, and written $\liminf_{n\to\infty} U_n$ or $\underline{\lim}_{n\to\infty} U_n$. In this sense (3), also called the **restricted li**[illegible] See SUPERIOR—limit superior. *Syn.* Lower limit.

IN'FI-NITE, *adj.* Becoming large beyond any fixed bound. *Tech.* A function is said

to *become infinite* as x approaches a if, for any number C, there is a neighborhood of a at all points of which the absolute value of the function is larger than C. It is said to become *positively infinite* (or to *approach plus infinity*) as x approaches a if, for any number C, there is a neighborhood of a at all points of which the value of the function is larger than C. It is said to become *negatively infinite* (or to *approach minus infinity*) as x approaches a if, for any number C, there is a neighborhood of a at all points of which the value of the function is less than C. For a function f, the above are written, respectively, as

$$\lim_{x\to a} f(x) = \infty, \lim_{x\to a} f(x) = +\infty, \lim_{x\to a} f(x) = -\infty$$

The above definitions also apply when a is one of the symbols $+\infty$ or $-\infty$, if a "neighborhood of $+\infty$" is the set of x satisfying $x > M$ for a specified number M, and "neighborhood of $-\infty$" is similarly defined. See UNBOUNDED—unbounded function.

infinite branch of a curve. A part of the curve which cannot be enclosed in any (finite) circle.

infinite integral. An integral at least one of whose limits of integration is infinite. The value of the integral is the limit it approaches as its limit (or limits) of integration becomes infinite. The integral exists only if this limit exists. An infinite integral is a type of **improper integral.** *E.g.*,

$$\int_1^\infty \frac{dp}{p^2} = \lim_{h\to\infty} \int_1^h \frac{dp}{p^2} = \lim_{h\to\infty} [-1/h+1] = 1.$$

infinite limit. A variable is said to have an *infinite limit* when it *becomes infinite.* See INFINITE.

infinite point. Same as IDEAL POINT.

infinite product. See PRODUCT.

infinite root. See ROOT—infinite root of an equation.

infinite sequence. See SEQUENCE.

infinite series. See SERIES.

infinite set. A set which is not finite; a set which has an unlimited number of members; a set which can be put into one-to-one correspondence with a part of itself. *E.g.*, All positive integers constitute an infinite set. It can be put into one-to-one correspondence, for instance, with the set of positive even integers. The rational fractions between 0 and 1 constitute an infinite set. It can be put into one-to-one correspondence with all rational fractions whose numerators are unity.

IN′FIN-I-TES′I-MAL, *adj., n.* A variable whose numerical value may become smaller than any assigned value; a variable which approaches zero as a limit.

infinitesimal analysis. The study of differentials, and of integration as a process of summing a set of n infinitesimals with $n\to\infty$ (see INTEGRAL—definite integral). Sometimes used for the calculus and all subjects making use of the calculus.

infinitesimal calculus. Ordinary calculus; so-called because it is based on the study of infinitesimal quantities.

order of an infinitesimal. A relative term used to compare infinitesimals. If the two infinitesimals u and v are functions of x and there are positive numbers A, B, and ϵ such that $A < |u/v| < B$, if $0 < |x| < \epsilon$, then u and v are said to be of the *same order*; if the limit of u/v is zero, the first infinitesimal, the numer at or,is said to be of *higher order* than the second, and the second of *lower order* than the first; if the limit is infinite, the first (the numerator) is said to be of lower order than the second and the second of higher order than the first. If the nth power of one infinitesimal is of the *same order* as a second infinitesimal, the second is said to be of the nth *order* relative to the first; $(1-\cos x)$ is of the second order relative to x, since $x^2/(1-\cos x)$ approaches 2 as x approaches zero.

IN-FIN′I-TY, *n.* **approach infinity.** See INFINITE.

line at infinity. See LINE—line at infinity.

order of infinities. Let $u(t)$ and $v(t)$ be functions of t and both become infinite as t approaches t_0. Then u and v are of the **same order** if there exist positive numbers M, A, and B such that $A < |u/v| < B$ whenever $t \neq t_0$ and $|t-t_0| < M$; u is said to be of **lower order** than v, or **higher order** than v, according as $\lim_{t\to t_0} u/v$ is zero, or infinite. If $\lim_{t\to t_0} u/v^r$ exists and is not zero, u is said to be of the rth *order* with respect to v.

Also, one says that u is **of the order of** v, written $u=O(v)$, if there are positive numbers M and K such that $|u/v|<K$ if $|t-t_0|<M$. If $\lim_{t\to t_0} u/v=0$, one writes $u=o(v)$. These concepts are also used when u and v become infinite as t becomes infinite, the only changes being that $|t-t_0|<M$ is replaced by $t>M$ and $\lim_{t\to t_0}$ is replaced by $\lim_{t\to\infty}$. In particular, if u_n and v_n are two sequences such that a number N exists for which $|u_n/v_n|<K$ whenever $n>N$, then u_n is said to be **of the order of** v_n, and written $u_n=O(v_n)$. If also $\lim_{t\to\infty} u_n/v_n=0$, it is written $u_n=o(v_n)$.

point at infinity. (1) See IDEAL—ideal point. (2) See below, point at infinity in the complex plane.

point at infinity in the complex plane. By hypothesis, there is a single "point at infinity" in the complex plane. Thus the complex plane is compact, whereas the Euclidean plane is not. The complex plane may be thought of as a sphere—for instance, a sphere which is mapped conformally on the complex plane by *stereographic projection.* The *pole* of the projection corresponds to the point at infinity.

IN-FLEC'TION, *n.* **point of inflection.** A point at which a plane curve changes from concavity toward any fixed line to convexity toward it; a point at which a curve has a *stationary tangent* and at which the tangent is changing from rotating in one direction to rotating in the opposite direction. The vanishing of the 2nd derivative, if it is continuous, is a *necessary* but not a *sufficient* condition for a point of inflection, because the second derivative may be zero without changing signs at the point. *E.g.*, the curve $y=x^3$ has its second derivative zero at the origin, and has a *point of inflection* there; the curve $y=x^4$ also has its second derivative zero at the origin, but has a minimum there; the curve $y=x^4+x$ has its 2nd derivative zero at the origin, but has neither a point of inflection nor a maximum or minimum there. A necessary and sufficient condition that a point be a point of inflection is that the second derivative change sign at this point, *i.e.*, have different signs for values of the independent variable slightly less and slightly greater than at the point.

IN-FLEC'TION-AL, *adj.* **inflectional tangent to a curve.** A tangent at a point of inflection. Such a tangent has contact of order 3, since dy/dx and d^2y/dx^2 each have the same value (at the point of inflection) for the curve as for the tangent. See CONTACT—order of contact.

IN'FLU-ENCE, *n.* **range of influence.** See RADIATION.

IN'FOR-MA'TION, *adj., n.* **information theory.** The branch of probability theory, founded in 1948 by C. E. Shannon, that is concerned with the likelihood of the transmission of messages, accurate to within specified limits, when the bits of information comprising the messages are subject to certain probabilities of transmission failure, distortion, and accidental additions called noise. If there are k pieces of information, one of which is to be chosen for transmission, then these pieces of information are called **messages.** The individual receiving the message is the **receiver** and the individual transmitting the message is the **sender.** A **channel** is a means of communication. Mathematically, it is described by the **input** set (the totality of possible elements from which the sender selects one for the purpose of transmitting a message to the receiver in accordance with a prearranged code), the **output** set (the totality of elements of which the receiver can observe just one), and the probability law that gives, for each element a of the input set and each element b of the output set, the probability that if a is sent then b is received. If there is a total of k messages that might be sent, a **code** is a sequence of k elements $a_1, \cdots, a_k$ of the input set and a division of the entire output set into k disjoint sets $E_1, \cdots, E_k$. Observing message b, the receiver determines the set E_j of which b is a member and concludes that the jth message was sent. A **probability law** is a rule specifying, for a given channel, for each element a of the input set, and for each element b of the output set, the probability that if element a is sent then element b is received. For a given code,

if the sender transmits element a_i representing the ith message, and the probability of the receiver's observing message b when a_i is transmitted is $p(b|a_i)$, then the probability of error when the ith message is sent is

$$P_e(i)=\sum_{b \text{ not in } E_i} p(b|a_i).$$

The **maximum error probability** for the given code is

$$\max_i P_e(i).$$

The **entropy** of a set of messages, each with a known probability of being sent, is roughly the number of binary digits needed to encode all long sequences of messages, except for the messages comprising a set of sufficiently small total probability of being sent. *Tech.* If the random variable X has k different possible values, with respective probabilities $p_1, \cdots, p_k$, then the entropy of X is

$$H(X)=-\sum_{i=1}^{k} p_i \log_2 p_i.$$

For random variables X and Y with entropies $H(X)$ and $H(Y)$, respectively, the **conditional entropy** of X when Y is given is $H(X|Y)=H(X,Y)-H(Y)$. It represents the additional number of bits of information (binary digits) required to identify an element of X when Y is given. For two sets of random numbers X and Y with entropies $H(X)$ and $H(Y)$, respectively, the **mutual information** is the number $R(X, Y)=H(X)+H(Y)-H(X, Y)$. It represents the number of bits of information, that is, the number of binary digits, of information that can be obtained concerning X by observing Y. For a channel with input set A and output set B, if X is the input variable and Y the output variable, with associated entropies $H(X)$, $H(Y)$, and $H(X, Y)$, and mutual information $R(X, Y)=H(X)+H(Y)-H(X, Y)$, the **capacity C of the channel** is defined to be the maximum of $R(X, Y)$ over all input distributions on A. The **fundamental theorem of information theory** states, roughly, that if a channel has capacity C, then in using the channel a sufficiently large number N of times, any one of about 2^{CN} messages can be transmitted with small error probability.

IN-I'TIAL, *adj.* **initial side of an angle.** See ANGLE.

INNER, *adj.* **inner product of tensors.** The *inner product* of two tensors $A^{a_1\cdots a_n}_{i_1\cdots i_m}$ and $B^{b_1\cdots b_p}_{j_1\cdots j_q}$ is the *contracted tensor* obtained from the product

$$C^{a_1\cdots a_m b_1\cdots b_p}_{i_1\cdots i_m j_1\cdots j_q}=A^{a_1\cdots a_n}_{i_1\cdots i_m}\, B^{b_1\cdots b_p}_{j_1\cdots j_q}$$

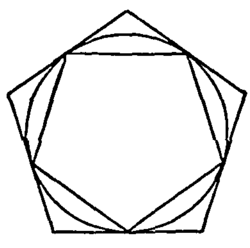
by putting a contravariant index of one equal to a covariant index of the other and summing with respect to that index. This inner multiplication of two tensors is also called **composition,** and the inner product is spoken of as the tensor **compounded** from the two given tensors.

inner product of two functions. For real-valued functions f and g, the inner product is $(f, g)=\int fg\,dx$, where the integral is taken over the common domain of definition. If f and g are complex-valued, then $(f, g)=\int f\bar{g}\,dx$. See HILBERT—Hilbert space, and ORTHOGONAL—orthogonal functions.

inner product of two vectors. See MULTIPLICATION—multiplication of vectors, VECTOR—vector space, and HILBERT—Hilbert space.

IN'PUT, *adj.*, *n.* **input component.** In a computing machine, any component that is used in introducing problems into the machine; for example, a numerical keyboard, typewriter, punched-card machine, or tape might be used for this purpose.

IN-SCRIBED', *adj.* A polygon (or polyhedron) is said to be inscribed in a closed



configuration composed of lines, curves, or surfaces, when it is contained in the configuration and every vertex of the polygon (or polyhedron) is incident upon the

configuration; a closed configuration is said to be inscribed in a polygon (or polyhedron) when every side (in the polygon) or every face (in the polyhedron) is tangent to it and the configuration is contained in the polygon (or polyhedron). If one configuration is inscribed in another, the latter is said to be **circumscribed** about the former (see CIRCUMSCRIBED). The figure shows a polygon inscribed in a circle, the circle inscribed in another polygon.

inscribed angle of a closed curve. An angle (in the interior of the curve) formed by two chords which intersect on the curve.

inscribed circle of a triangle. The circle tangent to the sides of the triangles. Its center is the intersection of the bisectors of the angles of the triangle. Also called **incircle.** Its center is called the **incenter** of the triangle. Its radius is

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

where a, b, c are the sides of the triangle and

$$s=\tfrac{1}{2}(a+b+c).$$

inscribed cone, cylinder, polygon, prism, pyramid. See CIRCUMSCRIBED.

IN-STALL'MENT (or IN-STAL'MENT), *n.* **installment payments.** Payments on notes, accounts, or mortgages at regular periods. Each payment may include a fixed payment on principal and interest on the balance over the preceding period, or the interest on the entire amount may have been added to the principal at the outset and the payments made equal. The latter practice makes the actual interest practically double the quoted rate. The former method is sometimes called the **long-end interest plan.** Sometimes fixed payments plus the interest on the amount of the payment from the beginning of the contract are made periodically (monthly). This is sometimes called the **short-end interest plan.** See AMORTIZE.

installment plan of buying. Any plan under which a debt is paid in installments.

installment policies. Insurance policies whose benefits are payable in installments beginning at the death (or end of the death year) of the insured, instead of in a single payment.

installment premium. See PREMIUM.

IN'STAN-TA'NE-OUS, *adj.* **instantaneous acceleration, speed, and velocity.** See ACCELERATION, SPEED, and VELOCITY.

IN'SUR'ANCE, *adj., n.* **extended insurance.** Insurance extended (because of default in payment or some other exigency) over a period of such length that the *net single premium* is equal to the *surrender value* of the policy.

insurance policy. A contract under which an insurance company agrees to pay the beneficiary a certain sum of money in case of death, fire, sickness, accident, or whatever contingency the policy provides against.

life insurance. A contract to pay a specified benefit under specified conditions when certain lives end (or periodically thereafter), or when the lives end if that is within a certain period. The former is called **whole life insurance**; the latter, **temporary (or term) insurance.** The money paid by the insured for his protection is called the **premium** and the formal contract between him and the company is called the **policy. Deferred life insurance** can refer to any type of life insurance which has the provision that benefits are not payable if the insured lives end before a specified future date. **Ordinary life insurance** is whole life insurance with premiums payable annually throughout life (a **limited payment policy** is one for which the annual payment is to be paid only for a stated number of years—it is called **insurance with return of premiums** if the company returns all premiums after payments are completed). Term insurance which permits the insured to reinsure (usually at the end of the period) without another medical examination is called **option term insurance.** Insurance payable upon the end of the life of the insured (the policy life) provided some other life (the counter life) continues is called **contingent life insurance.** This is a simple case of **compound survivorship insurance** which is payable at the end of a certain life provided a group of lives, including this one, end in a certain order. Insurance payable when the last of two or more lives end is called **last survivor insurance**; insurance payable when the first of two (or more) lives end is called **joint life insurance.** An **endowment insurance** policy provides that the benefits

be payable (usually to the insured) after a definite period or (to the beneficiary) at the death of the insured if that occurs before the end of the period; **double endowment insurance** provides twice the benefits at the close of the period as would be paid if the insured life ends during the period; **pure endowment insurance** provides payment of a benefit at the end of a given period if, and only if, the insured survives that period. Any type of life insurance policy is called an **installment policy** if the benefits are payable in installments instead of in a single payment. An endowment policy is a **continuous installment policy** if it provides that if the insured survives the period of the contract an annuity will be paid for a certain period and so long thereafter as the insured lives. Also see TONTINE—tontine insurance, DEBENTURE, and various headings above and below.

loan value of an insurance policy. See LOAN—loan value.

mutual insurance company. See MUTUAL.

participating insurance policy. A policy which entitles the holder to participate in the profits of the insurance company. The profits are usually paid in the form of a **dividend** or **bonus** at the time of payment of the policy, or at periods when the company invoices its insurance contracts. A bonus paid when the policy is paid is called a **reversionary bonus.** A reversionary bonus is a **uniform reversionary bonus** if it is determined as a percentage of the original amount of the insurance; it is a **compound reversionary bonus** if it is a sum of a percentage of the original amount of the insurance and any bonuses already credited to the policy. A **discounted bonus policy** is a policy in which the premiums are reduced by the estimated (anticipated) future bonuses. A **guaranteed bonus policy** is a policy which carries a guarantee of a certain rate of profit-sharing (bonus) (really non-participating since the bonuses assured are determined at the policy date). See MUTUAL—mutual insurance company.

reserve of an insurance policy. See RESERVE.

IN'TE-GER, *n.* Any of the numbers 1, 2, 3, etc. These are often spoken of as **positive integers** in contradistinction to the **negative integers,** -1, -2, -3, etc. The entire class of integers consists of 0, ± 1, $\pm 2, \cdots$. Peano defined the positive integers as a set of elements which satisfies the following postulates: (1) There is a positive integer 1; (2) every positive integer a has a *consequent* a^+ (a is called the *antecedent* of a^+); (3) the integer 1 has no antecedent; (4) if $a^+ = b^+$, then $a = b$; (5) every set of positive integers which contains 1 and the consequent of every number of the set contains all the positive integers. A positive integer (or zero) can also be thought of as describing the "manyness" of a set of objects in the sense of being a symbol denoting that property of a set of individuals which is independent of the natures of the individuals. That is, it is a symbol associated with a set and with all other sets which can be put into one-to-one correspondence with this set. The set is called a **number class.** A set consisting of part of the members of a number class is called a **subclass.** See SUM—sum of real numbers, PRODUCT—product of real numbers, and CARDINAL—cardinal number.

algebraic integer. See ALGEBRAIC—algebraic integer.

Gaussian integer. Any complex number of the form $a+bi$, where a and b are ordinary (real) integers. *Syn.* Complex integer.

indicator, totient, or ϕ-function of an integer. See EULER—Euler's ϕ-function.

IN'TE-GRA-BLE, *adj.* **integrable function.** See INTEGRAL—definite integral, and DARBOUX'S THEOREM.

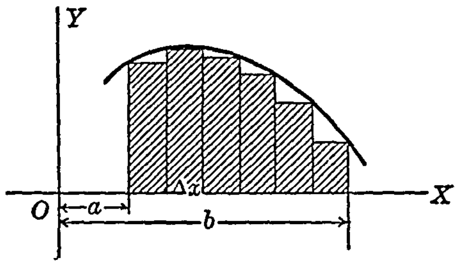
integrable differential equation. See DIFFERENTIAL—integrable differential equations.

IN'TE-GRAL, *adj., n.* **definite (Riemann) integral.** If the integrand is continuous, the definite integral is the difference between the values of the *indefinite integral* for two specified values of the independent variable. Geometrically, the definite integral is the limit, as Δx approaches zero, of the sum of the areas of the rectangles, Δx in width, formed with successive ordinates taken Δx apart throughout the interval between the two values of x (a and b in the figure). The *definite integral*

from a to b of $f(x)$ with respect to x is written:

$$\int_a^b f(x)\,dx$$

and spoken of as the integral of $f(x)$ with respect to x between the limits a and b. It



is the area between the curve $y=f(x)$, the x-axis, and the lines $x=a$ and $x=b$; $f(x)$ is called the **integrand** and a and b the **lower** and **upper limits of integration.** *Tech.* The *Riemann definite integral* of a function on a given interval is the limit obtained by dividing the given interval into nonoverlapping subintervals, multiplying the width of each subinterval by the value of the ordinate corresponding to some abscissa within that interval, adding all such products, and finding the limit of such sums as the number of subintervals becomes infinite in such a way that they all become arbitrarily small. The integral is said to exist if and only if this limit exists and is unique for all methods of summing which satisfy the stated restrictions. The definite integral always exists for a function which is continuous in the closed interval defined by the limits of integration; continuity is here a sufficient but not a necessary condition. A necessary and sufficient condition that a bounded function have a (Riemann) integral on a given interval is that the function be continuous *almost everywhere*. See DARBOUX'S THEOREM. The definite integral does not always represent area, but may be so interpreted when evaluating it. Some elementary properties of a definite integral are: (1) The value of a definite integral depends entirely upon the limits of integration and the form of the function; *i.e.*, the value of a definite integral is independent of the variable with respect to which the integration is performed;

$$\int_a^b f(x)\,dx$$

is the same as

$$\int_\alpha^\beta f[z(t)]\,dz(t),$$

where $a=z(\alpha)$ and $b=z(\beta)$. This property is very useful in reshaping the form of an integrand so that the value of the integral can be found; *e.g.*,

$$\int_0^{\pi/2} \sin x \cos x\,dx = \int_0^{\pi/2} \sin x\,d\sin x = \int_0^1 u\,du.$$

(2) Interchanging the limits of integration simply changes the sign of the result. (3)

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx,$$

if these exist. These facts are inherent in the definition of an integral as the limit of a sum. (4) The definite integral of a function over an interval of non-negative width is equal to or greater than the width of the interval times the least value of the function in that interval, and is equal to or less than the width of the interval times the greatest value of the function in that interval. (5) The definite integral of a continuous function over a given interval is equal to the product of the width of the interval by some value of the function within the interval. (Property (5) is called **the first law of the mean for integrals.**) See ELEMENT—element of integration. Such symbols as $\int_s$ indicate that the integral is a definite integral whose number of integration signs and limits are to be selected to fit the case at hand. See MEAN—mean value of a function, and various headings below.

derivative of an integral. See DERIVATIVE.

elliptic integral. See ELLIPTIC.

energy integral. See ENERGY.

Fredholm's integral equations and their solutions. See various headings under FREDHOLM.

Fresnel's integrals. See FRESNEL.

Hilbert-Schmidt theory of integral equations. See HILBERT.

homogeneous integral equation. See HOMOGENEOUS.

improper integral. An integral whose limits are not both finite or one whose integrand becomes infinite in the interval between the limits of integration; an integral in which the interval of integration and the integrand are not both bounded.

indefinite integral of a function of a single variable. Any function whose derivative is the given function. If $g(x)$ is an indefinite integral of $f(x)$, then $g(x)+c$, where c is an *arbitrary constant*, is also an integral of $f(x)$; c is called the **constant of integration.** The indefinite integral of $f(x)$ with respect to x is written $\int f(x)\,dx$; $f(x)$ is called the **integrand.** Many basic formulas for finding integrals are listed in the appendix. More extensive tables have been published, but the list of integrals is inexhaustible. *Syn.* Antiderivative.

infinite integral. See INFINITE—infinite integral.

integral calculus. See CALCULUS.

integral curves. The family of curves whose equations are solutions of a particular differential equation; the *integral curves* of the differential equation $y' = -x/y$ are the family of circles $x^2+y^2=c$, where c is an arbitrary parameter. See DIFFERENTIAL—solution of a differential equation.

integral domain. See DOMAIN.

integral equation. An equation in which the unknown function occurs under an integral sign. The (Fourier) equation

$$f(x)=\int_{-\infty}^{\infty} \cos(xt)\phi(t)\,dt,$$

where $f(x)$ is an even function, is the first integral equation that was solved. Under certain conditions, a solution is

$$\phi(x)=\frac{2}{\pi}\int_0^{\infty} \cos(ux)f(u)\,du.$$

An integral equation of the **third kind** is an integral equation of type

$$g(x)y(x)=f(x)+\lambda\int_a^b K(x,t)y(t)\,dt,$$

where $f(x)$, $g(x)$, and $K(x,t)$ are given functions and $y(x)$ is the unknown function. The function $K(x,t)$ is called the **kernel** or **nucleus** of the equation. The (Fredholm) integral equations of the **first** and **second kind** are special cases of this equation. See FREDHOLM—Fredholm's integral equations.

integral expression. An algebraic expression in which no variables appear in any denominator when the expression is written in a form having only positive exponents.

integral function. Same as ENTIRE FUNCTION.

integral of a function of a complex variable. If a *rectifiable curve* C in the z plane is given in parametric representation, $z=z(t)=x(t)+iy(t)$, $a \leqq t \leqq b$, and if $f(z)$ is continuous on C, then by definition

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_i)[z(t_j)-z(t_{j-1})],$$

where $\zeta_i = z(\tau_i)$ and

$$a=t_0<t_1<\cdots<t_n=b,\ t_{j-1} \leqq \tau_j \leqq t_j,$$

and $\lim_{n\to\infty} [\max(t_j - t_{j-1})]=0$. See CONTOUR—contour integral.

integral number. An INTEGER (positive, negative, or zero).

integral tables. Tables giving the primitives (indefinite integrals) of the more common functions and sometimes some of the more important definite integrals. See INTEGRATION TABLES in the appendix.

integral test for convergence. See CAUCHY—Cauchy's integral test.

iterated integral. An indicated succession of integrals in which integration is to be performed first with respect to one variable, the others being held constant, then with respect to a second, the remaining ones being held constant, etc.; the inverse of successive partial differentiation, if the integration is *indefinite integration.* When the integration is *definite integration*, the limits may be either constants or variables, the latter usually being functions of variables with respect to which integration is

yet to be performed. (Some writers use the term *multiple* for *iterated integrals.*) *E.g.*, (1) The **iterated integral,**

$$\iint xy\,dy\,dx,$$

may be written

$$\int\left\{\int xy\,dy\right\}dx.$$

Integrating the inner integral gives

$$(\tfrac{1}{2}xy^2+C_1)$$

where C_1 is any function of x, only. Integrating again gives

$$\tfrac{1}{4}x^2y^2+\int C_1\,dx+C_2$$

where C_2 is any function of y. The result may be written in the form $\frac{1}{4}x^2y^2+\phi_1(x)+\phi_2(y)$, where $\phi_1(x)$ and $\phi_2(y)$ are any differentiable functions of x and y, respectively. The order of integration is usually from the inner differential out, as taken here. The orders are not always interchangeable. (2) The **definite iterated integral,**

$$\int_a^b\int_z^{x+1} x\,dy\,dx,$$

is equivalent to

$$\int_a^b\left\{\int_x^{x+1} x\,dy\right\}dx$$

which is equal to

$$\int_a^b \{x(x+1)-x^2\}dx=\tfrac{1}{2}(b^2-a^2).$$

See below, multiple integral.

Lebesgue integral. See LEBESGUE.

Lebesgue-Stieltjes integral. See STIELTJES.

line integral. Let $f(x,y)$ be a given function of x and y and let A and B be points on a given curve. Form sums of type

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta_i s,$$

where $A=P_1, P_2, \cdots, P_n=B$ are successive points selected around (along) the curve from A to B, $\Delta_i s$ is the length of the straight line from P_{i-1} to P_i, for $i=1, 2, \cdots, n$, and (x_i, y_i) is a point on the piece of the curve from P_{i-1} to P_i. Let Δ be the largest of the numbers $\Delta_i s$. If the above sum has a limit as Δ approaches zero, this is defined to be the line integral of $f(x,y)$ from A to B along the curve:

$$\int_A^B f(x,y)ds=\lim_{\Delta\to 0}\sum_{i\to 1}^{n} f(x_i, y_i)\Delta_i s.$$

Let $\mathbf{F}(x,y)$ be a vector whose component tangent to the curve at (x,y) is equal to $f(x,y)$ and let $M(x,y)$ and $N(x,y)$ be the horizontal and vertical components of $\mathbf{F}(x,y)$ at (x,y). Then

$$\int_A^B f(x,y)ds=\int M(x,y)dx+\int N(x,y)dy,$$

which is also written as $\int(Mdx+Ndy)$, where x and y are related by the condition that (x,y) be a point on the curve and the limits of integration are to be determined so that the path of integration on the x (or y) axis corresponds to the projection of the curve onto the x (or y) axis. If $Mdx+Ndy$ is an *exact differential, i.e.,*

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},$$

then the line integral is a line integral of the **first kind** (otherwise, it is of the **second kind**), there is a function $u(x,y)$ such that $du=Mdx+Ndy$, and the line integral is equal to the change of $u(x,y)$ along the curve from A to B. Also, if $\partial M/\partial y=\partial N/\partial x$ in a *simply connected region,* the value of the line integral between two points of the region does not depend on the path of integration (see CONSERVATIVE—conservative field of force). In three dimensions, a line integral is defined in the same way and is equivalent to an integral of type

$$\int Pdx+Qdy+Rdz.$$

multiple integral. A generalization of the integral of a function of a single variable as the limit of a sum. The **double integral** of a function $f(x,y)$ is defined over a given region, A, as follows: Divide the region into n nonoverlapping subregions whose areas (or measures) are $\Delta_i A$, $i=1, 2, 3, \cdots, n$. Let Δ be the area of the smallest square in which each of these subregions can be embedded. Let (x_i, y_i) be a point in the ith subdivision. Form the sum

$$\sum_1^n f(x_i, y_i)\Delta_i A.$$

The double integral of f over A is defined as the limit of this sum as Δ approaches zero, if this limit exists, and is written:

$$\iint_A f(x, y)\, dA$$

If $f(x, y)$ is continuous throughout A, the double integral exists and is equal to the iterated integral of $f(x, y)$ over A. See above, iterated integral. The **triple integral** of a function $f(x, y, z)$ over a region R of space is defined in essentially the same way (Δ is then the volume of the smallest cube in which each subregion can be embedded) and is also equal to a *triple iterated integral* if $f(x, y, z)$ is continuous. Multiple integrals of higher order can be similarly defined. See FUBINI—Fubini's theorem.

rationalization of integrals. See RATIONALIZE.

Riemann-Stieltjes integral. See STIELTJES.

surface integral. See SURFACE—surface integral.

triple integral. See above, iterated integral, multiple integral.

Volterra's integral equations and their solutions. See various headings under VOLTERRA.

IN'TE-GRAND, *n.* See INTEGRAL—definite integral.

exact integrand. An integrand which is an exact differential. See DIFFERENTIAL—exact differential.

rationalization of integrands. Same as RATIONALIZATION OF INTEGRALS. See RATIONALIZATION.

IN'TE-GRAPH, *n.* A mechanical device for finding areas under curves; a mechanical device for performing *definite integration.*

IN'TE-GRAT'ING, *adj.* **integrating factor.** See FACTOR.

integrating machines. Mechanical instruments for use in evaluating definite integrals; such instruments as the *integraph* and *polar planimeter.*

IN'TE-GRA'TION, *adj., n.* The process of finding an indefinite or definite integral. See INTEGRAL, and the INTEGRAL TABLES in the appendix.

change of variables in integration. For an integral with a single variable of integration, this is also called *integration by substitution,* since it is usually used to transform the integral into a form that is more easily evaluated. Typical substitutions are those which replace the variable by the square of a variable in order to rationalize indicated square roots of the variable, and the *trigonometric substitutions* by which the square roots of quadratic binomials are reduced to monomials. *E.g.,* if one substitutes $\sin u$ for x in $\sqrt{1-x^2}$ the result is $\cos u$; similarly $\sqrt{1+x^2}$ and $\sqrt{x^2-1}$ lend themselves to the substitutions $\tan u = x$, and $\sec u = x$, respectively. In all cases when a substitution is made in the integrand, it must also be made in the differential and the proper changes must be made in the limits if the integrals are definite integrals (an equivalent and quite common practice is to integrate the new integrand as an indefinite integral, then make the inverse substitution and use the original limits). *E.g.,*

$$\int_0^1 \sqrt{1-x^2}\, dx,$$

transformed by the substitution $x = \sin u$, becomes

$$\int_0^{\pi/2} \cos^2 u\, du = [\tfrac{1}{2}u + \tfrac{1}{4}\sin 2u]_0^{\pi/2} = \tfrac{1}{4}\pi,$$

or one may write

$$\int \cos^2 u\, du = \tfrac{1}{2}u + \tfrac{1}{4}\sin 2u$$
$$= \tfrac{1}{2}\sin^{-1} x + \tfrac{1}{2}x\sqrt{1-x^2},$$

whence

$$\int_0^1 \sqrt{1-x^2}\, dx$$
$$= [\tfrac{1}{2}\sin^{-1} x + \tfrac{1}{2}x\sqrt{1-x^2}]_0^1 = \tfrac{1}{4}\pi.$$

The rule for change of variables is of the same form for multiple integrals of all orders when expressed in terms of a Jacobian and will therefore be given only for triple integrals (the conditions given are sufficient, but not necessary). Let T be a transformation which maps an open set W in xyz-space onto an open set W^* in uvw-space and let D be a subset of W which is the image in xyz-space of a closed bounded set D^* in uvw-space. If $f(x, y, z)$ is continuous on D; x, y, and z have continuous

first order partial derivatives with respect to u, v and w; and the Jacobian

$$J=\frac{\partial(x, y, z)}{\partial(u, v, w)}$$

is not zero on D^*, then

$$\iiint_D f(x, y, z)dx\,dy\,dz=$$

$$\iiint_{D^*} f(x, y, z)|J|du\,dv\,dw,$$

where x, y and z in the second integral are functions of u, v and w as determined by the transformation T. For spherical coordinates, this becomes:

$$\iiint f(x, y, z)dx\,dy\,dz=$$

$$\iiint f(x, y, z)\rho^2 \sin\phi d\rho d\phi d\theta.$$

For a single integral and the substitution $x=x(u)$, the Jacobian is dx/du and the analogous formula is

$$\int f(x)dx=\int f[x(u)]\frac{dx}{du}\,du,$$

which was used in the above illustrations of integration by substitution.

definite integration. The process of finding definite integrals. See INTEGRAL—definite integral.

element of integration. See ELEMENT—element of integration.

formulas of integration. Formulas giving the indefinite integrals or certain definite integrals of a few of the most commonly met functions.

integration of an infinite series. See SERIES—integration of an infinite series.

integration by partial fractions. A specific method of integration used when the integrand is a rational function with denominator of higher degree than the first. Consists of breaking the integrand up into *partial fractions* with simpler denominators. *E.g.*,

$$\int\frac{dx}{1-x^2}=\frac{1}{2}\int\frac{dx}{1-x}+\frac{1}{2}\int\frac{dx}{1+x}.$$

See PARTIAL—partial fractions.

integration by parts. A process of integrating by use of the formula for the differential of a product. The formula $d(uv)=u\,dv+v\,du$ is written $u\,dv=d(uv)-v\,du$; integrating both sides of this equation gives $\int u\,dv=uv-\int v\,du$. This last formula enables one to modify the form of an integrand and simplify the process of integration, or actually integrate functions whose exact integral could not otherwise be found directly. It is especially useful in integrating such functions as xe^x, $\log x$, $x\sin x$, etc.; *e.g.*,

$$\int xe^x\,dx=xe^x-\int e^x\,dx,$$

where $x=u$, $e^x\,dx=dv$, and $v=e^x$.

integration by substitution. See above, change of variables in integration.

integration by use of series. The expanding of the integrand in a series and integrating term by term. An upper bound to the numerical value of the remainder of the series, after any given number of terms, can be integrated to find limits to the error. See INTEGRAL—definite integral (4).

mechanical integration. Determining the area bounded by a curve without the use of its equation, by the use of some specific mechanical device such as the *integraph* or *polar planimeter.*

reduction formulas in integration. Formulas expressing an integral as the sum of certain functions and a simpler integral. Such formulas are most commonly derived by *integration by parts.* See INTEGRAL TABLES in the appendix, formulas 54, 66, 74, etc.

IN'TE-GRA'TOR, *n.* An instrument which approximates definite integrals mechanically, such as a *planimeter* for measuring areas. In a computing machine, an integrator is any *arithmetic component* that performs the operation of integration.

IN-TEN'SI-TY, *n.* **electrostatic intensity.** See ELECTROSTATIC.

IN'TER-AC'TION, *n.* (*Statistics.*) Let x be a function of y. If the variation in x associated with given changes in y is affected by the values assumed by a third variable z, there is **interaction** between y and z. Thus in the *analysis of variance*, if rows and columns represent different values of y and z, and the x values are the values in the cells, no interaction exists if the variation among the values from column to

column is the same for each row. **Interaction** exists if the pattern of variation is different among the rows. Mathematically, this means that interaction exists between the variables y and z if $\partial x/\partial y$ is a function of z, where $x=f(y, z)$. For example, in the following 2 by 2 table (assuming the values are true, nonsampling values) there is interaction, since the pattern of variation in values in row 1 is different from the pattern in row 2.

3	6
2	1

If no interaction exists in the population from which the sample of data is drawn in a two-way analysis of variance, the sampling fluctuations may be used as an estimate of the sampling variance, where the marginal totals are used as the expected values with which observed values are compared for the variance.

IN'TER-CEPT', *v.*, *adj.*, *n.* To cut off or bound some part of a line, plane, surface, or solid. Two radii intercept arcs of the circumference of a circle.

intercept form of the equation of a plane. See PLANE—equation of a plane.

intercept form of the equation of a straight line. See LINE—equation of a straight line.

intercept of a straight line, curve, or surface on an axis of coordinates. The distance from the origin to the point where the line, curve, or surface cuts the axis. The intercept on the axis of abscissas, or x-axis, is called the **x-intercept**, and that on the axis of ordinates, or y-axis, the **y-intercept.** (In space, the intercept on the z-axis is likewise called the **z-intercept.**) The intercepts of the line $2x+3y=6$ on the x-axis and y-axis, respectively, are 3 and 2.

IN'TER-DE'PEND'ENT, *adj.* **interdependent functions.** Same as DEPENDENT FUNCTIONS.

IN'TER-EST, *n.* Money paid for the use of money. The interest due at the end of a certain period is called **simple interest** if the interest is computed on the original principal during the entire period. In this case, the interest is equal to the product of the *time*, *rate of interest*, and *original principal*; *e.g.*, the interest upon \$100 at 6% for 5 years is $5\cdot(6/100)\cdot \$100=\$30$. If the interest when due is added to the principal and thereafter earns interest, the interest (calculated in this way) is called **compound interest**; the interest is computed upon the principal for the first period, upon the principal and the first period's interest for the second period, upon the new principal and the second period's interest for the third period, etc. Thus at 6%, the interest plus principal at the end of the first, second, and nth years is respectively $P(1.06)$, $P(1.06)^2$, and $P(1.06)^n$, where P denotes the principal (see CONVERSION—continuous conversion of compound interest). The interval of time between successive conversions of interest into principal is called the **interest or conversion period**; the total amount due at any time is called the **compound amount.** If 6% is the rate, the compound amount of \$1.00 at the end of 1 year is \$1.06, at the end of 2 years $\$(1.06)^2$, at the end of n years $\$(1.06)^n$. The **nominal rate of interest** is the stated yearly rate when interest is compounded over periods of less than a year. When interest is computed at the rate of 3% semi-annually, the nominal rate would be 6%. The annual rate which gives the same yield as the nominal computed over fractions of the year is called the **effective rate.** The effective rate for 6% nominal compounded semiannually is 6.09%.

exact interest. Interest computed upon the basis of the exact number of days in a year (365 days except for leap year, which has 366 days). Interest for 90 days at 6% would be 90/365 of 6% of the principal. In counting days between dates the last, but not the first, date is usually included. The number of days from Dec. 25th to Feb. 2nd would be counted as 39 under the customary practice. See below, ordinary interest.

force of interest. The nominal rate which converted *continuously* is equivalent to a certain effective rate.

interest rate. The ratio of interest to principal, times 100%. *Syn.* Rate of interest, rate, rate per cent.

long-end and short-end interest plan. See INSTALLMENT—installment payments.

ordinary interest. Interest computed on the basis of the commercial year of 360 days and 30 days to the month. Interest for 2 months at 6% would be 60/360 of 6% times the principal; when the time of a note is expressed in days, the exact number of days is counted. *E.g.*, a note dated July 26th and due in 30 days would be due Aug. 25th, whereas if due in one month it would be due Aug. 26th. See above, exact interest.

prevailing interest rate (for any given investment). The rate which is common, or generally accepted, for that particular type of investment at the time under consideration. *Syn.* Current rate.

sixty-day method for computing interest. A method for computing simple interest at 6%. The rate for 60 days is $(\frac{60}{360})\ \frac{6}{100}$ or 1%, so the interest for 60 days is $\frac{1}{100}$ of the principal and for 6 days $\frac{1}{1000}$ of the principal. The time over which interest is to be computed is expressed in terms of 6 days and 60 days and fractional parts thereof. Also used when the rate is other than 6%. See SIX—six per cent method.

six per cent method. A method of computing simple interest by computing it first for 6%, then for the given rate, if it is different. At 6%, the interest on \$1 for one year is \$.06, for one month \$.005, and for one day $\$\frac{1}{30}(.005)$. When the rate is other than 6%, one computes the interest for 6% as above and then takes the proper part of the result; for instance, if the rate is 5%, the interest is $\frac{5}{6}$ of the result obtained for 6%.

IN-TE′RI-OR, *adj., n.* **alternate interior angles.** See ALTERNATE—alternate interior angles.

interior and exterior-interior angles of two lines cut by a transversal. See ANGLE—angles made by a transversal.

interior angle of a polygon. An angle between any two sides of the polygon (not produced) and lying within the polygon. When the interior of the polygon is not defined, as when sides intersect at points other than vertices, this definition of interior angle does not apply.

interior angle of a triangle. An angle lying within the triangle.

interior content. See CONTENT—content of a set of points.

interior mapping (or transformation). Same as OPEN MAPPING. See OPEN.

interior measure. See MEASURE—exterior measure.

interior of a set. The interior of a set E is the set of all points of E that have a neighborhood contained in E. Each such point is called an **interior point.** The set of all points which belong to the closure of E and to the closure of the complement $C(E)$ of E is called the **frontier** (or **boundary**) of E and of $C(E)$. It contains all points which are not interior points of E or of $C(E)$. See EXTERIOR—exterior of a set.

IN-TER-ME′DI-ATE, *adj.* **intermediate differential.** See DIFFERENTIAL.

intermediate-value theorem. The theorem which states that if a function $f(x)$ is continuous for $a \leqq x \leqq b$, $f(a) \neq f(b)$, and k is between $f(a)$ and $f(b)$, then there is a number ξ between a and b for which $f(\xi)=k$.

IN-TER′NAL, *adj.* **internal ratio.** See POINT—point of division.

IN-TER′PO-LA′TION, *adj., n.* The process of finding a value of a function between two known values by a procedure other than the law which is given by the function itself. *E.g.*, in **linear interpolation** the procedure is based on the assumption that the three points having these values of the function for ordinates lie on a straight line. This is approximately true when the values of the arguments are close together and the function is continuous; *i.e.*, its graph between the points, whose abscissas are the three values of the arguments, is a single curve with no breaks in it. If the function is $f(x)$ and its value is known for $x=x_1$ and for $x=x_2$, the formula for linear interpolation is

$$f(x)=f(x_1)+[f(x_2)-f(x_1)]\,\frac{x-x_1}{x_2-x_1}.$$

correction by interpolation in a logarithmic (or trigonometric) table. The number added to a tabular logarithm (trigonometric function) to give the logarithm (trigonometric function) of a number

(angle) which is not in the tables; equally applicable when interpolating in any table. (When working with tables of trigonometric functions the *correction* is negative in the case of the decreasing functions, namely the cosine, cotangent, and cosecant.)

interpolation formulas of Gregory-Newton, Hermite, and Lagrange. See the respective names.

IN'TER-QUAR'TILE, *adj.* **interquartile range.** The distance between the first and third *quartiles* of a distribution. This covers the middle half of the values in the frequency distribution.

IN'TER-SEC'TION, *n.* The point, or locus of points, common to two (or more) geometric configurations. The **intersection of two curves** usually consists of a finite number of points, but may even contain a part of one of the curves if this also is part of the other curve; two distinct straight lines either have an empty intersection or their intersection is a single point. If two surfaces do not have an empty intersection, their intersection usually consists of curves, but may contain isolated points or pieces of the surfaces; the intersection of two distinct planes is either empty or a straight line. The term **imaginary intersection** is used to complete the analogy between discussions of equations and their graphs; it consists of the sets of imaginary values of the variables which are common solutions of the equations. The **intersection of two sets** consists of all the points which belong to each of the sets. The intersection of sets U and V is usually denoted by UV, $U \cdot V$, or $U \cap V$ and is also called the **product,** or **meet,** of U and V.

angle of intersection. See ANGLE—angle of intersection.

IN'TER·VAL, *n.* (1) A **closed interval** is a set of numbers which consists precisely of all the numbers simultaneously greater than or equal to one fixed number and less than or equal to another. If the two numbers are a and b, the interval is indicated by (a, b), or better, $[a, b]$, and the length of the interval is $b-a$; a and b are called the **end points** of the interval. Such an interval is called a **closed interval** in distinction from the sort of interval in which one or both of the end points are excluded. Intervals with no end points are called **open intervals.** In general, an **interval** is the set containing all numbers between two given numbers and one, both, or neither end point. (2) In n-dimensional space, a **closed interval** is a set of points containing those and only those points x whose coordinates satisfy inequalities $a_i \leqq x_i \leqq b_i$ (for each i), for some fixed numbers $a_1, a_2, \cdots a_n$ and $b_1, b_2, \cdots b_n$ with $a_i < b_i$ for all i. The set of points x satisfying the inequalities $a_i < x_i < b_i$ (for each i) is an **open interval.** An interval may be an open interval, a closed interval, or one that is partly open and partly closed (*i.e.*, some of the signs $\leqq$ may be replaced by $<$).

confidence interval. See CONFIDENCE.

interval of convergence. See CONVERGENCE.

open interval. See above, closed interval.

IN'TO, *prep.* See ONTO.

IN-TRAN'SI-TIVE, *adj.* **intransitive relation.** See TRANSITIVE.

IN-TRIN'SIC, *adj.* **intrinsic equations of a space curve.** Since a space curve is determined to within its position in space by its radii of curvature and torsion as functions of the arc length, $\rho=f(s)$, $\tau=g(s)$, these equations are called the intrinsic equations of the curve. *Syn.* Natural equations of a space curve.

intrinsic properties of a curve. Properties which are not altered by any change of coordinate systems. Some of the intrinsic properties of the conics are their eccentricity, distances from foci to directrices, length of latus rectum, length of the axes (of an ellipse or hyperbola), and their reflection properties.

intrinsic property of a surface. A property which pertains merely to the surface, not to the surrounding space; a property which is preserved under isometric transformations; a property expressible in terms of the coefficients of the first fundamental quadratic form alone. *Syn.* Absolute property of a surface.

IN-VA'RI-ANT, *adj.*, *n.* **invariant factor of a matrix.** One of the diagonal elements

when the matrix is reduced to *Smith's canonical form*, the elements of the matrix being polynomials. See CANONICAL—canonical form of a matrix. The invariant factors are unchanged by multiplication on either side by a matrix whose determinant does not involve the variable (λ). Each invariant factor is a product of the type $E_j(\lambda)=(\lambda-\lambda_1)^{p_{1j}}(\lambda-\lambda_2)^{p_{2j}}\cdots$, where λ_1, λ_2, $\cdots$ are distinct. Each factor $(\lambda-\lambda_i)^{p_{ij}}$ is called an **elementary divisor** of the matrix.

invariant property. A property of a function, configuration, or equation that is not altered by a particular transformation. The invariant property is used with reference to a particular transformation or type of transformation. *E.g.*, the value of a cross ratio is not changed by a projection, and hence is said to be invariant under projective transformations See TENSOR.

invariants of an algebraic equation. Algebraic expressions involving the coefficients which remain unaltered in value when any translation or rotation of the axes is made. For the general quadratic, $ax^2+bxy+cy^2+dx+ey+f=0$, $a+c$, b^2-4ac, and the **discriminant** are invariants. See DISCRIMINANT—discriminant of a quadratic equation in two variables.

invariant subgroup of a group. A subgroup H which is such that the transform of any element of H by any element of the group is in H. *Syn.* Normal divisor, normal subgroup. See FACTOR—factor group.

semi-invariants. (*Statistics.*) If a linear transformation $y=a+bx$ of a stochastic variable x affects the parameters of the distribution function of x by multiplying them by b^r, the parameters of the distribution are **semi-invariants**. *E.g.*, the moments and cumulants about the mean are semi-invariants.

IN-VERSE', *adj.*, *n.* **inverse of an element of a group.** See GROUP.

inverse function. The function obtained by expressing the independent variable explicitly in terms of the dependent and considering the dependent variable as an independent variable. If $y=f(x)$ results in $x=g(y)$, the latter is the inverse of the former (and vice versa). It is customary to interchange the variables in the latter, writing $y=g(x)$ as the inverse. If $y=\sin x$, $y=\sin^{-1}x$ is used to denote the inverse trigonometric sine. If *function* means *single-valued function* (a *multiple-valued function* being a *relation*), then a function has an inverse if and only if it is one-to-one; *e.g.*, the inverse of $y=\sin x$ is $y=\sin^{-1}x$ if the domains of these functions are $[-\frac{1}{2}\pi, +\frac{1}{2}\pi]$ and $[-1, +1]$ respectively and their ranges are $[-1, +1]$ and $[-\frac{1}{2}\pi, +\frac{1}{2}\pi]$, respectively.

inverse hyperbolic functions. See HYPERBOLIC—inverse hyperbolic functions.

inverse logarithm of a given number. The number whose logarithm is the given number. Log 100 is 2; hence 100 is inverse log 2. *Syn.* Antilogarithm.

inverse of an implication. The implication which results from replacing both the antecedent and the consequent by their negations. *E.g.*, the inverse of "If x is divisible by 4, then x is divisible by 2" is the (false) statement "If x is not divisible by 4, then x is not divisible by 2." The *converse* and the *inverse* of an implication are equivalent—they are either both true or both false.

inverse of a number. One divided by the number. *Syn.* Reciprocal of a number.

inverse of an operation. That operation which, when performed after a given operation, annuls the given operation. Subtraction of a quantity is the inverse of addition of that quantity. Addition is likewise the inverse of subtraction. See TRANSFORMATION—inverse transformation.

inverse of a point or curve. See INVERSION.

inverse or reciprocal proportion. A proportion containing one reciprocal ratio. See INVERSELY—inversely proportional quantities.

inverse or reciprocal ratio of two numbers. The ratio of the reciprocals of the numbers.

inverse trigonometric function. See TRIGONOMETRIC.

inverse variation. See VARIATION—inverse variation.

IN-VER'SION, *adj.*, *n.* **inversion formulas.** Such formulas as *Fourier transforms*, *Laplace transforms*, and the *Mellin inversion formulas* which give a pair of linear transformations T_1, T_2 such that T_2 applied to $T_1(f)$ produces f for any function of a certain class. See FOURIER, LAPLACE, and MELLIN.

inversion of a point with respect to a circle. The finding of the point on the radius through the given point such that the product of the distances of the two points from the center of the circle is equal to the square of the radius. Either of the points is called the **inverse** of the other and the center of the circle is called the **center of inversion.** Any curve whose points are the inverses of the points of a given curve is called the inverse of the given curve. *E.g.*, the inverse of a circle which passes through the center of inversion is a straight line; the inverse of any other circle is a circle. If the equation of a curve is $f(x, y) = 0$, the equation of its inverse relative to a circle with center at the origin is

$$f\left(\frac{k^2x}{x^2+y^2}, \frac{k^2y}{x^2+y^2}\right) = 0,$$

where k is the radius of the circle.

inversion of a point with reference to a sphere. The finding of a point on the radius through the given point such that the product of the distances of the two points from the center of the sphere is equal to the square of the radius of the sphere. *E.g.*, the inverse of every sphere with respect to a fixed sphere is another sphere, except that the inverse of a sphere passing through the center of the fixed sphere is a plane.

inversion in a sequence of objects. The interchange of two adjacent objects. The *number of inversions* in a sequence is the minimum number of inversions which can be performed in order to put the objects in a certain *normal* order. The permutation 1, 3, 2, 4, 5 has one inversion, if the normal order is 1, 2, 3, 4, 5; whereas the permutation 1, 4, 3, 2, 5 has three. A permutation is said to be **odd** or **even** according as it contains an *odd* or an *even* number of inversions.

proportion by inversion. See PROPORTION.

IN-VERSE'LY, *adv.* **inversely proportional quantities.** Two variable quantities having their product constant; *i.e.*, either of them is equal to a constant times the inverse (reciprocal) of the other. The term has no significance unless the quantities are variables.

IN-VER'SOR, *n.* A mechanical device which simultaneously traces out a curve and its *inverse.* A rhombus, with its sides pivoted at the vertices and a pair of opposite vertices each linked to a fixed point (the *center of inversion*) by equal links, is such a mechanism, called a *Peaucellier's cell.* When one of the unlinked vertices traces out a curve, the other traces out the inverse of the curve. See INVERSION.

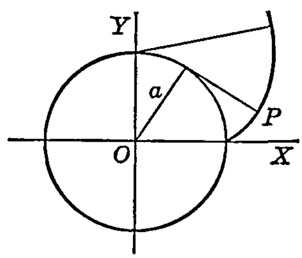
IN-VEST'MENT, *n.* Money used to buy notes, bonds, etc., or put into any enterprise for the purpose of making profit.

fixed investment. An investment which yields a fixed income; the amount which must be put into a sinking fund at the end of each year to accumulate to a given sum at the end of a given term.

investment, or investor's, rate. See YIELD.

mathematics of investment. Same as MATHEMATICS OF FINANCE.

IN'VO-LUTE, *n.* **involute of a curve.** For a **plane curve,** the locus of a fixed point on a nonflexible string as it is unwound, under tension, from the curve; the locus of any fixed point on a tangent line as this line rolls, but does not slide, around the curve; a curve orthogonal to the family of tangents to a given curve. Any two involutes of the same curve are parallel, *i.e.*, the segments cut off by the two involutes on a common normal are always of the same length. Also, an involute of a given curve is any curve whose *evolute* is



the given curve. The **evolute** of a curve is the locus of the centers of curvature of the given curve. The family of straight lines normal to a given curve are tangent to the evolute of this curve, and the change in length of the radius of curvature is equal to the change in length of arc of the evolute as the point on the curve moves continuously in one direction along the curve. The equation of the *evolute* is obtained by

eliminating the coordinates of the point on the curve between the equation of the curve and equations expressing the coordinates of the center of curvature in terms of the coordinates of the point on the curve. The **involute of a circle** is the curve described by the end of a thread as it is unwound from a stationary spool. The parametric equations of the involute of a circle in the position illustrated, where θ is the angle from the x-axis to the radius marked a, are:

$$x=a(\cos\theta+\theta\sin\theta)$$
$$y=a(\sin\theta-\theta\cos\theta).$$

An **involute of a space curve** is a curve orthogonal to the tangents of the given curve. The involutes of a space curve lie on its tangent surface. A given space curve has an infinitude of involutes; they constitute a family of geodesic parallels on the tangent surface (see GEODESIC—geodesic parallels on a surface). An **evolute of a space curve** is a curve of which the given curve is an involute. A given space curve C admits an infinitude of evolutes; when all the normals to C which are tangent to one of the evolutes are turned through the same angle in their planes normal to C, the resulting normals are tangent to another evolute of C.

involute of a surface. A surface of which the given surface is one of the two branches of the evolute. See EVOLUTE—evolute of a surface.

IN'VO-LU'TION, *n.* (1) Raising to a power; multiplying a quantity by itself a given number of times; the inverse of *evolution*. The process of squaring 2 is *involution*, of finding a square root of 4 is *evolution*. (2) A transformation which is its own inverse, *e.g.*, $x=1/x'$ is an involution. See TRANSFORMATION—inverse transformation.

involution on a line. A projective correspondence between the points of a line, which is its own inverse. Algebraically, the transformation

$$x'=\frac{ax+b}{cx-a},$$

where $a^2+bc\neq 0$. If $c\neq 0$, this can be written $x'=k/x$, by a proper choice of the origin.

involution of lines of a pencil. A correspondence between the lines, which is such that corresponding lines pass through corresponding points of an involution of points on a line which does not pass through the vertex of the pencil.

IR-RA'TION-AL, *adj.* **irrational algebraic surface.** The graph of an algebraic function in which the variable (or variables) appear irreducibly under a radical sign. The loci of $z=\sqrt{y+x^2}$ and $z=x^{1/2}+xy$ are irrational algebraic surfaces.

irrational equation. See EQUATION—irrational equation.

irrational exponent. See EXPONENT.

irrational number. A real number not expressible as an integer or quotient of integers; a nonrational number. The irrational numbers are those numbers defined by sets (A, B) of a *Dedekind cut* such that A has no greatest member and B has no least member. Also, the irrational numbers are precisely those infinite decimals which are not repeating. The irrational numbers are of two types, **algebraic irrational numbers** (irrational numbers which are roots of polynomial equations with rational coefficients; see ALGEBRAIC—algebraic number) and **transcendental numbers.** The numbers e and π are transcendental, as well as trigonometric and hyperbolic functions of any non-zero algebraic number and any power α^β, where α and β are algebraic numbers, α is not 0 or 1, and β is not a real rational number (see GELFOND—Gelfond-Schneider theorem). The set of all algebraic numbers (including the rational numbers) is less numerous than the transcendental numbers (a part of the irrational numbers) in two senses, of which the second is a consequence of the first: (1) the algebraic numbers are *countable*, while the transcendental numbers are not; (2) the set of algebraic numbers is of *measure zero*, while the *measure* of the transcendental numbers on an interval is the length of the interval. Also see LIOUVILLE—Liouville number, and NORMAL—normal number.

IR'RE-DUC'I-BLE, *adj.* **irreducible case** in Cardan's formula for the roots of a cubic. See CARDAN—Cardan's solution of the cubic.

irreducible equation. A rational integral equation of the form $f(x)=0$, where $f(x)$ is a polynomial irreducible in a certain field, usually the field of all rational numbers. See below, polynomials irreducible in a given field (domain).

irreducible matrices and transformations. See under REDUCIBLE.

irreducible radical. A radical which cannot be written in an equivalent rational form. The radicals $\sqrt{6}$ and $\sqrt{x}$ are irreducible, whereas $\sqrt{4}$ and $\sqrt[3]{x^3}$ are reducible since they are equal, respectively, to 2 and x.

polynomials irreducible in a given field (domain). Polynomials which cannot be factored into factors whose coefficients are in the given domain. Unless otherwise stated, *irreducible* means irreducible in the domain of the coefficients of the polynomial under consideration. The binomial x^2+1 is *irreducible* in the domain of real numbers, although in the domain of complex numbers it can be factored into $(x+i)(x-i)$, where $i^2=-1$. In elementary algebra an irreducible polynomial is a polynomial which cannot be factored into factors having rational coefficients.

IR-RO-TA'TION-AL, *n.* **irrotational vector in a region.** A vector whose integral around every *reducible* closed curve in the region is zero. The *curl* of a vector is zero at each point of a region if, and only if, the vector is irrotational, or if, and only if, it is the *gradient* of a scalar function (called a **scalar potential**); *i.e.*, $\nabla\times\mathbf{F}\equiv 0$ if and only if $\mathbf{F}=-\nabla\Phi$ for some scalar potential Φ. See CURL and GRADIENT.

I-SOG'O-NAL, *adj.* Having equal angles.

isogonal affine transformation. See AFFINE—affine transformation.

isogonal conjugate lines. See below, isogonal lines.

isogonal lines. Lines through the vertex of an angle and symmetric with respect to (making equal angles with) the bisector of the angle. The lines are called **isogonal conjugates.**

isogonal transformation. A transformation which leaves all the angles in any configuration unchanged. *E.g.*, the general similarity transformation is an isogonal transformation. *Syn.* Equiangular (or conformal) transformation.

I'SO-LATE, *v.* **isolate a root.** To find two numbers between which the root (and usually no other root) lies. See ROOT—root of an equation.

I'SO-LA-TED, *adj.* **isolated point.** See POINT—isolated point.

isolated set. A set which contains none of its limit points; a set consisting entirely of isolated points, a point of a set E being an isolated point of E if it has a neighborhood which contains no other point of E. If E has no limit points, it is said to be a **discrete set.** A discrete set is an isolated set, but the set $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$ is isolated and not discrete.

isolated singular point of an analytic function. See SINGULAR—singular point of an analytic function.

I'SO-MET'RIC, *adj.* **isometric family of curves on a surface.** A one-parameter family of curves on the surface such that the family together with its orthogonal trajectories forms an *isometric system* of curves on the surface.

isometric map. See ISOMETRY.

isometric parameters. See ISOMETRY.

isometric surfaces. Surfaces on which corresponding distances are equal and angles between corresponding lines are equal.

isometric system of curves on a surface. A system of two one-parameter families of curves (on the surface) which might be taken as parametric curves in an isometric map.

I-SOM'E-TRY, *n.* (1) An isothermic map. (2) A length preserving map. In the map given by $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, lengths are preserved if, and only if, the fundamental coefficients of the first order satisfy $E=G=1$, $F=0$. The coordinates u, v are called **isometric parameters.** The above functions and the functions $x=\bar{x}(u, v)$, $y=\bar{y}(u, v)$, $z=\bar{z}(u, v)$ give length preserving maps between corresponding surfaces S and $\bar{S}$ if, and only if, the corresponding fundamental coefficients of the first order satisfy $E=\bar{E}$, $F=\bar{F}$, $G=\bar{G}$. Then the surfaces S and $\bar{S}$ are said to be

applicable. *Syn.* Isometric map. (3) A one-to-one correspondence of a metric space A with a metric space B such that if x corresponds to x^* and y to y^*, then the "distances" $d(x, y)$ and $d(x^*, y^*)$ are equal. It is then said that A and B are **isometric.** If A and B are vector spaces with a norm, it is also required that the correspondence be an *isomorphism.* The preservation of distance is then equivalent to $\|x\|=\|x^*\|$ whenever x and x^* correspond. If A and B are Hilbert spaces, this is equivalent to the equality of the inner products (x, y) and (x^*, y^*) whenever x and x^* correspond and y and y^* correspond (see TRANSFORMATION — unitary transformation).

I-SO-MOR′PHISM, *n.* A one-to-one correspondence of a set A with a set B (the sets A and B are then said to be **isomorphic**). If operations such as multiplication, addition, or multiplication by scalars are defined for A and B, it is required that these correspond between A and B in the ways described in the following. If A and B are **groups** (or semigroups) with the operation denoted by $\cdot$, and x corresponds to x^* and y to y^*, then $x \cdot y$ must correspond to $x^* \cdot y^*$. An isomorphism of a set with itself is an **automorphism.** An automorphism of a group is an **inner automorphism** if there is an element t such that x corresponds to x^* if and only if $x^*=t^{-1}xt$; it is an **outer automorphism** if it is not an inner automorphism. The correspondence $\omega_1 \to \omega_2$, $\omega_2 \to \omega_1$, $1 \to 1$, is an outer automorphism of the group consisting of the cube roots of unity $(1, \omega_1, \omega_2)$. If a set S^* corresponds to a subgroup S by an automorphism (meaning that the elements of S^* and S correspond in pairs), then S^* is also a subgroup (if the automorphism is an inner automorphism, then S and S^* are said to be **conjugate subgroups**). If A and B are rings (or integral domains or fields) and x corresponds to x^* and y to y^*, then xy must correspond to x^*y^* and $x+y$ to x^*+y^*. If A and B are vector spaces, multiplication and addition must correspond as for rings and scalar multiplication must also correspond in the sense that if a is a scalar and x corresponds to x^*, then ax corresponds to ax^*. If the vector space is normed (*e.g.*, if it is a Banach or Hilbert space), then the correspondence must be continuous in both directions. This is equivalent to requiring that there be positive numbers c and d such that $c\|x\| \leqq \|x^*\| \leqq d\|x\|$ if x and x^* correspond. See HOMOMORPHISM and ISOMETRY.

I′SO-PER′I-MET′RIC or **I′SO-PER′I-MET′RI-CAL,** *adj.* **isoperimetric figures.** Figures having equal perimeters.

isoperimetric inequality. The inequality $A \leqq \frac{1}{4\pi} L^2$ between the area A of a plane region and the length L of its boundary curve. The equality sign holds if and only if the region is a circle. The inequality *holds also for regions on surfaces of non-positive total curvature*, and actually characterizes these surfaces. See below, isoperimetric problem in the calculus of variations.

isoperimetric problem in the calculus of variations. A problem in which it is required to make one integral a maximum or minimum while keeping constant the integral of a second given function (both integrals being of the general type indicated above under *calculus of variations*). An example is the problem of finding the closed plane curve of given perimeter and maximum area. In polar coordinates, the problem is that of finding the curve $r=f(\phi)$ for which $A=\int_0^{2\pi} \frac{1}{2}r^2\, d\phi$ is maximum and $P=\int_0^{2\pi} (r^2+r'^2)^{1/2}\, d\phi$ is constant. The solution can be found by maximizing the integral

$$A+\lambda P=\int_0^{2\pi} [\tfrac{1}{2}r^2+\lambda(r^2+r'^2)^{1/2}]\, d\phi$$

and determining the constant λ by the condition that P is a given constant. Most isoperimetric problems can be similarly reduced to the usual type of calculus of variations problem, a solution of the given problem necessarily being a solution of the reduced problem.

I-SOS′CE-LES, *adj.* **isosceles triangle.** See SPHERICAL—spherical triangle, and TRIANGLE.

isosceles trapezoid. See TRAPEZOID.

I'SO-THERM, *n.* (*Meteorology.*) A line drawn on a map through places having equal temperatures.

I'SO-THER'MAL, *adj.* Relating to equal temperatures.

isothermal change. (*Physics.*) A change in the volume and pressure of a substance which takes place at constant temperature.

isothermal-conjugate parameters. Parameters such that the parametric curves form an isothermal-conjugate system on the surface. See below, isothermal-conjugate system of curves on a surface.

isothermal-conjugate representation of one surface on another. See CONFORMAL—conformal-conjugate representation of one surface on another.

isothermal-conjugate system of curves on a surface. A system of two one-parameter families of curves on the surface S such that, when the curves are taken as parametric curves, the second fundamental quadratic form of S reduces to $\mu(u, v)(du^2 \pm dv^2)$. In particular, then, an isothermal-conjugate system is a conjugate system. See CONJUGATE—conjugate system of curves on a surface. An isothermal-conjugate system has a relation to the second fundamental quadratic form of S similar to that which an isothermic system has to the first fundamental quadratic form. See ISOTHERMIC—isothermic family of curves on a surface.

isothermal lines. Lines on a map connecting points which have the same mean (annual) temperatures. In *physics*, curves obtained by plotting pressure against volume for a gas kept at constant temperature.

I'SO-THER'MIC, *adj.* **isothermic family of curves on a surface.** A one-parameter family of curves on the surface such that the family together with its orthogonal trajectories forms an isothermic system of curves on the surface.

isothermic map. A map of a (u, v)-domain on a surface S in which the fundamental quantities of the first order satisfy $E=G=\lambda(u, v)$, $F=0$. The map is conformal except at the singular points where $\lambda=0$. The coordinates u, v are called **isothermic parameters.** See CONFORMAL—conformal map; and above, isothermic family of curves on a surface.

isothermic parameters. See above, isothermic map.

isothermic surface. A surface whose lines of curvature form an *isothermal system.* All surfaces of revolution are isothermic surfaces.

isothermic system of curves on a surface. A system of two one-parameter families of curves on a surface S such that there exist parameters u, v for which the curves of the system are the parametric curves of the surface, and for which the first fundamental quadratic form reduces to $\lambda(u, v)(du^2+dv^2)$. See above, isothermic map, and ISOTHERMAL—isothermal-conjugate system of curves on a surface.

I'SO-TROP'IC, *adj.* **isotropic curve.** Same as MINIMAL CURVE.

isotropic developable. An imaginary surface for which $EG-F^2$ vanishes identically. Such a surface is the tangent surface of a minimal curve. See SURFACE—fundamental coefficients of a surface.

isotropic elastic substances. Substances whose elastic properties are independent of the direction in the substances are said to be *isotropic.* This means that the elastic properties are the same in all directions.

isotropic matter. Matter which, at any point, has the same properties in any direction (such properties, for instance, as elasticity, density, and conductivity of heat or electricity).

isotropic plane. An imaginary plane with equation $ax+by+cz+d=0$, where $a^2+b^2+c^2=0$. *E.g.*, the osculating planes of minimal curves are isotropic.

IS'SUE, *n.* **issue of bonds, bank notes, money, or stock.** A set of bonds, bank notes, etc., which is (or has been) issued at a certain time.

IT'ER-ATE, *v.* Repeat; do or say over again.

iterated integral. See INTEGRAL—iterated integral.

J

JACOBI. Jacobi's elliptic functions. See ELLIPTIC.

Jacobi's polynomials. The polynomials $J_n(p, q; x) = F(-n, p+n; q; x)$, where $F(a, b; c; x)$ is the *hypergeometric function* and n is a positive integer. It follows that $J_n[1, 1, \frac{1}{2}(1-x)] = P_n(x)$ and

$$2^{1-n}J_n[0, \tfrac{1}{2}, \tfrac{1}{2}(1-x)] = T_n(x),$$

where $P_n(x)$ and $T_n(x)$ are Legendre's and Tchebycheff's polynomials, respectively.

Jacobi's theorem. See PERIODIC—doubly periodic function of a complex variable.

JA-CO'BI-AN, *n.* **Jacobian of two or more functions in as many variables.** For the n functions $f_i(x_1, x_2, \cdots x_n)$, $i = 1, 2, 3, \cdots n$, the Jacobian is the determinant

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_3} \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

which is often designated by:

$$\frac{D(f_1, f_2, f_3, \cdots, f_n)}{D(x_1, x_2, x_3, \cdots, x_n)},$$

or

$$\frac{\partial(f_1, f_2, f_3, \cdots, f_n)}{\partial(x_1, x_2, x_3, \cdots, x_n)}.$$

For two functions, $f(x, y)$ and $g(x, y)$, the Jacobian is the determinant

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}, \text{ designated by } \frac{D(f, g)}{D(x, y)}.$$

See INDEPENDENT—independent functions, and IMPLICIT—implicit function theorem. *Syn.* Functional determinant.

JENSEN. Jensen's inequality. For convex functions f, the inequality

$$f\left(\sum_1^n \lambda_i x_i\right) \leqq \sum_1^n \lambda_i f(x_i),$$

where the x_i are arbitrary values in the region on which f is convex and the λ_i are nonnegative numbers satisfying $\sum_1^n \lambda_i = 1$. The term "Jensen's inequality" is also applied to the inequality expressing the fact that for $t > 0$ the *sum of order t* is a nonincreasing function of t; *i.e.*, for positive numbers a_i and positive numbers s and t with $s > t$,

$$\left(\sum_1^n a_i^s\right)^{1/s} \leqq \sum_1^n (a_i^t)^{1/t}.$$

JOHNIAC. An automatic digital computing machine at the RAND Corporation. Named for the mathematician John von Neumann, the machine is similar to the one at the Institute for Advanced Study.

JOIN, *n.* See LATTICE, and SUM—sum of sets.

JOINT, *adj.* **joint expectation of life.** See EXPECTATION—expectation of life.

joint life annuity. See ANNUITY.

joint life insurance. See INSURANCE—life insurance.

joint variation. See VARIATION—joint variation.

JORDAN. Jordan content. See CONTENT—content of a set of points.

Jordan curve. Same as SIMPLE CLOSED CURVE. See SIMPLE.

Jordan curve theorem. A simple closed curve in a plane determines two regions, of which it is the common frontier (this theorem was incorrectly proved by Jordan; the first correct proof was given by Veblen in 1905).

Jordan matrix. A matrix having the elements of its principal diagonal equal and not zero, the elements immediately above those in the diagonal unity, and all other elements zero. *Syn.* Simple classical matrix. Such a matrix is also said to be in Jordan form.

Joukowski transformation. In complex variable theory, the transformation

$$w = z + 1/z.$$

It maps the points z and $1/z$ into the same point, so that the image of the exterior of the unit circle $|z| = 1$ is the same as the image of the interior of this circle. There are simple zeros of dw/dz at $z = \pm 1$, and otherwise $dw/dz \neq 0$; accordingly, the map is conformal except at these two points. The upper half of the z-plane, with its half of the unit circle deleted, is mapped on the

upper half of the w-plane. Under the Joukowski transformation, the exterior of a circle through the point $z=-1$ and having $z=+1$ in its interior is mapped on the exterior of a contour that, for some positions of the circle, bears a striking resemblance to the profile of an airplane wing. Such a contour is called a **Joukowski airfoil profile.**

JOULE, *n.* A unit of energy or work; the work done when a force of one *newton* produces a displacement of one meter in the direction of the force.

$$1 \text{ J}=10^7 \text{ erg}=.2390 \text{ calorie}.$$

JUMP, *n.* See DISCONTINUITY.

K

KAP'PA, *n.* The Greek letter κ, K.

kappa curve. The graph of the rectangular equation $x^4+x^2y^2=a^2y^2$. The curve has the lines $x=\pm a$ as asymptotes, is symmetrical about the coordinate axes and the origin, and has a double cusp at the origin. It is called the kappa curve because of its resemblance to the Greek letter K.

KEI, *adj.* **kei function.** See BER—ber function.

KEPLER'S LAWS of planetary motion. The three laws: (1) The orbits of the planets are ellipses having the sun at one focus. (2) The areas described by the radius vectors of a planet in equal times are equal. (3) The square of the period of revolution of a planet is proportional to the cube of its mean distance from the sun. These laws can be directly derived from the law of gravitation and Newton's laws of motion as applied to the sun and one planet.

KER, *adj.* **ker function.** See BER—ber function.

KER'NEL, *n.* **iterated kernels.** (*Integral Equations.*) The functions $K_n(x, y)$ defined by $K_1(x, y)=K(x, y)$ and $K_{n+1}(x, y) =\int_a^b K(x, t)K_n(t, y)\,dt$ $(n=1, 2, \cdots)$, where $K(x, y)$ is a given *kernel.* It follows that the *resolvent kernel* $k(x, t; \lambda)$ is equal to $(-1)\cdot \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x, t)$.

kernel of a homomorphism. If a homomorphism maps a **group** G onto a group G^*, then the kernel of the homomorphism is the set N of all elements which map onto the identity element of G^*. Then N is a *normal subgroup* of G and G^* is isomorphic with the *quotient group* G/N. If a homomorphism maps a **ring** R onto a ring R^*, then the kernel of the homomorphism is the set I of elements which map onto the zero element of R^*. The kernel I is an *ideal* and R^* is isomorphic with the *quotient ring* R/I (see IDEAL).

kernel of an integral equation. See VOLTERRA—Volterra's integral equations, and INTEGRAL—integral equation of the third kind. *Syn.* Nucleus.

resolvent kernel. See VOLTERRA—Volterra's reciprocal functions, and above, iterated kernels.

KHINTCHINE'S THEOREM. Let x_1, $x_2, \cdots$ be independent random variables having equivalent distribution functions $F(x)$, with mean u. Then the variable

$$\bar{x}=\sum_{i=1}^{n} x_i/n$$

converges in probability to u as $n\rightarrow\infty$.

KIL'O-GRAM, *n.* One thousand grams; the weight of a platinum rod preserved in Paris as the standard unit of the metric system of weights; approximately 2.2 lbs. avoirdupois. See DENOMINATE NUMBERS in the appendix.

KIL'O-ME'TER, *n.* One thousand meters; approximately 3280 feet. See DENOMINATE NUMBERS in the appendix.

KIL'O-WATT, *n.* A unit of measure of electrical power; 1000 watts. See WATT.

kilowatt-hour. A unit of energy; 1000 watt-hours; a kilowatt of power used for one hour; approximately $\frac{4}{3}$ horsepower acting for one hour.

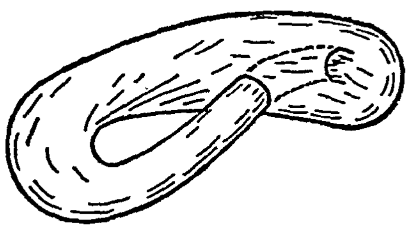
KIN'E-MAT'ICS, *n.* A branch of mechanics dealing with the motion of rigid bodies without reference to their masses or forces producing the motion. The ingre-

dients of kinematics are the concepts of space and time.

KI-NET'IC, *adj.* **kinetic energy.** See ENERGY.

KI-NET'ICS, *n.* That part of *mechanics* which treats of the effect of forces in changing the motion of bodies.

KLEIN. **Klein bottle.** A one-sided surface with no edges and no "inside" or "outside," which is formed by pulling the small end of a tapering tube through one side of the tube and spreading it so as to join with the other end.



KNOT, *n.* (*Naut.*) Nautical miles per hour. "A ship sails 20 knots" means it sails 20 nautical miles per hour.

knot in topology. A curve in space formed by looping and interlacing a piece of string in any way and then joining the ends together. Any two knots are topologically equivalent, but it may not be possible to continuously deform one into the other (*i.e.*, deform without breaking the string). *Tech.* A knot is a set of points in space which is topologically equivalent to a circle. The *theory of knots* consists of the mathematical analysis of possible types of knots and of methods for determining whether two knots can be continuously deformed into each other.

KRONECKER DELTA. The function δ_j^i of two variables i and j defined by $\delta_j^i=1$ if $i=j$, and $\delta_j^i=0$ if $i\neq j$. The *generalized Kronecker delta* ($\delta^{i_1 i_2 \cdots i_k}_{j_1 j_2 \cdots j_k}$) has k superscripts and k subscripts. If no two superscripts are equal and the subscripts are the same set of numbers as the superscripts, the value is said to be $+1$ or -1 according as an even or odd permutation is needed to arrange the subscripts in the same order as the superscripts. In all other cases its value is zero. See EPSILON. All of the Kronecker deltas are *numerical tensors.*

KUR-TO'SIS, *n.* (*Statistics.*) A descriptive property of distributions designed to indicate the general form of concentration around the mean. It is sometimes defined by the ratio $B_2=u_4/u_2^2$, where u_2 and u_4 are the 2nd and 4th moments with the arithmetic mean as the origin. In a normal distribution, $B_2=3$. It is called **mesokurtic, platykurtic** or **leptokurtic** according as $B_2=3$, $B_2<3$ or $B_2>3$. A platykurtic distribution often appears to be less heavily concentrated about the mean, a leptokurtic distribution to be more heavily concentrated, than the normal distribution.

L

LAC'U-NAR'Y, *adj.* **lacunary space relative to a monogenic analytic function.** A domain in the z-plane no point of which is covered by the domain of existence of the given function. See MONOGENIC—monogenic analytic function.

LAGRANGE. **Lagrange's form of the remainder for Taylor's theorem.** See TAYLOR'S THEOREM.

Lagrange's formula of interpolation. A formula for finding an approximation of an additional value of a function within a given interval of the independent variable, when certain values of the function within that interval are known. It is based upon the assumption that a polynomial of degree one less than the number of given values of the independent variable can be determined which will approximate the given function to the accuracy desired for the value sought. If $x_1, x_2, \cdots, x_n$ are the values of x for which the values of the function $f(x)$ are known, the formula is

$$f(x)=\frac{f(x_1)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}+\frac{f(x_2)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)}+\cdots, \text{ to } n \text{ terms.}$$

Lagrange's method of multipliers. A method for finding the maximum and minimum values of a function of several variables when relations between the variables

are given. If it is desired to find the maximum area of a rectangle whose perimeter is a constant, k, it is necessary to find the maximum value of xy for $2x+2y-k=0$. Lagrange's method of multipliers is to solve the three equations $2x+2y-k=0$, $\partial u/\partial x=0$, and $\partial u/\partial y=0$ for x and y, where $u=xy+t(2x+2y-k)$ and t is to be treated as an unknown to be eliminated. In general, given a function $f(x_1, x_2, \cdots, x_n)$ of n variables connected by h distinct relations, $\phi_1=0, \phi_2=0, \cdots, \phi_h=0$, in order to find the values of $x_1, x_2, \cdots, x_n$, for which this function may have a maximum or minimum, equate to zero the partial derivatives of the auxiliary function $f+t_1\phi_1 + \cdots + t_h\phi_h$, with respect to $x_1, x_2, \cdots, x_n$, regarding $t_1, t_2, \cdots, t_h$ as constants, and solve these n equations simultaneously with the given h relations, treating the t's as unknowns to be eliminated.

LAGRANGIAN FUNCTION. See POTENTIAL—kinetic potential.

LAGUERRE. associated Laguerre functions. The functions $y=e^{-\frac{1}{2}x}\, x^{\frac{1}{2}(k-1)} L_n^k(x)$, where $L_n^k(x)$ is an *associated Laguerre polynomial.* This function is a solution of the differential equation

$$xy''+2y'+[n-\tfrac{1}{2}(k-1)-\tfrac{1}{4}x -\tfrac{1}{4}(k^2-1)/x]y=0.$$

associated Laguerre polynomials. The polynomials

$$L_n^k(x)=\frac{d^k}{dx^k}L_n(x),$$

where $L_n(x)$ is a *Laguerre polynomial.* The differential equation $xy''+(k+1-x)y'+(n-k)y=0$ is satisfied by $y=L_n^k(x)$.

Laguerre polynomials. The polynomials

$$L_n(x)=e^x\frac{d^n}{dx^n}(x^ne^{-x}).$$

For all n,

$$(1+2n-x)L_n-n^2L_{n-1}-L_{n+1}=0,$$

and

$$(1-t)^{-1}e^{-xt/(1-t)}=\sum_{n=1}^{\infty}L_n(x)t^n/n!.$$

The Laguerre polynomial $L_n(x)$ is a solution of *Laguerre's differential equation* with the constant $\alpha=n$. The functions

$$e^{-x}L_n(x)$$

are *orthogonal functions* on the interval $(0, \infty)$.

Laguerre's differential equation. The differential equation $xy''+(1-x)y'+\alpha y=0$, where α is a constant.

LAMBDA-MATRIX. A matrix whose elements are polynomials in a variable λ.

LAMÉ'S CONSTANTS. Two positive constants λ and μ, introduced by Lamé, which completely characterize the elastic properties of an isotropic body. They are related to Young's modulus E and Poisson's ratio σ by the formulas

$$\lambda=\frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu=\frac{E}{2(1+\sigma)}.$$

The constant μ is called the **modulus of rigidity** (or **shearing modulus**), and it is equal to the ratio of the shearing stress to the change in angle produced by the shearing stress.

LAM'I-NA, *n.* A thin plate or sheet of uniform thickness and of constant density.

LAPLACE. Laplace's differential equation. The partial differential equation

$$\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}+\frac{\partial^2V}{\partial z^2}=0.$$

Under certain conditions, gravitational, electrostatic, magnetic, electric, and velocity potentials satisfy Laplace's equation. In general coordinates with the *fundamental metric tensor* g_{ij}, Laplace's equation takes the form

$$g^{ij}V_{,i,j}=0 \quad \text{or} \quad \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}g^{ij}\frac{\partial V}{\partial x^j}\right)}{\partial x^i}=0,$$

where g is the determinant $|g_{ij}|$, g^{ij} is $1/g$ times the cofactor of g_{ji} in g, $V_{,i,j}$ is the second *covariant derivative* of the scalar V, and the *summation convention* applies. In cylindrical and spherical coordinates, Laplace's equation takes the respective forms:

$$\frac{\partial^2V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\partial^2V}{\partial z^2}+\frac{1}{r^2}\frac{\partial^2V}{\partial\theta^2}=0,$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) +\frac{1}{r^2\sin^2\theta}\frac{\partial^2V}{\partial\phi^2}=0.$$

See DIRICHLET—Dirichlet characteristic properties of the potential function.

Laplace's expansion of a determinant. See DETERMINANT—Laplace's expansion of a determinant.

Laplace transform. The function $f(x)$ is the *Laplace transform* of $g(x)$ if

$$f(x)=\int e^{-xt}\,g(t)\,dt,$$

where the path of integration is some curve in the complex plane. It has become customary to restrict the path of integration to the real axis from 0 to $+\infty$. Suppose that $g(x)$ is defined for $x>0$, has only a finite number of infinite discontinuities, $\int |g(t)|\,dt$ exists for any finite interval, and $f(x)=\int_0^\infty e^{-xt}\,g(t)\,dt$, where this integral converges absolutely for $x>a$. Then this Laplace transformation has an inverse given by

$$g(x)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{xt}\,f(t)\,dt,$$

where the value of the integral is

$$\lim_{h\to 0}\tfrac{1}{2}[g(x+h)+g(x-h)]$$

if $g(x)$ is of *bounded variation* in the neighborhood of x and if $\alpha>a$. See FOURIER—Fourier transform.

LARGE, *adj., n.* **in the large.** See SMALL—in the small.

law of large numbers. (*Statistics.*) Same as TCHEBYCHEFF'S THEOREM or BERNOULLI'S THEOREM.

LA'TENT, *adj.* **latent root of a matrix.** See EIGENVALUE—eigenvalue of a matrix.

LAT'ER-AL, *adj.* **lateral area of a cone, cylinder, prism, etc.** See CONE, CYLINDER, PRISM, etc.

lateral face. See PYRAMID and PRISM.

LA'TIN, *adj.* **Latin square.** A method of ordering the observations in an experiment so as to control three sources of variability. The number of replications under each value of each of the sources of variability is the same as the number of different values under each source of variability. *E.g.*, a product produced by five different machines, operated by five different operators, with five different types of materials can be analyzed for variability.

LAT'I-TUDE, *n.* **latitude of a point on the earth's surface.** The number of degrees in the arc of a meridian from the equator to the point; the angle which the plane of the horizon makes with the earth's axis; the elevation of the pole of the heavens; the angle which a plumb line at the point makes with a plumb line on the same meridian at the equator.

middle latitude of two places. The arithmetic mean between the latitudes of the two places; one-half the sum of their latitudes if they are on the same side of the equator, one-half the difference (taken north or south according to which latitude was the larger) if the places are on different sides of the equator.

middle latitude sailing. See SAILING.

LAT'TICE, *n.* A partially ordered set in which any two elements have a greatest lower bound (g.l.b.) and a least upper bound (l.u.b.), the g.l.b. of a and b being an element c such that $c \leqq a$, $c \leqq b$, and there is no d for which $c<d\leqq a$ and $d \leqq b$, and the l.u.b. being defined analogously. The g.l.b. and the l.u.b. of a and b are denoted by $a \cap b$ and $a \cup b$, respectively, and called the meet and join, respectively, of a and b. The set of all subsets $U, V, \cdots$ of a given set is a lattice if $U \leqq V$ means that each element of U is contained in V. Then $U \cap V$ is the intersection of the sets U and V, and $U \cup V$ is the sum of U and V.

LA'TUS, *adj.* **latus rectum.** [*pl.* **latera recta.**] See PARABOLA, ELLIPSE, HYPERBOLA.

LAURENT. Laurent expansion of an analytic function of a complex variable. If $f(z)$ is analytic in the circular annulus $a<|z-z_0|<b$, then $f(z)$ can be represented by a two-way power series in the annulus. *I.e.*,

$$f(z)=\sum_{-\infty}^{\infty} a_n(z-z_0)^n.$$

The series is called the Laurent expansion

or the **Laurent series** of $f(z)$ about z_0. The coefficients a_n are given by

$$a_n = \frac{1}{2\pi i}\int_C (\zeta - z_0)^{-n-1} f(\zeta)\, d\zeta,$$

where C is a simple closed rectifiable curve lying in the annulus and inclosing the inner circle $|z-z_0| = a$.

Laurent series. See above, Laurent expansion of an analytic function of a complex variable.

LAW, *n.* A general principle or rule to which all cases to which it can apply must conform. For examples, see under ASSOCIATIVE, BOYLE'S LAW, COMMUTATIVE, EXPONENT, GOMPERTZ, BACTERIAL, COSINE, INERTIA, KEPLER, LOGARITHM, MAKEHAM, NEWTON, QUADRANT, SPECIES, and TANGENT.

LEAD'ING, *p.* **leading coefficient** of a polynomial in one variable. The coefficient of the term of highest degree.

LEAST, *adj.* **least common multiple.** See MULTIPLE—least common multiple.

least upper bound. See BOUND.

least value of a function. The smallest value the function takes on, if it takes on a smallest value.

method of least squares. See METHOD—method of least squares.

LEBESGUE. Lebesgue integral. First suppose that $f(x)$ is a *bounded* measurable function defined over a (Lebesgue) measurable set E of finite measure. If L and U are lower and upper bounds of $f(x)$, then the Lebesgue integral $\int_\Omega f(x)\, dx$ of $f(x)$ over Ω is defined as the unique limit of

$$\sum_{i=1}^{n} a_{i-1} m(e_i),$$

or of

$$\sum_{i=1}^{n} a_i m(e_i),$$

as the greatest of the numbers $a_i - a_{i-1}$ approaches zero, where the interval (L, U) is divided into n parts by the increasing sequence of numbers $a_0 = L, a_1, a_2, \cdots, a_n = U$, and where $m(e_i)$ is the measure of the set e_i, e_i consisting of all points x for which $a_{i-1} \leqq f(x) < a_i (i = 1, 2, \cdots n-1)$ and e_n of all x satisfying $a_{n-1} \leqq f(x) \leqq a_n$. If $f(x)$ is unbounded and $f_n^+(x)$ is defined by $f_n^+(x) = 0$ if $f(x) \leqq 0$, $f_n^+(x) = f(x)$ if $0 < f(x) \leqq n$, and $f_n^+(x) = n$ if $f(x) > n$, and $f_n^-(x)$ is defined by $f_n^-(x) = 0$ if $f(x) \geqq 0$, $f_n^-(x) = f(x)$ if $n \leqq f(x) < 0$, and $f_n^-(x) = n$ if $f(x) < n$, then $f(x)$ has the Lebesgue integral

$$\int_\Omega f(x)\, dx = \lim_{n\to\infty} \int_\Omega f_n^+(x)\, dx + \lim_{n\to-\infty} \int_\Omega f_n^-(x)\, dx,$$

provided these limits both exist. If the set Ω does not have finite measure and

$$\int_{\Omega\cdot I} f(x)\, dx$$

approaches a unique limit as the boundaries of an interval I all increase indefinitely, in any manner, then that limit is defined as $\int_\Omega f(x)\, dx$ ($\Omega\cdot I$ being the intersection of Ω and I). A function $\phi(x)$ defined on a set E contained in an interval I has a Lebesgue integral over E if, and only if, there exists a sequence of step functions (or of continuous functions) $f_n(x)$ such that

$$\lim_{n\to\infty} f_n(x) = \phi(x)$$

for almost all x of I (where $\phi(x)$ is taken as zero for points x not in E) and

$$\lim_{m,\, n\to\infty} \int_I |f_n(x) - f_m(x)|\, dx = 0.$$

In this case, $\lim_{n\to\infty} \int_I f_n(x)\, dx$ exists and is the Lebesgue integral of $f(x)$ over E. A function which has a Lebesgue integral over a set E is said to be **summable** over E. See MEASURABLE—measurable function. A function which has a Riemann integral necessarily has a Lebesgue integral, but not conversely.

Lebesgue measure. See MEASURABLE—measurable set.

LEFT, *adj.* **left-handed coordinate system.** See COORDINATE—left-handed coordinate system, and TRIHEDRAL.

left-handed curve. If the torsion of a directed curve C at a point P is positive, then a variable point P moving through the position P in the positive direction along C goes from the positive to the negative side of the osculating plane at P. See CANONICAL—canonical representation of a

space curve in the neighborhood of a point. Accordingly, C is said to be left-handed at P. See RIGHT-HANDED. *Syn.* Sinistrorsum [*Latin*], or sinistrorse curve.

left-handed trihedral. See TRIHEDRAL.

LEG, *n.* **leg of a right triangle.** Either one of the sides adjacent to the right angle.

LEGENDRE. associated Legendre functions. The functions

$$P_n^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x),$$

where $P_n(x)$ is a *Legendre polynomial;* $y = P_n^m(x)$ is a solution of the differential equation

$$(1-x^2)y'' - 2xy' + [n(n+1) - m^2/(1-x^2)]y = 0.$$

See HARMONIC—spherical harmonic, zonal harmonic.

Legendre's differential equation. The differential equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$. See below, Legendre's polynomials.

Legendre's necessary condition (*Calculus of Variations*). A condition, namely $f_{y'y'} \geqq 0$, that must be satisfied if the function y is to minimize $\int_{x_1}^{x_2} f(x, y, y')\,dx$. See CALCULUS—calculus of variations, EULER—Euler's equation, WEIERSTRASS—Weierstrass' necessary condition.

Legendre's polynomials. The coefficients $P_n(x)$ in the expansion.

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)h^n.$$

In particular, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$, $P_3(x) = \frac{1}{2}(5x^3-3x), \cdots$; $y = P_n(x)$ is a solution of *Legendre's differential equation.* For all n,

$$P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x),$$

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0,$$

and

$$P_n(\cos\theta) = \frac{(-1)^n}{n!} r^{n+1} \frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right),$$

where $\cos\theta = z/r$ and $r^2 = x^2+y^2+z^2$. The Legendre polynomials are a *complete* set of *orthogonal functions* on the interval $(-1, 1)$. Also called LEGENDRE'S COEFFICIENTS. See RODRIQUES' FORMULA, and SCHLÄFLI'S INTEGRAL.

Legendre's symbol $(c|p)$. The symbol $(c|p)$ is equal to 1 if the integer c is a quadratic residue of the odd prime p, and is equal to -1 if c is a quadratic nonresidue of p. *E.g.*, $(6|19) = 1$ since the congruence $x^2 \equiv 6 \pmod{19}$ has a solution, and $(39|47) = -1$ since the congruence $x^2 \equiv 39 \pmod{47}$ has no solutions.

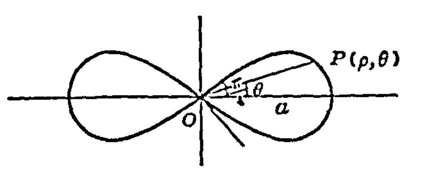
LEIBNIZ' THEOREM or FORMULA. A theorem for finding the nth derivative of the product of two functions; the theorem is as follows:

$$D_x^n(uv) = vD_x^n u + nD_x^{n-1}uD_xv + \tfrac{1}{2}n(n-1)D_x^{n-2}uD_x^2v + \cdots + uD_x^n v,$$

the numerical coefficients being the coefficients in the expansion of $(u+v)^n$ and the indicated derivatives being of the same order as the corresponding powers in this expansion. Analogously, the nth derivative of the product of k functions can be written out from the multinomial expansion of the nth power of the sum of k quantities.

LEM'MA, *n.* A theorem proved for use in the proof of another theorem.

LEM-NIS'CATE, *n.* The plane locus of the foot of the perpendicular from the origin to a variable tangent to the equilateral hyperbola; the locus of the vertex of a triangle when the product of the two adjacent sides is kept equal to one-fourth the square of the third side (which is fixed in length). Employing polar coordinates, if the node (see figure) is taken as the pole,



the axis of symmetry as the initial line, and the greatest distance from the pole to the curve as a, the equation of the lemniscate is $\rho^2 = a^2\cos 2\theta$. Its corresponding rectangular Cartesian equation is

$$(x^2+y^2)^2 = a^2(x^2-y^2).$$

This curve was first studied by Jacques Bernoulli, hence is frequently called *Bernoulli's lemniscate*, or the *lemniscate of Bernoulli*. See CASSINI.

LENGTH, *n.* **length of a curve.** Let A and B be points on a curve and choose points $A=P_1, P_2, P_3, \cdots, P_n=B$, starting at A and moving along the curve to B. If the sum of the lengths of the chords joining successive points,

$$\overline{P_1P_2}+\overline{P_2P_3}+\overline{P_3P_4}+\cdots+\overline{P_{n-1}P_n},$$

approaches a limit as the number of chords increases in such a way that the length of the longest chord approaches zero, this is said to be the length of the curve between A and B (otherwise the length is not defined, *i.e.*, does not exist). The length of some curves can be computed by integration (see ELEMENT—element of integration). *E.g.*, for a plane curve $f(x, y)=0$ in rectangular coordinates, the length between points (x_1, y_1) and (x_2, y_2) is given by

$$\int_{x_1}^{x_2} [1+(dy/dx)^2]^{1/2}\,dx$$

or

$$\int_{y_1}^{y_2} [1+(dx/dy)^2]^{1/2}\,dy.$$

length of a rectangle. The length of its longer side.

length of a straight line segment. If a straight line containing the line segment has real numbers associated with its points (in the usual way by which coordinates are defined), the length of the line segment is the absolute value of the difference of its end-points (whether or not the end-points belong to the segment). An equivalent definition is that the length is the number of times a unit interval will fit in the line segment, this being defined as the sum of the number of complete unit intervals that can be embedded in the line, $\frac{1}{2}$ the number of integrals of length $\frac{1}{2}$ that can be embedded in the remainder, $\frac{1}{4}$ the number of intervals of length $\frac{1}{4}$ that can be embedded in the remainder, etc. (an interval of length $\frac{1}{2}$ is one of two intervals which results from bisecting the unit interval, etc.). See MEASURABLE—measurable set.

LEPTOKURTIC, *adj.* **leptokurtic distribution.** See KURTOSIS.

LESS, *adj.* See GREATER.

LEV'EL, *adj.* **level lines.** See CONTOUR—contour lines.

net level premiums. See PREMIUM.

LE'VER, *adj., n.* A rigid bar used to lift weights by placing the bar against a support called the *fulcrum*, and applying a force or weight. A lever is said to be of the *first*, *second*, or *third* type, respectively, when the fulcrum is under the bar and between the weights, under the bar at one end, or above the bar at one end.

law of the lever. If there is equilibrium for two weights (forces), the weights (forces) are to each other inversely as their lever arms, or, what is equivalent, the products of the weights by their lever arms are equal, or the algebraic sum of the moments of all the forces about the fulcrum is equal to zero.

lever arm. The distance of a weight (or line of action of a force) from the fulcrum of the lever.

L'HOSPITAL'S RULE. A rule of evaluating *indeterminate forms*: If $f(x)/F(x)$ approaches one of the forms $0/0$ or ∞/∞ when x approaches a, and $f'(x)/F'(x)$, where $f'(x)$ and $F'(x)$ are the derivatives of $f(x)$ and $F(x)$, approaches a limit as x approaches a, then $f(x)/F(x)$ approaches the same limit. *E.g.*, if $f(x)=x^2-1$, $F(x)=x-1$, and $a=1$, $f(a)/F(a)$ takes the form $0/0$ and

$$\lim_{x\to 1} f'(x)/F'(x)=\lim_{x\to 1} 2x=2,$$

which is the limit approached by

$$(x^2-1)/(x-1)$$

as x approaches 1. L'Hospital's rule can be proved under the assumptions: (1) there is a neighborhood of $x=a$ in which $F(x)\neq 0$ if $x\neq a$; (2) $f(x)$ and $F(x)$ are continuous in some neighborhood of $x=a$ except perhaps at a; (3) $f'(x)$ and $F'(x)$ exist in some neighborhood of $x=a$ (except perhaps at $x=a$) and do not vanish simultaneously for $x\neq a$. See MEAN—mean value theorems for derivatives.

L'HUILIER'S THEOREM. A theorem relating the spherical excess of a spherical triangle to the sides:

$$\tan\tfrac{1}{4}E=\sqrt{\left\{\begin{matrix}\tan\frac{1}{2}s\,\tan\frac{1}{2}(s-a)\,\tan\frac{1}{2}(s-b)\\ \times\tan\frac{1}{2}(s-c)\end{matrix}\right\}}$$

where a, b, and c are the sides of the triangle, E is the spherical excess, and

$$s = \tfrac{1}{2}(a+b+c).$$

LI'A-BIL'I-TY, *n.* See ASSETS.

LIE. Lie group. A topological group which can be given an *analytic structure* for which the coordinates of a product xy are analytic functions of the coordinates of the elements x and y, and the coordinates of the inverse x^{-1} of an element x are analytic functions of x. See EUCLIDEAN—locally Euclidean space.

LIFE, *adj., n.* (1) (*Life Insurance.*) The difference between a policy date and the death of the insured. (2) The period during which something under consideration is effective, useful, or efficient, such as the life of a lease or contract, the life of an enterprise, or the life of a machine.

expectation of life. See EXPECTATION.

life annuity. See ANNUITY.

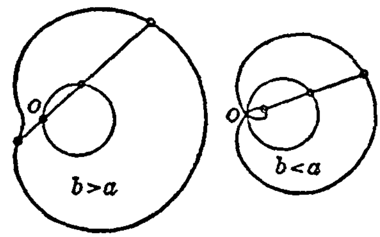
life insurance. See INSURANCE—life insurance.

LIFT, *n.* In aerodynamics, if the total force **F** that is applied to a body B gives B a motion with velocity vector **v**, then the component of **F** perpendicular to **v** is called *lift.* See DRAG.

LIGHT, *n.* **reflection and refraction of light.** See REFLECTION and REFRACTION.

LIKE'LI-HOOD, *adj.* **likelihood ratio.** The ratio of the probability of a random sample under a given hypothesis about the parameters of the population to the probability of the sample under the hypothesis that the sample was drawn from that population whose parameters are such that this probability is maximized.

LIM'A-ÇON, *n.* The locus of a point on a line, at a fixed distance from the intersection of this line with a fixed circle, as the line revolves about a point on the circle. If the diameter of the circle is taken as a (see figure), the fixed distance as b, the fixed point as the pole, the moving line as the radius vector, and the diameter through the fixed circle as the polar axis, the equation of the limaçon is $r = a\cos\theta + b$. This curve was first studied and named by Pascal, hence is usually called **Pascal's limaçon.** When b is less than the diameter of the fixed circle, the curve consists of two loops, one within the other; the outside loop is heart-shaped and the inside loop is pear-shaped, the curve having



a node at the origin. When b is equal to a, there is one heart-shaped loop, called the **cardioid.** When b is greater than a, there is one loop, whose shape tends toward that of a circle as b increases.

LIM'IT, *n.* **central limit theorem.** (*Statistics.*) See CENTRAL.

fundamental theorems on limits. (1) If a variable u approaches a limit l and c is a constant, then cu approaches the limit cl. (2) If u and v approach the limits l and m, respectively, then $u+v$ approaches the limit $l+m$. (3) If u and v approach the limits l and m, respectively, then uv approaches the limit lm. (4) If u and v approach the limits l and m, respectively, and if m is not zero, then u/v approaches the limit l/m. (5) If u steadily increases but never becomes greater than a given constant, A, then u approaches a limit, U, which is not greater than A. (6) If u steadily decreases but never becomes less than a given constant, B, then u approaches a limit, U, which is not less than B.

inferior and superior limits. See INFERIOR, SUPERIOR, and SEQUENCE—accumulation point of a sequence.

limit of a function. See below, limit of a variable.

limit on the left or right. The limit on the right of a function $f(x)$ at a point x_0 is a number M such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|M - f(x)| < \epsilon$ if

$$x_0 < x < x_0 + \delta,$$

and the limit on the left is a number N such that for any $\epsilon > 0$ there exists a $\delta > 0$

such that $|N-f(x)|<\epsilon$ if $x_0-\delta<x<x_0$. A function is continuous on the right (or left) at x_0 if, and only if, the limit on the right (or left) exists and is equal to $f(x_0)$. These limits are denoted by $\lim_{x\to x_0^+} f(x)$ or $f(x_0+0)$ and $\lim_{x\to x_0^-} f(x)$ or $f(x_0-0)$. *Syn.* Right-hand limit, left-hand limit.

limit point. Same as ACCUMULATION POINT.

limit of a product, quotient, sum. See above, fundamental theorems on limits.

limit of the ratio of an arc to its chord. Refers to the limit of this ratio when the chord (or arc) approaches zero. If the curve is a *circle*, this limit is 1, and it is also 1 for rectifiable curves.

limit of a sequence. See various headings under SEQUENCE, especially *limit of a sequence.*

limit of a variable. A quantity such that the difference between it and the variable can be made to become and remain as near zero as one pleases; *e.g.*, the limit of $1/x$ is 0, if x increases beyond all bounds; the same is true if x takes on numerically large negative values, and also if x takes on, alternately, large positive and numerically large negative values, such as $10, -10, 100, -100, 1000, -1000, \cdots$. A variable is said to *approach its limit* or to *approach* a certain quantity as *a limit.* The fact that a variable function, $f(x)$, approaches a certain limit k as x approaches a given number a is written

$$\lim_{z\to a} f(x)=k$$

and stated "limit of $f(x)$, as x approaches a, is k." *Tech.* A function $f(x)$ is said to approach k as a limit as x approaches a if, for every positive number ϵ, there is a number δ such that $|f(x)-k|<\epsilon$ if $0<|x-a|<\delta$; $f(x)$ is said to approach the limit k as x *becomes infinite* if, for every positive number ϵ, there is a number δ such that $|f(x)-k|<\epsilon$ if $x>\delta$.

limits of a class interval. (*Statistics.*) The upper and lower limits of the values of a class interval. *Syn.* Class bounds.

limits of integration. See INTEGRAL—definite integral.

problems of limit analysis and design. The problem of determining the carrying capacity, for a given type of loading, of a structure of which the geometry and the fully plastic moments of the members are known, is said to be a *problem of limit analysis.* A *problem of limit design* is the problem of determining the fully plastic moments of the members of a structure, of which the geometry and the loads it has to carry are known, in such a way as to minimize its weight.

LIM'IT-ING, *adj.* **limiting age in a mortality table.** The age which the last survivor of the group upon which the table is based would have attained had he lived to the end of the year during which he died.

limiting value. Same as LIMIT OF A VARIABLE.

LINDELÖF. Lindelöf space. A topological space T such that, for any class C of open sets whose union contains T, there is a countable class C^* of sets whose union contains T and such that each member of C^* is a member of C. A topological space which satisfies the *second axiom of countability* is a Lindelöf space (Lindelöf's theorem).

LINE, *n.* (1) Same as CURVE. (2) Same as STRAIGHT LINE. See below.

addition of line segments. See SUM—sum of directed line segments.

angle between two lines, or between a line and a plane. See ANGLE—angle of intersection.

bisection point of a line segment. Same as MID-POINT of the line segment.

broken line. A figure composed entirely of segments of straight lines, laid end to end.

concurrent lines. See CONCURRENT.

contour lines. See CONTOUR.

curved line. A line which is neither a broken nor a straight line; a line that continually turns (changes direction).

directed line. See DIRECTED.

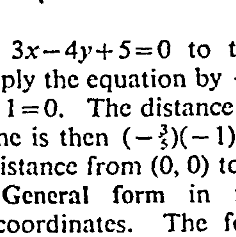
direction of a line. See DIRECTION—direction of a line.

equation of a straight line. A relation between the coordinates of a point which holds when, and only when, the point is on the straight line. The following are forms of the **equation of a straight line in the plane:** (1) **Slope-intercept form.** The equation (in rectangular Cartesian coordinates) $y=mx$

$+b$, where m is the slope of the line and b its intercept on the y-axis. (2) **Intercept form.** The equation $x/a+y/b=1$, where a and b are the x and y intercepts, respectively. (3) **Point-slope form.** The equation $y-y_1=m(x-x_1)$, where m is the slope and (x_1, y_1) is a point on the line. (4) **Two-point form.** The equation

$$(y-y_1)/(y_2-y_1)=(x-x_1)/(x_2-x_1)$$

where (x_1, y_1) and (x_2, y_2) are two points through which the line passes. This form can be written more elegantly by equating to zero the third-order determinant whose first, second, and third rows contain, in order, the sets of elements $x, y, 1; x_1, y_1, 1; x_2, y_2, 1$. (5) **Normal form.** The equation $x\cos\omega+y\sin\omega-p=0$, where ω is the angle from the x-axis to the perpendicular from the origin to the line and p is the length of the perpendicular from the origin to the line; any equation $ax+by+c=0$ with $a^2+b^2=1$ (some authors require that the sign of a be chosen so that $a=\cos\omega$). If $a^2+b^2=1$, then $ax+by+c$ is equal to the distance from the point (x, y) to the line $ax+by+c=0$ (positive on one side of the line and negative on the other). An equation $ax+by+c=0$ can be changed to normal form by dividing all coefficients by $\pm(a^2+b^2)^{1/2}$, the sign being the opposite of the constant term c (it is sometimes required that the angle in the normal form be less than 180°, in which case the sign of the coefficient of y is taken as positive, or the coefficient of x positive if y doesn't appear.)



To reduce $3x-4y+5=0$ to the normal form, multiply the equation by $-\frac{1}{5}$, getting $-\frac{3}{5}x+\frac{4}{5}y-1=0$. The distance from $(-1, 5)$ to the line is then $(-\frac{3}{5})(-1)+(\frac{4}{5})(5)-1=3\frac{3}{5}$; the distance from $(0, 0)$ to the line is -1. (6) **General form in rectangular Cartesian coordinates.** The form which includes all other forms in this system of coordinates as special cases. It is written $Ax+By+C=0$, where A and B are not both zero. (7) **Polar form.** The equation $r=p\sec(\theta-\omega)$, where p is the perpendicular distance from the pole to the line, ω is the inclination of this perpendicular to the polar axis, and r and θ are the polar coordinates of a variable point on the line; see figure, above. The **equation of a straight line in space** may be of the following types: (1) The equations of any two planes which intersect in the given line. (2) Equation of planes parallel to the coordinate axes are used as the **symmetric (standard) form** of the equation of a straight line, the equation being written

$$(x-x_1)/l=(y-y_1)/m=(z-z_1)/n,$$

where l, m, and n are *direction numbers* of the line and x_1, y_1, and z_1 are the coordinates of a point on it. (3) The **two-point form** of the equations of a line is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1},$$

where (x_1, y_1, z_1) and (x_2, y_2, z_2) are two points on the line. (4) The **parametric form** is derived by equating each fraction in the *symmetric form* to a parameter, say t, and solving for x, y, and z. This gives $x=x_1+lt$, $y=y_1+mt$, $z=z_1+nt$. The points determined by giving t any values desired lie on the line. If l, m, n are the direction cosines of the line, t is the distance between the points (x, y, z) and (x_1, y_1, z_1).

ideal line, or line at infinity. *Algebraically*, the locus of the equation $x_3=0$ in the system of *homogeneous coordinates* related to the Cartesian relations $x_1/x_3=x$, $x_2/x_3=y$. See COORDINATE—homogeneous coordinates. *Geometrically*, the aggregate of all *ideal* points in the plane. *Syn.* Ideal line.

level lines. See CONTOUR—contour lines.

line of best fit for a set of statistical values. (1) Usually the line determined by the method of least squares. (2) The trend line.

line integral. See INTEGRAL—line integral.

line segment. The part of a straight line between two points on the line (the points may, or may not, belong to the line segment).

material line. See MATERIAL.

midpoint of a line. See MIDPOINT.

nodal line. See NODAL.

oblique and parallel line (relative to another line or to a plane). See OBLIQUE, and PARALLEL.

perpendicular line (relative to another line or to a plane). See PERPENDICULAR.

plumb line. (1) The line in which a string hangs, when supporting a weight. (2) The string itself.

polar line (and pole of a line). See POLE—pole and polar of a conic.

projection of a line. See PROJECTION.

straight line. A curve such that if any part of it is placed so as to have two points in common with any other part, it will lie along the other part; a straight line is usually called simply a line. *Tech.* (1) The set of all "points" (x, y) which satisfy a given linear equation, $ax+by+c=0$, where a and b are not both zero. (2) An object called "line" in an axiomatic structure called "geometry". This may be an undefined element, which taken with some other element (or elements), such as point, satisfies certain assumptions; *e.g.*, two lines determine a point (including the ideal point), and two points determine a line.

trace of a line. See TRACE.

trend line. The line that represents the general drift of a set of data. See above, line of best fit.

LIN'E-AL, *adj.* **lineal element.** (*Differential Equations.*) A directed line segment through a point, whose slope taken with the coordinates of the point satisfy a given differential equation.

LIN'E-AR, *adj.* (1) In a straight line. (2) Along or pertaining to a curve. (3) Having only one dimension.

coefficient of linear expansion. See COEFFICIENT—coefficient of linear expansion.

consistency of a system of linear equations. See CONSISTENCY.

equation of linear regression. (*Statistics.*) $(y-\bar{y})/(x-\bar{x})=r(\sigma_y/\sigma_x)$, where σ_x and σ_y are the standard deviations of two sets of data (numbers) denoted by x's and y's, respectively, r is the correlation coefficient, and $\bar{x}$ and $\bar{y}$ are the means of the x's and y's. See DEVIATION—standard deviation, and COEFFICIENT—correlation coefficient.

linear combination. A linear combination of two or more quantities is a sum of the quantities, each multiplied by a constant (not all the constants being zero). See DEPENDENT—linearly dependent. For equations $f(x, y)=0$ and $F(x, y)=0$, a linear combination is $kf(x, y)+hF(x, y)=0$, where k and h are not both zero. The graph of the linear combination of any two equations passes through the points of intersection on their graphs and cuts neither in any other point. A **convex linear combination** of quantities x_i $(i=1, 2, \cdots, n)$ is an expression of the form $\sum_1^n \lambda_i x_i$, where $\sum_1^n \lambda_i=1$ and each λ_i is a non-negative real number. See BARYCENTRIC—barycentric coordinates.

linear congruence. A congruence in which all variable terms are of the first degree. *E.g.*, $12x+10y-6\equiv 0 \pmod{42}$ is a *linear congruence.*

linear differential equations. See DIFFERENTIAL—linear differential equations.

linear element. See ELEMENT—element of integration. For a surface S: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, and a curve $f(u, v)=0$ on S, the linear element ds is given by $ds^2=dx^2+dy^2+dz^2=E\,du^2+2F\,du\,dv+G\,dv^2$, where

$$E=\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2,$$

$$F=\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v},$$

$$G=\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2,$$

and du and dv satisfy $\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv=0.$ Also written $ds^2=g_{\alpha\beta}\,du^\alpha\,du^\beta$, in tensor notation. The **linear element of a surface** is the element of length ds, given by $ds^2=E\,du^2+2F\,du\,dv+G\,dv^2$, without necessary reference to any particular curve on the surface. In an n-dimensional Euclidean space there exist rectangular Cartesian coordinates y^i (not so in other Riemannian spaces) so that the linear element has the form

$$ds^2=(dy^1)^2+(dy^2)^2+\cdots+(dy^n)^2.$$

In other words, the fundamental Euclidean metric tensor $g_{ij}(x^1, \cdots, x^n)$ has the components ${}^*g_{ij}(y^1, \cdots, y^n)=\delta_{ij}$ in rectangular coordinates y^i, where δ_{ij} is Kronecker's

delta. If $y^i = f^i(x^1, \cdots, x^n)$ is the transformation of coordinates from any general coordinates x^i to rectangular coordinates y^i, then the components $g_{ij}(x^1, \cdots, x^n)$ in general coordinates can be computed by

$$g_{ij}(x^1, \cdots, x^n) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\alpha}{\partial x^j}.$$

Also called the **line element** and **element of length.**

linear equation or expression. An algebraic equation or expression which is of the first degree in its variable (or variables); *i.e.*, its highest degree term in the variable (or variables) is of the first degree. The equations, $x+2=0$ and $x+y+3=0$, are linear. An equation or expression is said to be **linear in a certain variable** if it is of the first degree in that variable. The equation $x+y^2=0$ is linear in x, but not in y.

linear expansion. Expansion in a straight line; expansion in one direction. The longitudinal expansion of a rod that is being heated is a linear expansion.

linear group. See GROUP—full linear group, real linear group.

linear hypothesis. See HYPOTHESIS—linear hypothesis.

linear interpolation. See INTERPOLATION.

linear programming. See PROGRAMMING.

linear space. Same as VECTOR SPACE.

linear theory of elasticity. See ELASTICITY.

linear transformation. (1) A transformation effected by an equality which is a linear algebraic equation in the old variables and in the new variables. The **general linear transformation** in *one dimension* is of the form

$$x' = (ax+b)/(cx+d),$$

or $\rho x_1' = ax_1 + bx_2$, $\rho x_2' = cx_1 + dx_2$, where ρ is an arbitrary constant and x_1, x_2 are *homogeneous coordinates* defined by $x_1/x_2 = x$. In *two dimensions* the general linear transformation is

$$x' = (a_1x + b_1y + c_1)/(d_1x + e_1y + f_1),$$
$$y' = (a_2x + b_2y + c_2)/(d_1x + e_1y + f_1),$$

or in *homogeneous coordinates*

$$\rho x_1' = a_1x_1 + b_1x_2 + c_1x_3,$$
$$\rho x_2' = a_2x_1 + b_2x_2 + c_2x_3,$$
$$\rho x_3' = a_3x_1 + b_3x_2 + c_3x_3.$$

General linear transformations in more than two dimensions are defined similarly. Called singular or nonsingular according as the determinant of the coefficients on the right side is or is not zero. (2) A transformation which takes $ax+by$ into $ax'+by'$ for all a and b if it takes vectors x and y into x' and y'. It is sometimes also required that the transformation be continuous. Here x and y may be vectors in n-dimensional Euclidean space or in a vector space. The numbers a and b may be real, complex, or of any field for which multiplication with elements of the vector space is defined. In Euclidean space, such a transformation is of the form

$$y_i = \sum_{j=1}^{n} a_{ij}x_j \; (i = 1, 2, \cdots, n),$$

or $y = Ax$, where x and y are one-column *matrices (vectors) with elements* $(x_1, x_2, \cdots x_n)$ and $(y_1, y_2, \cdots, y_n)$, A is the matrix (a_{ij}), and multiplication is matrix multiplication (see MATRIX—matrix of a linear transformation). A linear transformation T between normed vector spaces is said to be **bounded** if there exists a constant M such that $\|T(x)\| \leqq M\|x\|$ for each x. The smallest such number M is called the **norm** of the linear transformation and is denoted by $\|T\|$. If such a number M does not exist, the linear transformation is said to be **unbounded.** A linear transformation is continuous if and only if it is bounded.

linear velocity. See VELOCITY.

solution of a system of linear equations. See ELIMINATION, CRAMER'S RULE, and CONSISTENCY—consistency of linear equations.

LIN'E-AR-LY, *adv.* **linearly dependent and linearly independent.** See DEPENDENT—linearly dependent.

linearly ordered set. See ORDERED—simply ordered set.

LIOUVILLE. Liouville function. The function $\lambda(n)$ of the positive integers defined by $\lambda(1)=1$ and $\lambda(n)=(-1)^{a_1+\cdots+a_r}$ if $n = p_1^{a_1}\cdots p_r^{a_r}$, where $p_1, \cdots, p_r$ are prime numbers.

Liouville-Neumann series. (*Integral Equations.*) The series

$$y(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \phi_n(x),$$

where

$$\phi_1(x) = \int_a^b K(x, t)f(t)\,dt$$

and

$$\phi_n(x)=\int_a^b K(x, t)\phi_{n-1}(t)\,dt\ (n=2, 3, \cdots).$$

This function $y(x)$ is a solution of the equation

$$y(x)=f(x)+\lambda\int_a^b K(x, t)y(t)\,dt$$

if (1) $K(x, y)$ is real, continuous, and not identically zero in the square $a \leqq x \leqq b$, $a \leqq y \leqq b$; (2) $|\lambda| < 1/[M(b-a)]$, where M is the l.u.b. of $K(x, y)$ in this square; (3) $f(x) \neq 0$ and is real and continuous for $a \leqq x \leqq b$. See KERNEL—iterated kernels.

Liouville number. An irrational number X such that, for any integer n, there is a rational number p/q such that $q>1$ and

$$|X-p/q|<1/q^n.$$

All Liouville numbers are *transcendental* (see IRRATIONAL—irrational number). For any *irrational number* I, there exist infinitely many rational numbers p/q such that

$$|I-p/q|<1/(\sqrt{5}\,q^2),$$

but $\sqrt{5}$ is the largest number that can be used for every I; for an *algebraic number* of degree n, there is a positive number c for which there are infinitely many rational numbers p/q with

$$|A-p/q|<c/q^n,$$

but the degree n is the largest exponent that can be used on q. There is a Liouville number between any two real numbers. In fact, the set of Liouville numbers is a set of *second category* (although it is of *measure zero*).

Liouville's theorem. If $f(z)$ is an *entire analytic function* of the complex variable z and is bounded, then $f(z)$ is identically constant.

LIPSCHITZ. Lipschitz condition. A function $f(x)$ is said to satisfy a **Lipschitz condition** (with constant K) at a point x_0 if $|f(x)-f(x_0)| \leqq K|x-x_0|$ for all x in some neighborhood of x_0. It is said to satisfy a **Hölder condition** of order p at x_0 if $|f(x)-f(x_0)| \leqq K|x-x_0|^p$ for all x in some neighborhood of x_0 (this is sometimes called a Lipschitz condition of order p). A function $f(x)$ is said to satisfy a Lipschitz condition on the interval $[a, b]$ if

$$|f(x_2)-f(x_1)| \leqq K|x_2-x_1|$$

for all x_1 and x_2 on $[a, b]$. A function having a continuous derivative at each point of a closed interval satisfies a Lipschitz condition.

LI'TER (LI'TRE), *n.* One cubic decimeter. Approximately equal to 61.026 cubic inches or 1.056 quarts. See DENOMINATE NUMBERS in the appendix.

LIT'ER-AL, *adj.* **literal constant.** A letter which denotes any one of certain constants (say any real number, or any rational number), as contrasted to a specific constant like 1, 2, or 3. Letters from the first part of the alphabet are usually used (however, see SUBSCRIPT).

literal expression, or equation. An expression or equation in which the constants are represented by letters; $ax^2+bx+c=0$ and $ax+by+cz=0$ are *literal* equations, whereas $3x+5=7$ is a *numerical* equation.

literal notation. The use of letters to denote numbers, either unknown numbers or any of a set of numbers under discussion. *E.g.*, algebra uses letters in discussing the fundamental operations of arithmetic in order to make statements regarding all numbers, such as $a+a=2a$.

LIT'U-US, *n.* [*pl.* **litui**]. A plane curve shaped like a trumpet, from which it gets its name; the locus of a point such that the square of the radius vector varies inversely as the vectorial angle. Its equation in polar coordinates is $r^2=a/\theta$. The curve is asymptotic to the polar axis and winds around infinitely close to the pole but never touches it. Only positive values of r are used in the figure. Negative values

would give an identical branch of the curve in such a position that the two branches would be symmetrical about the pole.

LOAD'ING, *v.* (*Insurance.*) The amount added to the net insurance premiums to cover agents' fees, company expenses, etc.

LOAN, *adj., n.* **building and loan association.** See BUILDING.

loan value. A term used in connection with an insurance policy. It is an amount, usually somewhat less than the cash surrender value, which the insurance company agrees to loan the policy holder at a stipulated rate as long as the policy is in force.

LO'CAL, *adj.* **local value.** Same as PLACE VALUE.

LO'CAL-LY, *adv.* **locally compact, locally connected, locally convex, locally Euclidean, and locally finite.** See COMPACT—compact set, CONNECTED, CONVEX—convex set, EUCLIDEAN and FINITE—locally finite family of sets.

LO-CA'TION, *adj., n.* **location theorem (or principle)** for the roots of an equation. See ROOT—root of an equation.

LO'CUS, *n.* [*pl.* **loci**]. Any system of points, lines, or curves which satisfies one or more given conditions. If a set of points consists of those points (and only those points) whose coordinates satisfy a given equation, then the set of points is said to be the **locus of the equation** and the equation is said to be the **equation of the locus.** *E.g.*, the locus of the equation $2x+3y=6$ is a straight line, the line which contains the points $(0, 2)$ and $(3, 0)$. The locus of points which satisfy a given condition is the set which contains all the points which satisfy the condition and none which do not; *e.g.*, the locus of points equidistant from two parallel lines is a line parallel to the two lines and midway between them; the locus of points at a given distance r from a given point P is the circle of radius r with center at P. The locus of an inequality consists of those points whose coordinates satisfy the given inequality. Thus in a one-dimensional space the locus of the inequality $x>2$ is the x-axis to the right of 2. In a two-dimensional space, the locus of the inequality $2x+3y-6<0$ is that portion of the (x, y)-plane which is below the line $2x+3y-6=0$.

LOG'A-RITHM, *n.* The logarithm of a number is the exponent indicating the power to which it is necessary to raise a given number, called the base, to produce the number; the logarithm, with base a, of M is equal to x if $a^x=M$. Since $10^2=100$, 2 is the logarithm of 100 to the base 10, written $\log_{10} 100=2$; likewise, $\log_{10} .01=-2$ and $\log_9 27=\frac{3}{2}$. Logarithms which use 10 as a base are called **common (or Briggs') logarithms.** Logarithms which use the base $e=2.71828\cdots$ are called **natural (or Napierian) logarithms,** and sometimes **hyperbolic logarithms** (see e); $\log_e x$ is often written as $\ln x$. *Common logarithms* are particularly useful for performing multiplication, division, evolution, and involution, because of the following **fundamental laws of logarithms** (valid for logarithms to any base) together with the fact that shifting the decimal point n places to the right (or left) changes the logarithm of the number by the addition (or subtraction) of the integer n (see below, characteristic and mantissa of a logarithm). (1) The logarithm of the product of two numbers is the sum of the logarithms of the numbers [$\log (4\times 7)=\log 4+\log 7=.60206+.84510=1.44716$ (see TABLE I in the appendix)]. (2) The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor [$\log \frac{4}{7}=\log 4-\log 7=10.60206-10-.84510=9.75696-10$]. (3) The logarithm of a power of a number is equal to the product of the exponent and the logarithm of the number [$\log 7^2=2\cdot\log 7=1.69020$]. (4) The logarithm of a root of a number is equal to the quotient of the logarithm of the number and the index of the root [$\log \sqrt{49}=\log (49)^{1/2}=\frac{1}{2}\log 49=\frac{1}{2}(1.69020)=.84510$]. *Natural logarithms* are particularly adapted to analytical work. This originates from the fact that the *derivative* of $\log_e x$ is equal to $1/x$, while the derivative of $\log_a x$ is $(1/x)\log_a e$. Change of base of logarithms can be accomplished by use of the formula:

$$\log_b N=\log_a N\cdot\log_b a.$$

In particular, $\log_{10} N=\log_e N\cdot\log_{10} e$ and $\log_e N=\log_{10} N\cdot\log_e 10$. The number by which logarithms in one system are multiplied to give logarithms in a second system is called the **modulus of the second system with respect to the first.** Thus the *modulus of common logarithms* (with respect

to natural logarithms) is $\log_{10} e = .434294 \cdots$ and the *modulus of natural logarithms* (with respect to common logarithms) is $\log_e 10 = 2.302585 \cdots$. The calculation of logarithmic tables is usually based on infinite series, such as:

$$\log_e (N+1) = \log_e N + 2\left[\frac{1}{2N+1} + \frac{1}{3}\frac{1}{(2N+1)^3} + \frac{1}{5}\frac{1}{(2N+1)^5} + \cdots\right],$$

which is convergent for all values of N.

characteristic and mantissa of logarithms. Due to the fundamental laws of logarithms (see above, LOGARITHMS) and the fact that $\log_{10} 10 = 1$, common logarithms have the property that

$$\log_{10}(10^n \cdot K) = \log_{10} 10^n + \log_{10} K = n + \log_{10} K.$$

I.e., the common logarithm of a number is changed by adding (or subtracting) n if the decimal point is moved n places to the right (or left). Thus when the logarithm is written as the sum of an integer (the *characteristic*) and a positive decimal (the *mantissa*), the characteristic serves to locate the decimal point and the mantissa determines the digits in the number. The characteristic of the logarithm of a number can be determined by either of the following rules: (1) The characteristic is the number of places the decimal point is to the right of standard position, or the negative of the number of places the decimal point is to the left of standard position (standard position of the decimal point is the position to the right of the first nonzero digit of the number). (2) When the number is greater than or equal to 1, the characteristic is always one less than the number of digits to the left of the decimal point. When the number is less than 1, the characteristic is negative and numerically one greater than the number of zeros immediately following the decimal point; *e.g.*, .1 has the characteristic -1, .01 has the characteristic -2. If a logarithm has the mantissa .7519 and the characteristic 2, it is written either as 2.7519, or $12.7519-10$, or $12.7519\ \overline{10}$; if it has the characteristic -1, it is written as $\bar{1}.7519$, or $9.7519-10$, or $9.7519\ \overline{10}$. The mantissa of a common logarithm is the same regardless of where the decimal point is located in the number. Only mantissas are put in tables of logarithms, since the characteristics can be found by the above rules. The mantissa can be found in TABLE I of the appendix as follows: When the number whose logarithm is sought has not more than four digits, find the first three in the *column* headed N and the fourth in the row at the top of the table and take the mantissa which is common to the *row* and the *column* headed, respectively, by the first three digits and the fourth. The mantissas for numbers with more digits than those listed in the tables can be found (*i.e.*, closely approximated) by interpolation (see INTERPOLATION). *E.g.*, to find the mantissa of $\log_{10} 10134$, we find the mantissa of log 1013 to be .00561. In the next column, we find the mantissa of log 1014 to be .00604. The difference, $.00604-.00561$, is .00043; .4 of this difference is .00017, which we add to .00561, getting, for the mantissa of log 10134,.00578. See below, proportional parts in a table of logarithms.

logarithm of a complex number. The number w is said to be the *logarithm* of z to base e if $z = e^w$. Writing z in the form

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

it is seen that $\log(x+iy) = i\theta + \log r$, where θ is an argument of z and r is the absolute value of z; *i.e.*, $\log z = \log|z| + i \arg z$. See COMPLEX—polar form of a complex number, and EULER—Euler's formula. The logarithm of a complex number is a many-valued function since the argument of a complex number is many-valued. Since $e^{i\pi} = \cos\pi + i\sin\pi = -1$, $\log(-1) = i\pi$. For any number, $-n$, $\log(-n) = i\pi + \log n$, thus providing a definition for the **logarithm of a negative number.** More generally, $\log(-n) = (2k+1)\pi i + \log n$, where k is any integer. When $\log_e z$ is known, the logarithm of z to any base can be found. See LOGARITHMIC—logarithmic function of a complex variable, and above, LOGARITHM.

proportional parts in a table of logarithms. The numbers to be added to the next smaller mantissa to produce a desired mantissa (see TABLE I in the appendix). The proportional parts are the products of the differences between successive mantissas (written above them) and the numbers $.1, .2, \cdots, .9$ (written in the table without decimal points). These *proportional parts*

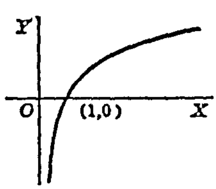
tables are multiplication (and division) tables to aid in interpolating for logarithms of numbers not in the tables (and for numbers whose logarithms are not in the tables).

LOG'A-RITH'MIC, *adj.* **logarithmic convexity.** See CONVEX—logarithmically convex function.

logarithmic coordinates. See COORDINATE—logarithmic coordinates.

logarithmic coordinate paper. Coordinate paper on which the rulings corresponding (for instance) to the numbers 1, 2, 3, etc., are at distances from the coordinate axes proportional to the logarithms of these numbers; *i.e.*, the markings on the graph are not the distances from the axes, but the antilogarithms of the actual distances. This scale is called a **logarithmic scale,** where the ordinary scale, which marks the actual distances, is called a **uniform scale.**

logarithmic curve. The plane locus of the rectangular Cartesian equation $y = \log_a x$, $a > 1$. This curve passes through the point whose coordinates are (1, 0) and is asymptotic to the negative y-axis. The ordinates of the curve increase arithmetically while the abscissas increase geometrically; *i.e.*, if the ordinates of three points are 1, 2, 3, respectively, the corresponding abscissas are a, a^2, a^3. When the base a of the logarithmic system is given different values, the general characteristics of the curve are not altered. The figure shows the graph of $y = \log_2 x$.



logarithmic derivative of a function. The ratio $f'(z)/f(z)$. *I.e.*, $d \log f(z)/dz$.

logarithmic differentiation. See DIFFERENTIATION—logarithmic differentiation.

logarithmic equation. See EQUATION—logarithmic equation.

logarithmic function of a complex variable. The function $\log z$ can be defined as the inverse of the exponential function; *i.e.*, if $z = e^w$, then by definition $w = \log z$. It can also be defined by $\log z = \int_1^z \frac{d\zeta}{\zeta}$ with the path of integration restricted away from the branch-point $z = 0$, or by the function-element

$$f(z) = (z-1) - \tfrac{1}{2}(z-1)^2 + \cdots + \frac{(-1)^{n-1}}{n}(z-1)^n + \cdots$$

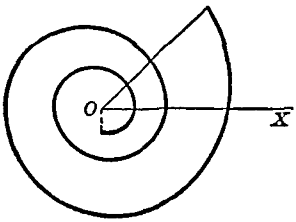
and its analytic continuations. The logarithmic function is infinitely multiple-valued; if its principal branch is denoted by Log z, then all of its values are given by Log $z = $ Log $z + 2k\pi i$, $k = 0, \pm 1, \pm 2, \cdots$. The **principal branch** of the function $\log z$ is the single-valued analytic function of the complex variable $z = x + iy$ defined in the z-plane cut along the negative real axis, and coinciding with the real function $\log x$ along the positive real axis.

logarithmic plotting (or graphing). A system of graphing such that curves whose equations are of the form $y = kx^n$ are graphed as straight lines. The logarithms of both sides of the equation are taken, giving an equation of the form $\log y = \log k + n \log x$; $\log y$ and $\log x$ are then treated as the variables, and a straight line plotted whose abscissas are $\log x$ and ordinates $\log y$. Points can be found on this straight line whose coordinates are ($\log x$, $\log y$), just as the coordinates of points on any line are found. It is then a matter of taking antilogarithms of the coordinates to find x and y, and even this is not necessary if *logarithmic coordinate paper* is used.

logarithmic potential. Potential based on a force which varies inversely as the first power of the distance instead of inversely as the square, as is the case in the Newtonian law of gravitation, Coulomb's law for point charges, and the law of force for isolated magnetic poles. An example of such a force field is furnished by a uniformly charged straight wire of infinite length. If we take the z-axis along this wire, then the force experienced by a unit charge at a point r units from the wire is given by $(k/r)\rho_1$, where k is a constant and ρ_1 is a unit vector having the direction of the perpendicular from the wire to the point. In this case the field depends on two variables only (say r and θ). Hence we are dealing with a two-dimensional situation. Consequently, we may replace the

uniformly charged wire with a particle that supposedly exerts an attractive or repulsive force which is inversely proportional to the first power of the distance r. The potential corresponding to such a particle (the *logarithmic potential*) is given by ($a \log r+b$), where a and b are constants.

logarithmic spiral. The plane curve whose vectorial angle is proportional to the logarithm of the radius vector. Its polar equation is $\log r=a\phi$. The angle between the radius vector to a point on the spiral and the tangent at this point is always equal to the modulus of the system of logarithms being used. Also called logistic spiral and equiangular spiral.



logarithmic sine, cosine, tangent, cotangent, secant, or cosecant. The logarithms of the corresponding sine, cosine, etc.

logarithmic solution of triangles. Solutions using logarithms and formulas adapted to the use of logarithms, formulas which essentially involve only multiplication and division.

logarithmic transformation. (*Statistics.*) The *logarithm* of a variable x is often normally distributed (where x is not). Hence the transformation of a variable into its *log* may be used to permit application of normal distribution theory. See GIBRAT'S DISTRIBUTION.

semilogarithmic coordinate paper. Coordinate paper on which the logarithmic scale is used on one axis and the uniform scale on the other. It is adapted to graphing equations of the type $y=ck^x$. When the logarithms of both sides are taken, the equation takes the form

$$\log y=\log c+x \log k.$$

Log y is now treated as one variable, say u, and the linear equation $u=\log c+\log k$ graphed. Useful in *statistics* for showing a series in which fluctuations in the rate of change are of interest and for comparing two or more greatly divergent series, or one series which fluctuates widely. *Syn.* Ratio paper. See above, logarithmic coordinate paper.

LO-GIS'TIC, *adj., n.* (1) Logical. (2) Skilled in or pertaining to computation and calculation. (3) Proportional; pertaining to proportions. (4) The art of calculation. (5) Sexagesimal arithmetic.

logistic curve. A curve whose equation is of the form $y=k/(1+e^{a+bx})$, where $b<0$. The value of y at $x=0$ is

$$k/(1+e^a),$$

and as $x\to\infty$, $y\to k$. The increments in y as x increases are such that the difference of increments of $1/y$ is proportional to the corresponding difference in $1/y$. Also known as the **Pearl-Reed curve.** This is one of the types of curves known as **growth curves.**

logistic spiral. Same as LOGARITHMIC SPIRAL.

LON'GI-TUDE, *n.* The number of degrees in the arc of the equator cut off by the meridian through the place under consideration and the meridian through some established point (Greenwich, England, unless otherwise stated). See MERIDIAN—principal meridian.

LOOP, *n.* **loop of a curve.** A section of the curve which completely encloses an area. See CLOSED—closed curve.

LOW'ER, *adj.* **lower bound.** See BOUND.

lower limit of an integral. See INTEGRAL—definite integral.

LOW'EST, *adj.* **fraction in lowest terms.** A fraction in which all common factors have been divided out of numerator and denominator; $\frac{1}{2}$, $\frac{2}{3}$ and $1/(x+1)$ are in their lowest terms, but $\frac{2}{4}$, $\frac{6}{9}$ and

$$(x-1)/(x^2-1)$$

are not.

lowest common multiple. *Syn.* Least common multiple. See MULTIPLE—least common multiple.

LOX'O-DROME, *n.* Same as LOXODROMIC SPIRAL.

LOX-O-DROM'IC, *adj.* **loxodromic spiral.** The path of a ship which cuts the meridians at a constant angle not equal to a right angle; more generally, any curve on a surface of revolution which cuts the meridians at a constant angle. See SURFACE—surface of revolution. *Syn.* Rhumb-line, loxodromic line, or loxodromic curve.

LUNE, *n.* A portion of a sphere bounded by two great semicircles. The angle at which the great circles intersect is the **angle of the lune.** The area of a lune is equal to the area of the sphere multiplied by the ratio of its angle to 360°; *i.e.*,

$$\text{area} = [(\text{angle of lune})/360°]4\pi r^2.$$

LUSIN. Lusin's theorem. Let $f(x)$ be defined on the real line (or an n-dimensional space), finite *almost everywhere*, and *measurable*. Then, for any positive number ϵ, there is a function $g(x)$ which is continuous on the line (or space) and is such that $f(x) = g(x)$ except for points of a set of measure less than ϵ.

M

MACH, *adj.* **mach number.** The ratio v/a of v, the speed at which a body is traveling, to a, the local velocity of sound in air.

MACLAURIN. Maclaurin's series (theorem). See TAYLOR—Taylor's theorem.

trisectrix of Maclaurin. See TRISECTRIX.

MAG'IC, *adj.* **magic square.** A square array of integers such that the sum of the numbers in each row, each column, and each diagonal are all the same, such as:

17	3	13
7	11	15
9	19	5

and

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

MAG'NI-FI-CA'TION, *adj.* **magnification ratio.** Same as DEFORMATION RATIO.

MAG'NI-TUDE, *n.* (1) Greatness, vastness. (2) Size, or the property of having size; length, area, or volume.

geometrical magnitude. See GEOMETRIC—geometric magnitude.

magnitude of a star. Two stars differ by one magnitude if one is $(100)^{1/5}$, or 2.512+, times as bright as the other. The faintest stars seen with the naked eye on a clear moonless night are said to be of the 6th magnitude. The pole star (Polaris) is nearly of the 2nd magnitude.

MAG'NUS, *adj.* **magnus effects.** In aerodynamics, those forces and moments on a rotating shell that acount for such phenomena as right-hand drift, etc.

MA'JOR, *adj.* **major arc.** The longer of two arcs in a circle, subtended by a secant. See SECTOR—sector of a circle.

major axis. See ELLIPSE and ELLIPSOID.

major and minor segments of a circle. See SEGMENT—segment of a circle.

MAKEHAM. Makeham's formula for bonds. The price to be paid for a bond n periods before redemption equals $Cv^n + (j/i)F(1-v^n)$, where C is the redemption price, F the par value, j the dividend rate, i the investment rate, and $v = (1+i)^{-1}$.

Makeham's law. The force of mortality (risk of dying) is equal to the sum of a constant and a multiple of a constant raised to a power equal to the age, x, of the life: $M = A + Be^x$. Makeham's law is a closer approximation to statistical findings than Gompertz's law. From the age of 20 to the end of life it very nearly represents the data of most tables.

MANIAC. An automatic digital computing machine at the Los Alamos Scientific Laboratory. MANIAC is an acronym for Mathematical Analyzer, Numerical Integrator, and Computer.

MAN'I-FOLD, *n.* In general, manifold may mean any collection or set of objects. *E.g.*, a Riemannian space is also called a Riemannian manifold; a subset of a vector space is said to be a *linear set* or a *linear manifold* if it contains all linear combinations of its members. However, manifold frequently has technical meaning beyond being a mere set, as illustrated by the following definitions. **A topological manifold** of dimension n (frequently called simply an

***n*-manifold**) is a topological space such that each point has a neighborhood which is homeomorphic to the interior of a sphere in Euclidean space of dimension n. Such a manifold M is said to be **differentiable of order r** (or to have a **differentiable structure of class C^r**) if there is a family of neighborhoods which cover M and which are such that each neighborhood is homeomorphic to the interior of a sphere in Euclidean space of dimension n; no point of M belongs to more than a finite number of the neighborhoods; and when x belongs to two neighborhoods U and V the $2n$ functions $u_k = u_k(v_1, v_2, \cdots, v_n)$ and $v_k = v_k(u_1, u_2, \cdots, u_n)$, $k = 1, \cdots, n$, have continuous partial derivatives of order r, where $(u_1, \cdots, u_n)$ and $(v_1, \cdots, v_n)$ are coordinates given to the same point in the intersection of U and V. A manifold which is compact and differentiable of order 1 is a *polyhedron* (*i.e.*, homeomorphic to the point-set union of the simplexes of a simplicial complex). A *manifold* is sometimes defined to be a *topological manifold* which is also a *polyhedron*. A connected manifold of this type is also a **manifold** (sometimes called a **pseudomanifold**) in the sense that it is an n-dimensional simplicial complex $(n \geqq 1)$ such that (i) each k-simplex $(k < n)$ is a face of at least one n-simplex; (ii) each $(n-1)$-simplex is a face of exactly two n-simplexes; and (iii) any two n-simplexes can be connected by a sequence whose members are alternatively n-simplexes and $(n-1)$-simplexes, each $(n-1)$-simplex being a face of the two adjacent n-simplexes. Such a manifold is said to be **orientable** if its n-simplexes can be *coherently oriented*; *i.e.*, oriented so that no $(n-1)$-simplex can be oriented so as to be coherently oriented with each of the n-simplexes of which it is a face (see SIMPLEX). Otherwise it is **nonorientable.** Any topological space which is homeomorphic to a manifold is also called a manifold and is orientable or nonorientable according as the manifold is orientable or nonorientable. A one-dimensional manifold is a simple closed curve. A two-dimensional manifold is also called a closed surrace. The closed surfaces can be classified by use of certain topological invariants (see SURFACE). No such classification is known for three-dimensional manifolds.

MAN-TIS'SA, *n.* See LOGARITHM—characteristic and mantissa of a logarithm.

MAN'Y, *adj.* **many-valued function.** See MULTIPLE—multiple-valued function.

MAP, *n.* If to each element x of a set (or space) A there corresponds a unique element $f(x)$ of a space B, then there is said to be a **mapping** or **map** f of the set A in the set B and the point $f(x)$ is said to be the **image** of the point x. If every point of B is the image of a point of A, then f is said to be a map of A onto B. If S is a subset of B, then the **inverse image** of S is the set of all those points of A whose images are in S. See CONTINUOUS—continuous correspondence of points, HOMOMORPHISM, ISOMETRY, and ISOMORPHISM. *Syn.* Transformation, correspondence, function.

area-preserving map. A *map* which preserves areas. The map $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ of the (u, v)-domain of definition D on a surface S is area-preserving if, and only if, the *fundamental quantities of the first order* satisfy $E \cdot G - F^2 \equiv 1$. The induced map between the above surface S and the surface $\bar{S}: x = \bar{x}(u, v)$, $y = \bar{y}(u, v)$, $z = \bar{z}(u, v)$ is area-preserving if, and only if, $EG - F^2 \equiv \bar{E}\bar{G} - \bar{F}^2$. *Syn.* Equivalent map, equiareal map.

cylindrical map. See CYLINDRICAL.

MAP'PING, *n.* Same as MAP.

MAR'GIN, *n.* (*Finance.*) (1) The difference between the selling price and the cost of goods. (2) A sum of money deposited with a broker by a client to cover any losses that may occur in the broker's dealings for him.

MARIOTT'S LAW. Same as BOYLE'S LAW.

MARK, *n.* (*Statistics.*) The value or name given to a particular class interval. Often the mid-value, or the integral value nearest the midpoint.

MAR'KET, *n.* **market value.** The amount a commodity sells for on the open market. *Syn.* Market price.

MARKOFF. **Markoff process.** Any probalistic process in which the future develop-

ment is completely determined by the present state and not at all by the way in which the present state arose.

MAR′TIN-GALE, *n.* A stochastic sequence $x_1, x_2, \cdots$, or its continuous analogue, such that, given $x_1, x_2, \cdots, x_n$, the conditional expected value of x_{n+1} is equal to x_n. In particular, a system of betting planned so that in a sequence of bets losses are recovered by progressively increasing the stakes (*e.g*,, by doubling the stakes after each loss, but reducing to the original amount after a win). This is not an effective system, since it is to be expected that a sequence of successive losses will soon carry the stake beyond the player's resources or above the limit of the game.

MASCHERONI'S CONSTANT. See EULER—Euler's constant.

MASS, *n.* The measure of the tendency of a body to oppose changes in its velocity. Mass can be defined, with the aid of Newton's Second Law of Motion, as the ratio of the magnitudes of the force and acceleration which the force produces. This amounts to defining the mass in terms of force. At speeds small compared with the speed of light, the masses m_1 and m_2 of two bodies may be compared by allowing the two bodies to interact. Then

$$m_1/m_2 = |a_2|/|a_1|,$$

where $|a_1|$ and $|a_2|$ are the magnitudes of the respective accelerations of the two bodies as a result of the interaction. This permits the measurement of the mass of any particle with respect to a standard particle (for example, the standard kilogram). At higher speeds, the mass of a body depends on its speed relative to the observer according to the relation

$$m = m_0/\sqrt{1 - v^2/c^2},$$

where m_0 is the mass of the body as found by an observer at rest with respect to the body, v is the speed of the body relative to the observer who finds its mass to be m, and c is the speed of light in empty space (*theory of relativity*). Equal masses at the same location in a gravitational field have equal weights. Because of this, masses may be compared by weighing. Mass is particularly important because it is a conserved quantity, which can neither be created nor be destroyed. Thus, the mass of any isolated system is a constant. When relativistic mechanics is appropriate, *e.g.*, when speeds comparable to the speed of light are involved, mass may be converted into energy and *vice versa*, hence the energy of the system must be converted into mass through the Einstein equation

$$E = mc^2,$$

where c is the speed of light in empty space, before the conservation law may be applied.

center of mass. See CENTER—center of mass, and CENTROID.

differential (or element) of mass. See ELEMENT—element of integration.

moment of mass. See MOMENT—moment of mass.

point-mass. Same as PARTICLE.

unit mass. The standard unit of mass, or some multiple of this unit chosen for convenience. There are several such standard units. In the c.g.s. system, one gram-mass is defined as $\frac{1}{1000}$ part of the mass of a certain block of platinum-iridium alloy preserved in the Bureau of Weights and Measures at Sèvres, France. The corresponding unit in the British system is the standard pound of mass which is a block of platinum alloy preserved in the Standards Office, London.

MATCHED, *adj.* **matched groups.** (*Statistics.*) Several groups are matched if the mean values (or some other characteristic) of some outside associated variable are the same for all the groups. Essentially a method of controlling the variation due to some outside factor.

matched pairs. (*Statistics.*) Pairs are matched if the paired individuals are equated with respect to some variable other than the one under immediate study; *e.g.*, in a study of heights of two groups of ten persons each, the individuals may be paired one from each group so that the two persons in a pair have the same age.

MA-TE′RI-AL, *adj.* **material point, line, or surface.** A point, line, or surface thought of as having mass. (If one thinks of a lamina with a fixed mass whose thickness approaches zero and density increases proportionally, the limiting situation can be thought of as an area with the fixed mass.)

MATH'E-MAT'I-CAL, *adj.* **mathematical expectation.** See EXPECTATION.

mathematical induction. See INDUCTION.

MATH'E-MAT'ICS, *n.* The logical study of shape, arrangement, and quantity.

applied mathematics. A branch of mathematics concerned with the study of the physical, biological, and sociological worlds. It includes mechanics of rigid and deformable bodies (elasticity, plasticity, mechanics of fluids), theory of electricity and magnetism, relativity, theory of potential, thermodynamics, biomathematics, and statistics. Broadly speaking, a mathematical structure utilizing, in addition to the purely mathematical concepts of space and number, the notions of time and matter belongs to the domain of applied mathematics. In a restricted sense, the term refers to the use of mathematical principles as tools in the fields of physics, chemistry, engineering, biology, and social studies.

mathematics of finance. The study of the mathematical practices in brokerage, banking, and insurance. *Syn.* Mathematics of investment.

pure mathematics. The study and development of the principles of mathematics as such (for their own sake and possible future usefulness) rather than for their immediate usefulness. *Syn.* Abstract mathematics. See above, applied mathematics.

MATHIEU. Mathieu's differential equation. A differential equation of the form

$$y''+(a+b\cos 2x)y=0.$$

The general solution can be written in the form $y=Ae^{rx}\phi(x)+Be^{-rx}\phi(-x)$, for some constant r and function $\phi(x)$ which is periodic with period 2π. There are periodic solutions for some *characteristic values* of a, but no Mathieu equation (with $b\neq 0$) can have two independent periodic solutions.

Mathieu function. Any solution of Mathieu's differential equation which is periodic and is either an even or an odd function, the solution being multiplied by an appropriate constant. The solution which reduces to $\cos nx$ when $b\to 0$ and $a=n^2$, and for which the coefficient of $\cos nx$ in its Fourier expansion is unity, is denoted by $ce_n(x)$; the solution which reduces to $\sin nx$ when $b\to 0$, and in which the coefficient of $\sin nx$ in its Fourier expansion is unity, is denoted by $se_n(x)$.

MA'TRIX, *adj., n.* [*pl.* **matrices**]. A rectangular array of terms called **elements** (written between parentheses or double lines on either side of the array), as

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \quad \text{or} \quad \begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Vmatrix}.$$

Used to facilitate the study of problems in which the relation between these elements is fundamental, as in the study of the existence of solutions of simultaneous linear equations. Unlike determinants, a matrix does not have quantitative value. It is not the symbolic representation of some polynomial, as is a determinant (see below, rank of a matrix). If the elements of a matrix are all real, the matrix is a **real matrix.** A **square matrix** is a matrix for which the number of rows is equal to the number of columns. The number of rows (or columns) is called the **order** of the matrix. The diagonal from the upper left corner to the lower right corner is called the **principal** (or **main**) **diagonal.** The diagonal from the lower left corner to the upper right corner is called the **secondary diagonal.** The **determinant** of a square matrix is the determinant gotten by considering the array of elements in the matrix as a determinant. A square matrix is **singular** or **nonsingular** according as the determinant of the matrix is zero or nonzero. A **diagonal matrix** is a square matrix all of whose nonzero elements are in the principal diagonal. If, in addition, all the diagonal elements are equal, the matrix is a **scalar matrix.** An **identity** (or **unit**) **matrix** is a diagonal matrix whose elements in the principal diagonal are all unity. For any square matrix A of the same order as I, $IA=AI=A$.

adjoint of a matrix. See ADJOINT.

associate matrix. See HERMITIAN—Hermitian conjugate of a matrix.

augmented matrix of a set of simultaneous linear equations. The matrix of the coefficients, with an added column consisting of the constant terms of the equations. The augmented matrix of

$$\begin{matrix} a_1x+b_1y+c_1z+d_1=0 \\ a_2x+b_2y+c_2z+d_2=0 \end{matrix} \quad \text{is} \quad \begin{Vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{Vmatrix}.$$

canonical (or normal) form of a matrix. See CANONICAL—canonical form of a matrix.

characteristic equation and function of a matrix. See CHARACTERISTIC.

complex conjugate of a matrix. The matrix whose elements are the complex conjugates of the corresponding elements of the given matrix.

derogatory matrix. See CHARACTERISTIC—characteristic equation of a matrix.

determinant of a matrix. See DETERMINANT—determinant of a matrix.

eigenvalue and eigenvector of a matrix. See EIGENVALUE—eigenvalue of a matrix.

elementary divisor of a matrix. See INVARIANT—invariant factor of a matrix.

equivalent matrices. See EQUIVALENT—equivalent matrices.

Hermitian matrix. See HERMITIAN—Hermitian matrix.

inverse of a matrix. The quotient of the *adjoint* of the matrix and the *determinant* of the matrix; the transpose of the matrix obtained by replacing each element by its cofactor divided by the determinant of the matrix. If A^{-1} is the inverse of A, then $AA^{-1}=A^{-1}A=I$, where I is the identity matrix. The inverse is defined only for square matrices.

Jordan matrix. See JORDAN.

matrix of the coefficients of a set of simultaneous linear equations. The rectangular array left by dropping the variables from the equations when they are written so that the variables are in the same order in all equations and are in such a position that the coefficients of like variables are in the same columns, zero being used as the coefficient if a term is missing. When the number of variables is the same as the number of equations, the matrix of the coefficients is a square array. The matrix of the coefficients of

$$\begin{matrix} a_1x+b_1y+c_1z+d_1=0 \\ a_2x+b_2y+c_2z+d_2=0 \end{matrix} \quad \text{is} \quad \begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Vmatrix}.$$

See below, rank of a matrix.

matrix of a linear transformation. The matrix of a linear transformation defined by $y_i=\sum_{j=1}^{n} a_{ij}x_j$ $(i=1, 2, \cdots, n)$ is the matrix $A=(a_{ij})$, where a_{ij} is the element in the ith row and jth column. Two linear transformations, T_1 and T_2, applied in this order are equivalent to the linear transformation with matrix BA, where A is the matrix of T_1 and B is the matrix of T_2.

norm of a matrix. See NORM.

normal matrix. A square matrix A such that $A^*A=AA^*$, where A^* is the *Hermitian conjugate* of A (the *transpose* if A is real). A matrix is normal if, and only if, it is the *transform* of a diagonal matrix by a *unitary matrix* (*i.e.*, it can be changed to diagonal form by a unitary transformation), whereas any nonsingular matrix can be written as the product of two normal matrices.

orthogonal matrix. A matrix that is equal to the *inverse* of its *transpose*; a matrix such that $\sum_{s=1}^{n} a_{is}a_{js}=\sum_{s=1}^{n} a_{si}a_{sj}=\delta_{ij}$ for all i and j, where δ_{ij} is *Kronecker's delta* and a_{ij} is the element in the ith row and jth column. Thus any two distinct rows or any two distinct columns are *orthogonal vectors* (if the components are real). For matrices whose elements are real numbers, this is equivalent to being a unitary matrix. See ORTHOGONAL—orthogonal transformation.

payoff matrix. See PAYOFF.

permutation matrix. See PERMUTATION—permutation matrix.

product of matrices. See PRODUCT—product of matrices, direct product of matrices.

product of a scalar and a matrix. See PRODUCT—product of a scalar and a matrix.

rank of a matrix. The order of the nonzero determinant of greatest order that can be selected from the matrix by taking out rows and columns. The concept *rank* facilitates, for instance, the statement of the condition for consistency of simultaneous linear equations: m linear equations in n unknowns are consistent when, and only when, the *rank* of the matrix of the coefficients is equal to the *rank* of the augmented matrix. In the system of linear equations

$$x+y+z+3=0$$
$$2x+y+z+4=0,$$

the matrix of the coefficients is

$$\begin{Vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{Vmatrix}$$

and the augmented matrix is

$$\begin{Vmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 4 \end{Vmatrix}.$$

The rank of both is *two*, because the determinant

$$\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

is not zero. Hence these equations are satisfied by some set of values of x and y and z. See CONSISTENCY—consistency of linear equations.

reducible matrix representation of a group. See REPRESENTATION—reducible matrix representation of a group.

skew symmetric matrix. See SYMMETRIC—skew symmetric matrix.

sum of matrices. See SUM—sum of matrices.

symmetric matrix. See SYMMETRIC—symmetric matrix.

trace of a matrix. See TRACE—trace of a matrix.

transpose of a matrix. See TRANSPOSE—transpose of a matrix.

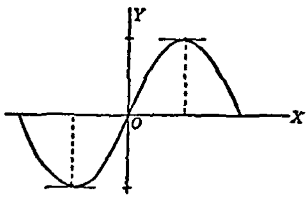
unitary matrix. A matrix which is equal to the inverse of its *Hermitian conjugate*; a matrix such that

$$\sum_{s=1}^{n} a_{is}\bar{a}_{js} = \sum_{s=1}^{n} a_{si}\bar{a}_{sj} = \delta_{ij}$$

for all i and j, where δ_{ij} is *Kronecker's delta* and a_{ij} is the element in the ith row and jth column. Thus any two distinct rows or any two distinct columns are *orthogonal vectors* in Hermitian vector space. For real matrices this is equivalent to being an orthogonal matrix. See TRANSFORMATION—unitary transformation.

MAX'I-MAL, *adj.* **maximal member of a set.** In a set which is *partially ordered*, a maximal element is any element x for which there is no y which follows x in the ordering. For a family of sets, partial ordering can be defined by means of set inclusion and a maximal member is a set which is not properly contained in any other set. *E.g.*, a maximal *connected* subset of a set S is a subset which is connected and is not contained in any other connected subset of S.

MAX'I-MUM, *n.* [*pl.* **maxima**]. The maximum (minimum) of a **function of one real variable** is the greatest (least) value which the function takes on in a given interval, if it takes on such an extreme value. Several situations arise: A function may have just one, or several (equal) such values, or several (not necessarily equal) such values each of which is a maximum (minimum) relative to the values of the function in a corresponding interval. These three cases are illustrated, respectively, by a path over a mountain (through a valley), a path over several mountains of equal heights (through the valleys), and a path over several mountains of *different heights* (through valleys of different depths). In the first case the function is said to have an **absolute maximum (minimum).** In the second and third cases each hilltop (valley) is said to be a **relative maximum (minimum)** and in the third case the highest hill top is said to be an **absolute maximum (minimum).** At points where a function takes on such maximum (minimum) values it is said to have a maximum (minimum) point, an absolute maximum (minimum) point, etc.



A test for maxima (minima) under this definition can be made by examining values of the function very near the point (value) under investigation. In cases such as we have illustrated by paths the slope of the graph of a function changes from positive to negative (negative to positive) at a maximum (minimum) point as one passes from left to right, being zero at the point if the derivative is continuous. The elementary condition for testing for maximum (minimum) is that the first derivative be zero at the point and the second derivative be negative (positive) at the point. This rule fails when the second derivative is zero or when the curve has a cusp at the point. For instance, the functions $y=x^3$ and $y=x^4$ both have their first and second derivatives zero at the origin, but the first has a point of inflection there and the second a minimum. For the exceptional

cases in which this rule fails, an elementary test can be made by examining the sign of the derivative on either side of the point, or finding the values of the function on either side. A general test for a maximum (minimum) of an analytic function is that the *first derivative* be zero at the point and the lowest order derivative not zero at the point be of even order and negative (positive). A maximum (minimum) of a function of two variables (*i.e.*, of a surface) is a point such that the function does not have a larger (smaller) value near the point. *Tech.* $f(x, y)$ has a maximum (minimum) at the point (a, b) if $f(a+h, b+k)—f(a, b)$ is not greater than (not less than) zero for all sufficiently small values of h and k different from zero. A necessary condition for a maximum (or minimum) at a point (a, b) in a region in which $f(x, y)$ and its partial derivatives are continuous is that the latter be zero at the point. If these partial derivatives are zero and the expression

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right) - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}$$

is greater than zero at the point, there is no maximum (or minimum); if this expression is less than zero at the point, there is a maximum if the derivatives $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$ are both negative (a minimum if both are positive). If the above expression is zero, the test fails. Relative and absolute maximum of functions of two variables are used in the same sense as in the case of a function of one variable. The function $F(x_1, x_2, \cdots, x_n)$ of the n independent variables $x_1, x_2, \cdots, x_n$ is said to have a maximum (minimum) at the point $P(x_1, x_2, \cdots, x_n)$ if the difference $F(x_1', x_2' \cdots, x_n') - F(x_1, x_2, \cdots, x_n)$ is not positive (not negative) for all points in a sufficiently small neighborhood of P. If F and its first partial derivatives exist in the neighborhood of a point, a necessary condition for F to have a maximum (minimum) at the point is that all of its first partial derivatives be zero at the point. For the case when the arguments of F are not independent, see LAGRANGE—Lagrange's method of multipliers.

principle of the maximum. See PRINCIPLE.

MAZUR. **Mazur-Banach game.** Let I be a given closed interval and A and B be any two disjoint subsets whose union is I. Two players (A) and (B) alternately choose closed intervals $I_1, I_2, \cdots$ with each interval contained in the previous one. Player (A) chooses the intervals with odd subscripts, (B) those with even subscripts. Player (A) wins if there is no point which belongs to A and all of the chosen intervals; otherwise (B) wins (and there is a point which belongs to B and all of the chosen intervals). There is a strategy by which (B) can win for any strategy chosen by (A) if and only if A is of *first category* in I; there is a strategy by which (A) can win for any strategy chosen by (B) if and only if B is of *first category at some point of I.* The first of these statements can be extended to an arbitrary topological space and the second to a complete metric space, provided the players choose sets from a specified collection G of subsets which have non-empty interiors and which have the property that each nonempty open set contains a member of G. See NESTED—nested intervals.

MEAN, *adj., n.* **arithmetic (and geometric) mean.** See AVERAGE.

arithmetic-geometric mean. For two positive numbers p_1, q_1, the arithmetic-geometric mean is the common limit of the two sequences $\{p_n\}$, $\{q_n\}$, where

$$p_n = \tfrac{1}{2}(p_{n-1}+q_{n-1}) \quad \text{and} \quad q_n = (p_{n-1}q_{n-1})^{1/2}.$$

This mean is used in particular in Gauss' determination of the potential due to a homogeneous circular wire.

mean axis of an ellipsoid. See ELLIPSOID.

mean curvature of a surface. See CURVATURE.

mean density. See DENSITY.

mean deviation. See DEVIATION.

mean ordinate. See below, mean value of a function.

mean proportional. See PROPORTIONAL.

mean terms of a proportion. The second and third terms; the consequent in the first ratio and the antecedent in the second. Usually called simply the means of the proportion. In the proportion $a/b = c/d$, b and c are the *mean terms.*

mean value of a function. For a function of one variable, the mean value (or mean ordinate) on the interval (a, b) is the quotient of the area bounded by the curve, the ordinates corresponding to a and b,

and the axis of the variable, divided by the length of the interval (a, b); this is the side of the rectangle whose other side is of length $b-a$ and whose area is equal to the above area. *Tech.* The *mean value* of $f(x)$ for

$$a \leqq x \leqq b \quad \text{is} \quad \frac{1}{b-a}\int_a^b f(x)\,dx.$$

In geometric language, this means that the mean value of the ordinates of a curve is the quotient of the area under the curve and the length of the interval, $(b-a)$, where the area under the curve is bounded by the curve, the x-axis, and the lines $x=a$ and $x=b$; this is the other side of the rectangle whose area is equal to the given area and one of whose sides is equal to $(b-a)$. The mean value of a function over a region (of one or more dimensions) is the integral of the function over the region, divided by the magnitude of the region. *Tech.*

$$\frac{1}{s}\int_s f\,ds,$$

where ds is an element of the region and s denotes both the region and the magnitude of the region. *E.g.*, the mean of xy over the rectangle whose vertices are (0, 0), (2, 0), (2, 3), (0, 3) is

$$\frac{1}{s}\int_s xy\,ds = \frac{1}{6}\int_0^3\int_0^2 xy\,dx\,dy = \frac{3}{2}.$$

The **mean square ordinate** of a curve $y=f(x)$ in the interval (a, b) is the mean value of y^2 in the interval, *i.e.*,

$$\frac{1}{b-a}\int_a^b y^2\,dx.$$

mean value theorems (or **laws of the mean**) **for derivatives.** For a single **function of one variable**, the mean value theorem states that an arc of a smooth, single-valued curve has at least one tangent parallel to its secant. When the secant is on the x-axis this is the same as ROLLE'S THEOREM. *Tech.* If $f(x)$ is single-valued and continuous for $a \leqq x \leqq b$ and if $f'(x)$ is defined for $a < x < b$, then

$$f(b) - f(a) = (b-a)f'(c),$$

for some c between a and b. The **second mean value theorem** [also called the **double law of the mean, Cauchy's mean value formula,** and **generalized** (or **extended**) **mean value theorem,** although the generalized (or extended) mean value theorem sometimes means Taylor's theorem] states that if the functions $f(x)$ and $g(x)$ are continuous on the closed interval $[a, b]$ and have first derivatives on the open interval (a, b), and if $g(b)-g(a) \neq 0$ and $f'(x)$ and $g'(x)$ are not simultaneously zero at any point of the open interval (a, b), then there exists at least one value of x, say x_1, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_1)}{g'(x_1)},$$

where $a < x_1 < b$. The **mean value theorem for a function of two variables** states that if $f(x, y)$ is continuous and has continuous first partial derivatives for $x_1 \leqq x \leqq x_2$, $y_1 \leqq y \leqq y_2$, there exist numbers ξ and η such that

$$f(x_2, y_2) - f(x_1, y_1) = (x_2 - x_1)f_x(\xi, \eta) + (y_2 - y_1)f_y(\xi, \eta),$$

where f_x and f_y denote the partial derivatives of $f(x, y)$ with respect to x and y, respectively, and $x_1 < \xi < x_2, y_1 < \eta < y_3$ [actually, it is possible for (ξ, η) to be an interior point of the line segment joining (x_1, y_1) and (x_2, y_2)]. This theorem can be extended to any number of variables. See DIFFERENTIAL.

mean value theorems (or **laws of the mean**) **for integrals.** The **first law of the mean for** integrals states that the definite integral of a continuous function over a given interval is equal to the product of the width of the interval by some value of the function within the interval (see INTEGRAL—definite integral). The **second law of the mean for** integrals means one of the following: (1) If $f(x)$ and $g(x)$ are both integrable on the interval (a, b) and $f(x)$ is always of the same sign, then

$$\int_a^b f(x)g(x)\,dx = K\int_a^b f(x)\,dx,$$

where K is between the greatest and the least values of $g(x)$, or possibly equal to one of them. If $g(x)$ is continuous in the interval (a, b), then K may be replaced by $g(k)$, where k is a value of x in the interval (a, b). (2) If in addition to the above conditions $g(x)$ is a *positive monotonically decreasing function*, the theorem can be written in *Bonnet's form*

$$\int_a^b f(x)g(x)\,dx = g(a)\int_a^p f(x)\,dx,$$

where $a \leqq p \leqq b$, or, if $g(x)$ is only *monotonic*, in the form

$$\int_a^b f(x)g(x)\,dx = g(a)\int_a^p f(x)\,dx + g(b)\int_p^b f(x)\,dx.$$

weighted mean. See AVERAGE.

MEAS'UR-A-BLE, *adj.* **measurable function.** A real-valued function f is (Lebesgue) measurable if for any real number a the set of all x for which $f(x) > a$ is measurable. Equivalent definitions result if the set of all x satisfying $f(x) \geqq a$, or the set of all x satisfying $a \leqq f(x) \leqq b$ for arbitrary a and b, are required to be measurable (and either of the signs $\leqq$ could be replaced by $<$). A bounded function defined on a set of finite measure is summable if it is measurable; any summable function defined on a measurable set is measurable; and if $g(x)$ is summable and $|f(x)| \leqq g(x)$ for all x, then $f(x)$ is summable if it is measurable. See below, measurable set, BAIRE—Baire function, and SUMMABLE—summable function.

measurable set. (1) See MEASURE—measure of a set. (2) A *bounded* point set of Euclidean space is (Lebesgue) measurable if its *exterior* and *interior measures* are equal, the common value being called the (Lebesgue) measure of the set. For an *unbounded* point set S, let W_I be the set of all points belonging to both S and a bounded interval I. Then S is (Lebesgue) measurable if, and only if, W_I is measurable for each I, and the (Lebesgue) measure of S is the least upper bound of the measures of the sets W_I if these are bounded, S being of infinite measure otherwise. See MEASURE—exterior measure, measure zero, INTERVAL—closed interval (2), and above, measurable function. The class of all measurable sets is *completely additive*, and all closed sets and all open sets are measurable. Other equivalent definitions of measurable sets are (1) a set whose *characteristic function* $f(x)$ is such that there exists a sequence of step functions (or continuous functions) f_n for which $\lim_{n\to\infty} f_n(x) = f(x)$ *almost everywhere*; (2) a set E such that the equation $m_e[E] = m_e[E\cdot S] + m_e[E\cdot C(S)]$ is satisfied for all sets S, where m_e denotes exterior measure and $C(S)$ is the complement of S. The class of all Lebesgue measurable sets is the *σ-algebra* generated by the open sets (or the closed sets) together with the sets of measure zero. A set is measurable if and only if it can be represented as an F_σ set plus a set of measure zero, or as a G_δ set minus a set of measure zero (see BOREL—Borel set). See SIERPINSKI—Sierpinski set, VITALI—Vitali set.

MEAS'URE, *n.* Comparison to some unit recognized as a standard.

angular measure. A system of measuring angles. See DEGREE, MIL, RADIAN, and SEXAGESIMAL.

board measure. A system of measure in which boards one inch or less in thickness are measured in terms of square feet on the side, the thickness being neglected; those thicker than one inch are measured in terms of the number of square feet one inch thick to which they are equivalent.

Caratheodory measure. A function which assigns a nonnegative number $\mu^*(M)$ to each subset of a set M is a **Caratheodory outer measure** if (i) $\mu^*(R) \leqq \mu^*(S)$ if R is a subset of S; (ii) $\mu^*(\cup R_i) \leqq \Sigma\mu^*(R_i)$ for any sequence of sets $\{R_i\}$; (iii) $\mu^*(R \cup S) = \mu^*(R) + \mu^*(S)$ if the distance between R and S is positive. A set R is then said to be **measurable** if $\mu^*(E) = \mu^*(R \cap E) + \mu^*(R' \cap E)$ for any set E, where R' is the complement of R. A bounded set R of real numbers (or a set in n-dimensional Euclidean space) is Lebesgue measurable if and only if

$$m^*(E) = m^*(R \cap E) + m^*(R' \cap E)$$

for any bounded set E, where m^* is the exterior Lebesgue measure. This is called the *Caratheodory test* for measurability. See below, exterior measure.

circular measure. (1) Same as angular measure. (2) The measure of angles by means of radians. See RADIAN.

common measure. Same as COMMON DIVISOR.

convergence in measure. See CONVERGENCE.

cubic measure. The measurement of volumes in terms of a cube whose edge is a standard linear unit, a *unit cube*. *Syn.* Volume measure.

decimal measure. See DECIMAL—decimal measure.

dry measure. The system of units used in measuring dry commodities, such as grain, fruit, etc. In the United States, the system is based on the bushel. See DENOMINATE NUMBERS in the appendix.

exterior and interior measure. Let E be a set of points and S be a finite or countably infinite set of intervals (open or closed) such that each point of E belongs to at least one of the intervals (*interval* is used in a generalized sense described below). The **exterior measure** of E is the greatest lower bound of the sum of the measures of the intervals of S, for all such sets S. Let E be contained in an interval I and let E' be the complement of E in I. Then the **interior measure** of E is the difference beteween the measure of I and the exterior measure of E'. If a set E is either open or closed, its exterior and interior measures are equal and the common value is its measure (see MEASURABLE—measurable set). Also, the exterior measure of a set E is the greatest lower bound of the measures of open sets which contain E and the interior measure of E is the least upper bound of the measures of closed sets which are contained in E. The measure of an interval of a straight line is its length. An **interval** I in n-dimensional space is a "generalized rectangular parallelepiped" consisting of all points $x=(x_1, x_2, \cdots, x_n)$ for which $a_i \leqq x_i \leqq b_i$ for each i, where (a_i, b_i), $i=1, \cdots, n$, are fixed constants. The measure of I is the product

$$(b_1-a_1)(b_2-a_2)\cdots(b_n-a_n),$$

the same definition being used if the interval is open, or partly open and partly closed. *Syn.* Lebesgue exterior (interior) measure. See INTERVAL—closed interval, INTERVAL, CONTENT, MEASURABLE—measurable set, and above, Caratheodory measure.

Haar measure. Let G be a locally compact topological group. A *Haar measure* is a measure which assigns a nonnegative real number $m(E)$ to each set E of the σ-ring S generated by the *compact* (*i.e., bicompact*) subsets of G and which has the properties: (i) there is a member of S for which m is not zero; (ii) m is either left-invariant or m is right-invariant—*i.e.*, $m(aE)=m(E)$ for each element a and member E of S, or $m(Ea)=m(E)$ for each a and E, where aE and Ea are, respectively, the set of all ax for x in E and the set of all xa for x in E. Any locally compact topological group has a left-invariant Haar measure and a right-invariant Haar measure and each is unique to within a multiplicative constant.

land measure. See ACRE, and DENOMINATE NUMBERS in the appendix.

Lebesgue measure. See MEASURABLE—measurable set.

linear measure. Measurement along a line, the line being either straight or curved.

liquid measure. The system of units ordinarily used in measuring liquids. See DENOMINATE NUMBERS in the appendix.

measure of central tendency. (*Statistics.*) The *mean*, the *mode*, the *median*, and the *geometric mean* are commonly used.

measure of dispersion. (*Statistics.*) Same as DEVIATION.

measure of a set. Let R be a class of sets which constitute a *ring* (or a *semiring*) *of sets*. A *measure* is a set function m which associates a nonnegative real number (or the symbol ∞) with each set of R, for which $m(\theta)=0$, where θ is the empty set, and for which $m(\bigcup_1^\infty S_n)=\sum_1^\infty m(S_n)$ if S_n is a member of R for n any positive integer, the sets S_n and S_m have no common points if $m\neq n$, and the union $\bigcup_1^\infty S_n$ of all the sets $\{S_n\}$ is a member of R. It is understood that $\sum_1^\infty m(S_n)=\infty$ if $\sum_1^k m(S_n)$ is not bounded as a function of k, and that any sum is equal to ∞ if one of the summands is ∞. The measure of a set S of R is said to be **σ-finite** if there is a sequence $\{S_n\}$ of sets in R such that $S\subset\bigcup_1^\infty S_n$ and $m(S_n)\neq\infty$ (for each n). If the measure of each set of R is σ-finite, the measure m is said to be **σ-finite.** See MEASURABLE—measurable set, and MEASURE—Caratheodory measure, product measure.

measure of a spherical angle. The plane angle formed by the tangents to the sides of the spherical angle at their points of intersection.

measure ring and **measure algebra.** If a measure is defined on a σ-ring of subsets of a space X, then these measurable subsets of X are a **measure ring** if equality is defined

by the definition that $A=B$ if the measure of the symmetric difference of A and B is zero (see RING—ring of sets). That is, the measure ring is the *quotient ring* of the σ-ring modulo the ideal of sets of measure zero. A measure ring is a measure algebra if there is a measurable set which contains all the measurable sets (it is then a Boolean algebra). The set of elements of finite measure in a measure ring is a metric space if the distance between sets A and B is defined to be the measure of the symmetric difference of A and B.

measure zero. A point set is said to be of measure zero if for any positive number ϵ there exists a finite or countably infinite set of intervals (open or closed) such that each point of the set is contained in at least one of the intervals, and the sum of the measures of the intervals is less than ϵ. The point set may be points of a line or of n-dimensional Euclidean space (see INTERVAL (2)). A property of points x is said to hold **almost everywhere, a.e.,** or for **almost all points,** if it holds for all points except those of a set of measure zero. *E.g.*, a function is continuous *almost everywhere* if the set of points at which it is discontinuous is of measure zero. A set of points of measure zero is necessarily measurable. See MEASURABLE—measurable set.

product measure. Let m_1 and m_2 be measures defined on σ-rings of subsets of spaces X and Y, respectively, and let $X\times Y$ be the Cartesian product whose points consist of all pairs (x, y) with x in X and y in Y. The *product measure* of m_1 and m_2 is the measure defined on the σ-ring generated by the "rectangles" $A\times B$ of $X\times Y$ for which A and B are measurable and the measure of $A\times B$ is the product of the measures of A and B.

square measure. The measure of areas of surfaces in terms of a square whose side is a standard linear unit, a *unit square*. *Syn.* Surface measure.

surveyor's measure. See DENOMINATE NUMBERS in the appendix.

wood measure. See DENOMINATE NUMBERS in the appendix.

MEAS'URE-MENT, *n.* The act of measuring.

median of a group of measurements. See MEDIAN.

ME-CHAN'IC, *n.* **mechanic's rule.** A rule for extracting square roots. The rule is as follows: Make an estimate of the root, divide the number by this estimate, and take for the approximate square root the arithmetic mean (average) of the estimate and the quotient thus obtained. If a more accurate result is desired, repeat the process. *Algebraically*, if a is the estimate, e the error, and $(a+e)$ the required number, divide $(a+e)^2$, that is $a^2+2ae+e^2$, by a and take the average between a and the quotient. This gives $a+e+e^2/(2a)$. The error is $e^2/(2a)$, which is very small if e is small. *E.g.*, if one estimates $\sqrt{2}$ to be 1.5, the mechanic's rule gives the root as 1.4167. The error in this is less than .003. The error in a second application of the rule is less than $(.003)^2/2$ or .0000045.

ME-CHAN'I-CAL, *adj.* **mechanical integration.** See INTEGRATION—mechanical integration.

ME-CHAN'ICS, *n.* The mathematical theory of the motions and tendencies to motion of particles and systems under the influence of forces and constraints; the study of motions of masses and of the effect of forces in causing or modifying these motions. Usually divided into *kinematics* and *dynamics*.

analytical mechanics. A mathematical structure of mechanics whose present-day formulation is largely due to J. L. Lagrange (1736–1813) and W. R. Hamilton (1805–1865). This structure is also known as *theoretical mechanics*. It uses differential and integral calculus as tools.

mechanics of fluids. A body of knowledge including the theory of gases, hydrodynamics, and aerodynamics.

theoretical mechanics. See above, analytical mechanics.

ME'DI-AN, *adj., n.* **median deviation.** See DEVIATION.

median of a group of measurements. The middle measurement when the items are arranged in order of size; or, if there is no middle one, then the average of the two middle ones. If five students make the grades 15%, 75%, 80%, 95%, and 100%, the median is 80%.

median point of a triangle. The point

of intersection of the medians of a triangle; the centroid of a triangle.

median of a trapezoid. The line joining the midpoints of the nonparallel sides. *Syn.* Midline.

median of a triangle. A line joining a vertex to the middle point of the opposite side.

MEET, *n.* See LATTICE and INTERSECTION.

MELLIN. **Mellin inversion formulas.** The associated formulas

$$g(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s}f(s)\,ds,$$

$$f(s)=\int_0^{\infty} x^{s-1}g(x)\,dx,$$

each of which gives the inverse of the other under suitable conditions of regularity. See FOURIER—Fourier transform, LAPLACE—Laplace transform.

MEM′BER, *n.* **member of an equation.** The expression on one (or the other) side of the equality sign. The two members of an equation are distinguished as the *left* or *first* and the *right* or *second* member.

MEM′O-RY, *adj.* **memory component.** See STORAGE—storage component.

MENELAUS' THEOREM. If, in the triangle ABC, points P_1, P_2, P_3 are on the sides AB, BC, CA, respectively, then P_1, P_2, P_3 are collinear if, and only if,

$$\frac{AP_1}{P_1B}\frac{BP_2}{P_2C}\frac{CP_3}{P_3A}=-1.$$

MEN′SU-RA′TION, *n.* The measuring of geometric magnitudes, such as the lengths of lines, areas of surfaces, and volumes of solids.

MER′CAN-TILE, *adj.* **mercantile rule.** Same as MERCHANT'S RULE. See MERCHANT.

MERCATOR. **Mercator chart.** A map made by use of *Mercator's projection.* A straight line on the plane corresponds to a curve on the sphere cutting meridians at constant angle. Magnification of area on the sphere increases with increasing distance from the equator. See below, mercator's projection.

Mercator's projection. A correspondence between points of the (x, y)-plane and points on the surface of the sphere, given by $x=k\theta$,

$$y=k\operatorname{sech}^{-1}(\sin\phi)=k\log\tan\frac{\phi}{2},$$

where θ is the angle of longitude and ϕ is the angle of colatitude. The correspondence is a conformal one, except for singular points at the poles.

MER′CHANT, *n.* **merchant's rule.** A rule for computing the balance due on a note after partial payments have been made. The method is to find the amount of each partial payment at the settlement date and subtract the sum of these from the value of the face of the note at the same date. *Syn.* Mercantile rule.

ME-RID′I-AN, *n.* On the *celestial sphere,* a meridian is a great circle passing through the zenith and the north and south line in the plane of the horizon (see HOUR—hour angle and hour circle). On the *earth,* a meridian is a great circle on the surface of the earth passing through the geographic poles. The **local meridian** of a point on the earth is the meridian which passes through that point. The **principal meridian** is the meridian from which longitude is reckoned (usually the meridian through the transit-circle of the Royal Observatory of Greenwich, England, although observers frequently use the meridian through the capital of their country). The principal meridian is also called the *first, prime, zone,* and *zero meridian.*

meridian curve on a surface. A curve C on a surface such that the spherical representation of C lies on a great circle on the unit sphere.

meridian sections, or meridians. See SURFACE—surface of revolution.

MER-O-MOR′PHIC, *adj.* **meromorphic function.** A function of the complex variable z is said to be *meromorphic* in a domain D if it is *analytic* in D except for a finite number of poles.

MERSENNE NUMBERS. Numbers of the type $M_p=2^p-1$, where p is a prime.

Mersenne asserted that the only primes for which M_p is a prime are 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257. Actually, M_{67} and M_{257} are not primes. It is now known that M_{61}, M_{89}, and M_{107} are primes.

MESOKURTIC, *adj.* **mesokurtic distribution.** See KURTOSIS.

ME'TER (ME'TRE), *n.* The basic unit of linear measure of the metric system; the distance between two marks on a platinum bar preserved in Paris. It is equal to 39.37+ inches.

METH'OD, *n.* **method of geometric exhaustion.** See GEOMETRIC.

method of least squares. A method based upon the principle that the best value of a quantity that can be deduced from a set of measurements or observations is that for which the sum of the squares of the deviations of the observed values (from it) is a minimum. In the case of a single set of measurements this principle gives the arithmetic mean of the several measurements as the best value. The *method of least squares* consists in using this principle to determine the arbitrary constants in the equation of a curve which has been assumed to be the type that best fits a given set of data. Suppose $y=mx$ is such a type and that for $x=1, 2, 3, 4$ the observed values of y are 2, 4, 7, 6. Then the method of least squares would take for m that value which makes

$$(m-2)^2+(2m-4)^2+(3m-7)^2+(4m-6)^2$$

a minimum. This value is 1.8+ and the straight line through the origin whose graph best fits the points (1, 2), (2, 4), (3, 7), (4, 6) is $y=1.8x$. *Tech.* Let y be a function of x such that, for fixed values of x, the mean value of y is given by a linear function of x, and the variances of y around the mean values are constant. Then the *minimum variance unbiased estimates* of the parameters of the linear function of x are obtained by determining those values of the parameters which minimize the sum of squares of the observed y around the estimated y (based on the estimated parameters). Specifically, the quantity

$$\sum_{i=1}^{n}(y_i-\sum_{j=1}^{r}a_jx_{i,j})^2$$

is minimized with respect to the parameters a_j. This does not exclude the use of polynomial functions of x of degree greater than one, since the polynomials in x may be considered the new variables which are linearly related in the function. In addition to justifying the use of least squares via the criterion of *minimum variance unbiased estimates*, as the preceding does, it is possible to use another criterion, *viz.*, that of obtaining those parameter estimates which maximize the probability of obtaining the observed values y_i, under the assumption that the observed values differ from the true values in that the deviations from the true values are normally and independently distributed. This leads to the use of the minimized least squares. The criterion is known as the **maximum likelihood criterion.**

MET'RIC, *adj., n.* **metric density.** Let E be a measurable subset of the line (or of n-dimensional Euclidean space). The metric *density* of E at a point x is the limit (if it exists) of $m(E\cap I)/m(I)$ as the length (or measure) $m(I)$ of I approaches zero, where I is an interval which contains x. The metric density of E is 1 at all points of E except a set of measure zero; it is 0 at all points of the complement of E except a set of measure zero.

metric space. A set T such that to each pair x, y of its points there is associated a nonnegative real number $\rho(x, y)$, called their *distance*, which satisfies the conditions: (1) $\rho(x, y)=0$ if, and only if, $x=y$; (2) $\rho(x, y)=\rho(y, x)$; (3) $\rho(x, y)+\rho(y, z)\geq\rho(x, z)$. The function $\rho(x, y)$ is said to be a metric for T. The plane and three-dimensional space are metric spaces with the usual distance. Hilbert space is also a metric space. A topological space is said to be metrizable if a distance between points can be defined so the space is a metric space such that open sets in the original space are open sets in the metric space, and conversely; *i.e.*, there exists a topological transformation between the given space and a metric space. A compact (or bicompact) Hausdorff space is metrizable if and only if it satisfies the *second axiom of countability*; a regular T_1 topological space is metrizable if it satisfies the second axiom of countability (Urysohn's theorem). A topological

space is metrizable if and only if it is a regular T_1 topological space whose topology has a base B which is the union of a countable number of classes $\{B_n\}$ of open sets which have the property that, for each point x and class B_n, there is a neighborhood of x which intersects only a finite number of the members of B_n.

metric system. The system of measurement in which the meter is the fundamental unit. It was first adopted in France and is in general use in most other civilized countries, except the English-speaking countries, and is now almost universally used for scientific measurements. The unit of surface is the *are* (100 square meters) and the theoretical unit of volume is the *stere* (one cubic meter), although the *liter* (one cubic decimeter) is most commonly used. See DENOMINATE NUMBERS in the appendix. The prefixes *deca-*, *hecto-*, *kilo-*, and *myria-* are used on the above units to designate 10 times, 100 times, 1000 times and 10,000 times the unit. The prefixes *deci-*, *centi-*, and *milli-* are used to designate $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$ parts of the respective unit.

MET-RIZ′A-BLE, *adj.* **metrizable space.** See METRIC—metric space.

MEUSNIER'S THEOREM. This theorem states that the center of curvature of a curve on a surface S is the projection, on its osculating plane, of the center of curvature of the normal section tangent to the curve at the point. More graphically, if a segment equal to twice the radius of normal curvature for a given direction at a point of S is laid off from the point on the normal to S, and a sphere is drawn with the segment as diameter, then the circle of intersection of the sphere by the osculating plane of a curve on S in the given direction at the point is the circle of curvature of the curve.

MIDAC. An automatic digital computing machine at the Willow Run Research Center. MIDAC is an acronym for *Michigan Digital Automatic Computer.*

MIDLINE, *n.* **midline of a trapezoid.** See MEDIAN—median of a trapezoid.

MIDPERPENDICULAR, *n.* The perpendicular bisector of a line segment.

MIDPOINT, *n.* **midpoint of a line segment.** The point that divides the line segment into two equal parts; the point that bisects the line. See BISECT.

MIL, *n.* A unit of angle measure, equal to $\frac{1}{6400}$ of a complete revolution, .05625°, and nearly $\frac{1}{1000}$ of a radian. Used by U. S. artillery.

MILE, *n.* A unit of linear measure equal to 5280 feet or 320 rods.

geographical mile. Same as NAUTICAL MILE. See NAUTICAL.

nautical mile. See KNOT.

MIL′LI-ME′TER, *n.* One thousandth part of a meter.

MIL′LION, *n.* One thousand thousands (1,000,000).

MINAC. An automatic digital computing machine at the California Institute of Technology. MINAC is an acronym for *Minimal Automatic Computer.*

MIN′I-MAL, *adj.* **adjoint minimal surfaces.** Two associate minimal surfaces with parameters α_1 and α_2 differing by $\pi/2$. See below, associate minimal surfaces.

associate minimal surfaces. When the minimal curves of a minimal surface are parametric, the coordinate functions are of the form $x=x_1(u)+x_2(v)$, $y=y_1(u)+y_2(v)$, $z=z_1(u)+z_2(v)$. The related equations $x=e^{i\alpha}x_1(u)+e^{-i\alpha}x_2(v)$, $y=e^{i\alpha}y_1(u)+e^{-i\alpha}y_2(v)$, $z=e^{i\alpha}z_1(u)+e^{-i\alpha}z_2(v)$ define a family of minimal surfaces, called **associate minimal surfaces**, with parameter α.

double minimal surface. A one-sided minimal surface; a minimal surface S such that, through each of its points P, there exists a closed path C on S having the property that, when a variable point traverses C returning to P, the positive direction of the normal is reversed. See SURFACE—surface of Henneberg. *Syn.* One-sided minimal surface.

minimal curve. A curve for which the linear element ds vanishes identically. With a Euclidean metric, $ds^2=dx_1^2+dx_2^2+\cdots+dx_n^2$, this can occur only if the curve reduces to a point or if at least one of the coordinate functions is imagi-

nary. See below, minimal straight line. *Syn.* Isotropic curve, curve of zero length.

minimal (or minimum) equation. See ALGEBRAIC—algebraic number, CHARACTERISTIC—characteristic equation of a matrix.

minimal straight line. A minimal curve which is an imaginary straight line. There is an infinitude of them through each point of space; their direction components are $(1-a^2)/2$, $i(1+a^2)/2$, a, where a is arbitrary. See above, minimal curve.

minimal surface. A surface whose mean curvature vanishes identically; a surface for which the first variation of the area integral vanishes. A minimal surface does not necessarily minimize the area spanned by a given contour; but if a smooth surface S minimizes the area, then S is a minimal surface.

one-sided minimal surface. Same as DOUBLE MINIMAL SURFACE.

MIN'I-MAX, *n.* Same as SADDLE POINT. See SADDLE.

minimax theorem. For a finite two-person zero-sum game, with *payoff matrix* (a_{ij}), where $i=1, 2, \cdots, m$ and $j=1, 2, \cdots, n$; if the maximizing player uses *mixed strategy* $X=(x_1, x_2, \cdots, x_m)$ and the minimizing player uses *mixed strategy* $Y=(y_1, y_2, \cdots, y_n)$, then the expected value of the payoff is given by the expression

$$\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}x_iy_j = v_{X,Y} = E(X, Y).$$

It is easy to see that $\max_X(\min_Y v_{X,Y}) \leqq \min_Y(\max_X v_{X,Y})$. The *minimax theorem*, fundamental in the theory of games, states that the sign of equality must hold for every finite two-person zero-sum game:

$$\max_X(\min_Y v_{X,Y}) = \min_Y(\max_X v_{X,Y}) = v.$$

This result can be extended to the class of continuous two-person zero-sum games with continuous payoff functions and to certain other classes of infinite games. In particular, if a two-person zero-sum game has a *saddle point* at (x_0, y_0), then v is the value of the payoff function at (x_0, y_0). See GAME—value of a game, SADDLE—saddle point of a game.

MIN'I-MUM, *adj., n.* [*pl.* minima]. See various headings under MAXIMUM.

MINKOWSKI, **Minkowski distance function.** Relative to a convex body B of which the origin O is an interior point, the Minkowski *distance function* $F(P)$ is defined to be the ratio of distances OP/OQ, where P is any point of the space (other than O) and Q is the point of B on the ray OP that is farthest from O. We then define $F(O)$ to be O and have $F(P)<1$ if P is interior to B, $F(P)=1$ if P is on the boundary of B, and $F(P)>1$ if P is exterior to B. The function $F(P)$ is a convex function of P. Two **polar reciprocal convex bodies** are any two convex bodies, each containing the origin in its interior, such that the *support function* of each is the *distance function* of the other. See SUPPORT—support function.

Minkowski's inequality. Either of the inequalities

$$(1)\quad \left[\sum_1^n |a_i+b_i|^p\right]^{1/p} \leqq \left[\sum_1^n |a_i|^p\right]^{1/p} + \left[\sum_1^n |b_i|^p\right]^{1/p}$$

or

$$(2)\quad \left[\int_\Omega |f+g|^p\,dx\right]^{1/p} \leqq \left[\int_\Omega |f|^p\,dx\right]^{1/p} + \left[\int_\Omega |g|^p\,dx\right]^{1/p},$$

which are valid if $p \geqq 1$ and the integrals involved exist for the interval or region of integration Ω. The numbers in (1) or the functions in (2) may be real or complex. Either of these inequalities is easily deduced from the other or from *Hölder's inequalities*. The inequalities are reversed for $p \leqq 1$, if a_i, b_i, f, and g are nonnegative for each i and x.

MI'NOR, *adj., n.* **minor arc of a circle.** See SECTOR—sector of a circle.

minor axis of an ellipse. The smaller axis of the ellipse.

minor of an element in a determinant. The determinant, of next lower order, obtained by striking out the row and column in which the element lies. This is sometimes called the complementary minor. The minor, taken with a positive or negative sign according as the *sum* of the position numbers of the row and column stricken out of the original determinant is even or odd, is called the signed minor, or

cofactor, of the element. *E.g.*, the minor of b_1 in the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

is the determinant

$$\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}$$

and the cofactor of b_1 is

$$-\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}.$$

MIN'U-END, *n.* The quantity from which another quantity is to be subtracted.

MIN'US, *adj.* A word used between two quantities to state that the second is to be subtracted from the first. The statement 3 minus 2, written $3-2$, means 2 is to be subtracted from 3. *Minus* is also used as a synonym for *negative.*

MIN'UTE, *n.* (1) The sixtieth part of an hour. (2) The sixtieth part of a degree (in the sexagesimal system of measuring angles). See SEXAGESIMAL.

MIXED, *adj.* **mixed decimal.** See DECIMAL.

mixed number, quantity. In *arithmetic,* the sum of an integer and a fraction, as $2\frac{3}{4}$. In *algebra,* the sum of a polynomial and a rational algebraic fraction, as

$$2x+3+\frac{1}{x+1}.$$

MNE-MON'IC, *adj.* Assisting the memory; relating to the memory.

mnemonics or mnemonic devices. Any scheme or device to aid in remembering certain facts. *E.g.*, (1) $\sin(x+y)$ is equal to the sum of two mixed terms in sine and cosine, whereas $\cos(x+y)$ is equal to the difference of unmixed terms; (2) the derivatives of all cofunctions (*trigonometric*) take a negative sign.

MÖBIUS. Möbius function. The function of the positive integers defined by $\mu(1)=1$; $\mu(n)=(-1)^r$ if $n=p_1p_2\cdots p_r$, where $p_1,\cdots,$ p_r are distinct positive prime numbers, and $\mu(n)=0$ for all other positive integers. It follows that $\mu(n)$ is also the sum of the primitive nth roots of unity. See RIEMANN—Riemann's hypothesis about the zeta function.

Möbius strip. The one-sided surface formed by taking a long rectangular strip of paper and pasting its two ends together after giving it half a twist. See SURFACE—one-sided surface.

Möbius' transformation. A transformation (in the complex plane) of the form $\omega=(az+b)/(cz+d)$, with $ad-bc\neq 0$.

MODE, *n.* (*Statistics.*) The most frequent value. If more students (of a given group) make 75% than any other one grade, then 75% is the mode.

MOD'U-LAR, *adj.* **(elliptic) modular function.** A function which is *automorphic* with respect to the modular group (or a subgroup of the modular group) and which is single-valued and analytic in the upper half of the complex plane, except for poles. Usually suitable additional restrictions are made, so that such a function is a rational function of the modular function $J(\tau)$ defined by

$$J(\tau)=\frac{4}{27}\frac{(\vartheta_3{}^8-\vartheta_2{}^4\vartheta_4{}^4)^3}{(\vartheta_2\vartheta_3\vartheta_4)^8},$$

where ϑ_i denotes the function of the parameter τ obtained by setting $z=0$ in the theta function $\vartheta_i(z)$. Also:

$$J(\tau)=\frac{g_2{}^3}{g_2{}^3-27g_3{}^2},$$

where $g_2=60\Sigma'(m\omega+n\omega')^{-4}$, $g_3=140\Sigma'$ $(m\omega+n\omega')^{-6}$ and $\tau=\omega'/\omega$ (the primes indicate that the summations are for all integral values of m and n except $m=n=0$). The modular functions $\lambda(\tau)=\vartheta_2{}^4/\vartheta_3{}^4$ [also denoted by $f(\tau)$], $g(\tau)$, and $h(\tau)=-f(\tau)/g(\tau)$ are also of considerable interest; J and λ are related by

$$27J\lambda^2(1-\lambda)^2=4(1-\lambda+\lambda^2)^3.$$

modular group. The transformation group which consists of all transformations

$$w=\frac{az+b}{cz+d}$$

with $ad-bc=1$, where a, b, c and d are real integers. Such transformations map the upper half (and the lower half) of the complex plane onto itself, and map real points into real points.

MOD′I-FIED, *p.* **modified Bessel functions.** See BESSEL.

MOD′ULE, *n.* Let S be a set such as a ring, integral domain, or algebra, which is a group with respect to an operation called addition (it may have other operations defined, such as multiplication and scalar multiplication). A module M of S is a subset of S which is a group with respect to addition (this is equivalent to stating that $x-y$ belongs to M whenever x and y belong to M). See IDEAL.

MOD′U-LUS, *n.* **bulk modulus.** (*Elasticity.*) The ratio of the compressive stress to the cubical compression. It is connected with Young's modulus E and Poisson's ratio σ by the formula $k=\dfrac{E}{3(1-2\sigma)}$. The bulk modulus k is positive for all physical substances. *Syn.* Compression modulus.

modulus of logarithms. See LOGARITHM.

modulus of a complex number. The numerical length of the vector representing the complex number (see COMPLEX—complex number). The modulus of a complex number $a+bi$ is $\sqrt{a^2+b^2}$, or, if the number is in the form $r(\cos\beta+i\sin\beta)$ with $r\geqq 0$, the modulus is r. The modulus of $4+3i$ is 5; the modulus of $1+i=\sqrt{2}(\cos\frac{1}{4}\pi+i\sin\frac{1}{4}\pi)$ is $\sqrt{2}$. *Syn.* Absolute value.

modulus of a congruence. See CONGRUENT—congruent numbers.

modulus of an elliptic integral and of an elliptic function. See ELLIPTIC—elliptic integral, and Jacobian elliptic functions.

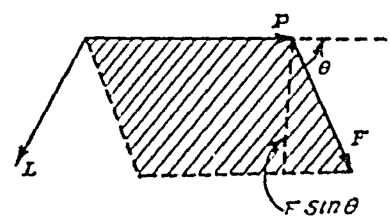
modulus of rigidity. See RIGIDITY.

Young's modulus. The constant introduced in the theory of elasticity by Thomas Young in 1807. It characterizes the behavior of elastic substances. If the stress acting in the cross section of a thin rod is T and the small elongation produced by it is e, then $T=Ee$, where E is Young's modulus in tension. For many substances the modulus in tension differs from the corresponding compression modulus. Young's modulus and Poisson's ratio (*q.v.*) are found to be sufficient to completely characterize the elastic state of an isotropic substance.

MOEBIUS. See MÖBIUS.

MO′MENT, *n.* **method of moments.** (*Statistics.*) A method of estimating the parameters of a frequency distribution by means of a function which relates the parameters to moments. The moments may be estimated from samples. As many moments as there are parameters to be estimated are computed and the parameters are estimated by solving the inverse of the moment-generating function. This method does not in general yield the *minimum-variance unbiased estimates.* Nevertheless, it is of practical importance because of its expedience. See below, moment of a frequency distribution.

moment of a force. The moment of a force about a line is the product of the projection of the force on a plane perpendicular to the line and the perpendicular distance from the line to the line of action of the force. The moment of a force F about a point O is the vector product of the position vector $\mathbf{r}$ (from O to the point of application of the force) by the force. In symbols of vector analysis, the moment $\mathbf{L}$ is given by $\mathbf{L}=\mathbf{r}\times\mathbf{F}$. The magnitude of the moment of force is $L=rF\sin\theta$, where θ is the angle between the vectors $\mathbf{r}$ and $\mathbf{F}$. Numerically it is equal to the area of the parallelogram constructed on the vectors $\mathbf{r}$ and $\mathbf{F}$ as sides. *Syn.* Torque.



moment of a frequency distribution. (*Statistics.*) A set of parameters of a distribution which measures its properties and frequently specifies it exactly;

$$\mu_r=\int_{-\infty}^{\infty}(x-a)^r f(x)\,dx$$

is the rth moment of x around the point a, where x is a random variable with frequency function $f(x)$ and a is any given point of distribution. See DISTRIBUTION—F distribution, and VARIANCE—analysis of variance.

moment generating function. (*Statistics.*) The function

$$\phi(t)=\int_{-\infty}^{\infty} e^{tx}f(x)\,dx,$$

where $f(x)$ is the frequency function. By taking the jth derivative of $\phi(t)$ and evaluating for $t=0$, the jth moment is obtained. Differs from a characteristic function only in that $\phi(it)$ is the characteristic function, where $i^2=-1$. *E.g.*, for the *binomial distribution*,

$$\phi(t)=\sum_{k=0}^{n} e^{tk}C_n{}^k p^k q^{n-k}=(q+pe^t)^n.$$

The first derivative of $\phi(t)$ is

$$npe^t(q+pe^t)^{n-1},$$

which evaluated at $t=0$ yields np, the first moment. The second derivative evaluated at $t=0$ yields the second moment about zero, $np+n(n-1)p^2$.

moment of inertia of a particle about a point, line, or plane. The product of the mass and the square of the distance from the particle to the point, line, or plane. The moment of inertia of a system of discrete particles about an axis is the sum of the products of the masses of the particles by the squares of their distances from the axis. Thus $I=\sum_i m_i r_i^2$, where r_i is the distance of the particle of mass m_i from the axis. For a continuous body, referred to a system of rectangular Cartesian axes x, y, z, one can define the moments of inertia about the axes x, y, and z, respectively, by the formulas

$$I_x=\int_s (y^2+z^2)\,dm,$$

$$I_y=\int_s (z^2+x^2)\,dm,$$

$$I_z=\int_s (x^2+y^2)\,dm,$$

where the integrations are extended over the entire body. The quantities

$$I_{xy}=\int_s xy\,dm,\quad I_{yz}=\int_s yz\,dm,$$

and

$$I_{xz}=\int_s xz\,dm$$

are called the **products of inertia.** If the coordinate axes x, y, z are so chosen that the products of inertia vanish, the axes are called the **principal axes of inertia.**

moment of mass about a point, line, or plane. The sum of the products of each of the particles of the mass and its distance from the point, line, or plane. *Tech.* The integral, over the given mass, of the element of mass times its perpendicular distance from the point, line, or plane (this product is called the **element of moment of mass**; see ELEMENT—element of integration). Algebraic (signed) distances are to be used in computing moments. A moment is essentially the sum of the moments of individual particles (elements of integration). For a mass on a line (the x-axis), the moment about a point on the line is

$$\int (x-a)\rho(x)\,dx,$$

where $\rho(x)$ is the density (mass per unit length) at the point x [this is the same as the *first moment of a frequency distribution* for which $\rho(x)$ is the frequency function (see above)]. For mass in the plane, moment about the y-axis is

$$\int x\rho(x,y)\,dA,$$

where $\rho(x, y)$ is the density (mass per unit area) at the point (x, y) (if ρ is a function of x alone, then dA may be a strip parallel to the y-axis) [similar formulas can be given for other lines in the plane (*e.g.*, by using signed distances from the line, as given by the *normal form* of the equation of the line). For a mass in space, the moment with respect to the (x, y)-plane is given by

$$\int z\rho(x,y,z)\,dV,$$

where $\rho(x, y, z)$ is the density (mass per unit volume) at the point (x, y, z) in the element of volume dV. **Moment of a curve** means the moment obtained by regarding the curve as having unit mass per unit length. **Moment of area** means the moment obtained by regarding the area as having unit mass per unit area.

moment of momentum. See MOMENTUM.

product moment. (*Statistics.*) The function

$$\frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{n}$$

is the product moment of two variables x and y, with n values each which may be paired on the basis of some given criterion, where $\bar{x}$ and $\bar{y}$ are their arithmetic means. If the product moment is divided by both the standard deviations of the variables x and y, the product moment correlation coefficient is obtained. The product moments about the origin are defined by

$$u_{pq}' = \int_{a_2}^{b_2} \int_{a_1}^{b_1} x^p y^q f(x, y)\, dx\, dy,$$

and about the mean by

$$u_{pq} = \int_{a_2}^{b_2} \int_{a_1}^{b_1} (x-\bar{x})^p (y-\bar{y})^q f(x, y)\, dx\, dy,$$

u_{11} is the integral counterpart of the above (summation) special definition.

second moment. See above, moment of inertia.

MO-MEN'TUM, *n.* The "quantity of motion" measured by the product of the mass and the velocity of the body. Sometimes called the *linear momentum* to distinguish it from the angular momentum or moment of momentum (*q.v.*). The *linear momentum* of a particle of mass m moving with the velocity $\mathbf{v}$ is the vector $m\mathbf{v}$. The linear momentum of a system of particles of masses $m_1, m_2, \cdots, m_n$, moving with the velocities $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$, is the vector

$$\mathbf{M} = \sum_i m_i \mathbf{v}_i,$$

representing the vector sum of the linear momenta of individual particles. The momentum of a continuous distribution of mass is defined by the integral

$$\mathbf{M} = \int_s \mathbf{v}\, dm$$

evaluated over the body occupied by the mass.

angular momentum. See below, moment of momentum.

moment of momentum. The moment of momentum relative to a point O of a particle of mass m moving with velocity $\mathbf{v}$ is the vector product of the position vector $\mathbf{r}$ of the particle relative to O and the momentum $m\mathbf{v}$. In symbols of vector analysis, where a cross denotes the vector product, the moment of momentum is $\mathbf{H} = \mathbf{r} \times m\mathbf{v}$. The moment of momentum relative to O of a system of particles is defined as the sum of the moments of momentum of the individual particles. For a continuous distribution of mass,

$$\mathbf{H} = \int_s (\mathbf{r} \times \mathbf{v})\, dm,$$

where the integration is extended over the entire body. *Syn.* Angular momentum.

principle of linear momentum. A theorem in mechanics stating that the time rate of change of linear momentum of a system of particles is equal to the vector sum of external forces.

MON'IC, *adj.* **monic equation.** A polynomial equation whose coefficients are integers, the coefficient of the term of highest degree being $+1$.

MO-NOD'RO-MY, *n.* **monodromy theorem.** The theorem states that, if the function $f(z)$ of the complex variable z is analytic at the point z_0 and can be continued analytically along every curve issuing from z_0 in a finite simply connected domain D, then $f(z)$ is a function-element of an analytic function which is single valued in D; in other words, analytic continuation around any closed curve in D leads to the original function element. See DARBOUX—Darboux's monodromy theorem.

MON'O-GEN'IC, *adj.* **monogenic analytic function.** The totality of all pairs $z_0, f(z)$, where

$$f(z) = \sum a_n (z - z_0)^n,$$

which can theoretically be obtained directly or indirectly by analytic continuation from a given function element $f_0(z)$. The function $f_0(z)$ is called the primitive element of the monogenic function. The Riemann surface of the values z_0 is called the domain of existence of the monogenic function and the boundary of the domain of existence is called the natural boundary of the monogenic analytic function. For example, the unit circle $|z| = 1$ is a natural boundary for the function

$$f(z) = \sum_{n=1}^{\infty} z^{n!}.$$

See ANALYTIC—analytic continuation of an analytic function of a complex variable.

MO-NO′MI-AL, *n.* A single term.

monomial factor of an expression. A single term that may be divided out of every member of the expression; x is a monomial factor of $2x+3xy+x^2$.

MON′O-TONE, MON′O-TON′IC, *adj.* A **monotonic** (or **monotone**) **increasing** quantity is a quantity which never decreases (the quantity may be a function, sequence, etc., which either increases or remains the same, but never decreases). A sequence of sets $E_1, E_2, \cdots$ is monotonic increasing if E_n is contained in E_{n+1} for each n. A **monotonic** (or **monotone**) **decreasing** quantity is a quantity which never increases (the quantity may be a function, sequence, etc., which either decreases or remains the same, but never increases). A sequence of sets $E_1, E_2, \cdots$ is monotonic decreasing if E_n contains E_{n+1} for each n. A **monotonic** (or **monotone**) **system of sets** is a system of sets such that, for any two sets of the system, one of the sets is contained in the other. A mapping of a topological space A onto a topological space B is said to be *monotone* if the inverse image of each point of B is a continuum. A mapping of an ordered set A onto an ordered set B is *monotone* provided x^* precedes (or equals) y^* whenever x^* and y^* are the images in B of points x and y of A for which x precedes y.

MONTE CARLO METHOD. Any procedure that involves statistical sampling techniques in obtaining a probabilistic approximation to the solution of a mathematical or physical problem.

MOORE-SMITH CONVERGENCE. A **directed set** (also called a **directed system** or a **Moore-Smith set**) is a set D which is ordered in the sense that there is a relation which holds for some of the pairs (a, b) of D (one then writes $a \geqq b$) and for which (i) if $a \geqq b$ and $b \geqq c$, then $a \geqq c$; (ii) $a \geqq a$ for all a of D; (iii) if a and b are members of D, there is a c of D such that $c \geqq a$ and $c \geqq b$. A **net** (also called a **Moore-Smith sequence**) of a set S is a mapping of a directed set into S (onto a subset of S). *E.g.*, the set of positive integers is a directed set and a sequence $x_1, x_2, x_3, \cdots$ is a net; the set of all open subsets of a space T is a directed set if $U \geqq V$ means $U \subset V$. Let D be a directed set and ϕ a net, which is a mapping of D into a topological space T. Then ϕ is **eventually in** a subset U of T if there is an a of D such that if b belongs to D and $b \geqq a$, then $\phi(b)$ is in U; ϕ is **frequently in** U if for each a in D there is a b in D such that $b \geqq a$ and $\phi(b)$ is in U. The set E of all elements a of D for which $\phi(a)$ is in U is then a **cofinal subset** of D, meaning that, if b is in D, there is an a in E for which $a \geqq b$. The net ϕ **converges to a point** x of D if and only if it is eventually in each neighborhood of x. It follows that a point x is an *accumulation point* of a set V if and only if there is a net of V which converges to x. Also, a topological space is a *Hausdorff space* if and only if no net in the space converges to more than one point. See FILTER.

MORERA. Morera's theorem. The theorem states that if the function $f(z)$ of the complex variable z is continuous in a finite simply connected domain D, and satisfies $\int_C f(z)\,dz=0$ for all closed rectifiable curves in D, then $f(z)$ is an analytic function of z in D. This theorem is the converse of the Cauchy integral theorem.

MOR′RA, *n.* A game in which each of two players shows one, two, or three fingers and at the same time states his guess as to the number of fingers simultaneously being shown by his opponent. A player guessing correctly wins an amount proportional to the sum of the numbers of fingers shown by the two players, and the opponent loses this amount. Morra is an example of a *two-person zero-sum game* with chance moves. See MOVE.

MOR-TAL′I-TY, *adj., n.* **American experience table of mortality.** (1) A table of mortality (mortality table) based upon the lives of Americans constructed from insurance records about 1860. (2) A mortality table constructed from data obtained from American insurance companies and census records. See the APPENDIX.

force of mortality. The annual rate of mortality under the assumption that the intensity of mortality is constant throughout the age-year which is under consideration and has the value it had at the moment after the beginning of the age-year.

mortality table. A table showing the number of deaths that are likely to occur during a given year among a group of persons of the same age (the table being based upon past statistics). The number of lives at the age with which the table starts is called the **radix** of the mortality table. If a mortality table is based on lives of people who have had medical examinations, or are chosen from special groups, or otherwise selected so as to constitute a better risk than the general run of persons, it is called a **select mortality table**; the period during which the selection has effect upon the table is called the **select period** of the table (the effect of selection often wears off with the passing of years). An **ultimate life table** is a mortality table either based on the years after the select period (the latter not entering into consideration in the table), or using as a basis all the lives of a given age that are available.

rate of mortality. The probability that a person will die within one year after attaining a certain age; d_x/l_x, where d_x is the number dying during the year x and l_x is the number attaining the age x in the group on which the mortality table is based.

MORT'GAGE *n.* A conditional conveyance of, or lien upon, property as a security for money lent.

MO'TION, *n.* **constant (or uniform) motion.** See CONSTANT—constant speed and velocity.

curvilinear motion. Motion along a curve; motion which is not in a straight line.

curvilinear motion about a center of force. Motion such as that of celestial bodies about the sun; the motion of a particle whose initial velocity was not directed toward the center of force and which is attracted by a force at the given center. If this force is gravitational the path is a conic whose focus (or one of whose foci) is at the center of force.

Newtonian laws of motion. See NEWTON—Newton's laws of motion.

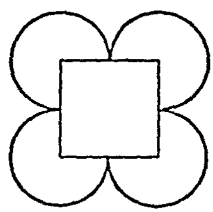
rigid motion. See RIGID.

simple harmonic motion. See HARMONIC.

MOVE, *n.* A component element of a game; a particular performance made at the choice of one of the players, or determined by a random device. A **personal move** is a move elected by one of the players, as contrasted with a **chance move** determined by a random device. See GAME, HER, and MORRA.

MUL'TI-AD-DRESS', *adj.* **multiaddress system.** A method of coding problems for machine solution, whereby a single instruction might involve more than one address, memory position, or command. See SINGLE—single-address system.

MUL'TI-FOIL. A plane figure made of congruent arcs of a circle arranged on a regular polygon so that the figure is symmetrical about the center of the polygon and the ends of the arcs are on the polygon. The name is sometimes restricted to the cases in which the polygon has six or more sides. When the polygon is a square, the figure is called a *quatrefoil* (as illustrated); when it is a regular hexagon, a *hexafoil*; when a triangle, a *trefoil*.



MUL'TI-LIN'E-AR, *adj.* **multilinear form.** See FORM.

MUL-TI-NO'MI-AL, *adj., n.* An algebraic expression containing more than one term. Compare POLYNOMIAL.

multinomial theorem. A theorem for the expansion of powers of multinomials. It includes the *binomial theorem* as a special case. The formula for the expansion is:

$$(x_1+x_2+\cdots+x_m)^n = \sum \frac{n!}{a_1!a_2!\cdots a_m!} x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m}$$

where $a_1, a_2, \cdots, a_m$ are any selection of m numbers from $0, 1, 2, \cdots, n$ such that $a_1+a_2+\cdots+a_m=n$, and $0!=1$.

MUL'TI-PLE, *adj., n.* In *arithmetic*, a number which is the product of a given

number and another factor; 12 is a *multiple* of 2, 3, 4, 6, and trivially of 1 and 12. In general, a product, no matter whether it be arithmetic or algebraic, is said to be a *multiple* of any of its factors.

common multiple. A quantity which is a multiple of each of two or more given quantities. The number 6 is a common multiple of 2 and 3; x^2-1 is a common multiple of $x-1$ and $x+1$.

least common multiple of two or more quantities. Denoted by l.c.m. or L.C.M. The least quantity that is exactly divisible by each of the given quantities; 12 is the l.c.m. of 2, 3, 4, and 6. The l.c.m. of a set of algebraic quantities is the product of all their distinct prime factors, each taken the greatest number of times it occurs in any one of the quantities; the l.c.m. of

$$x^2-1 \quad \text{and} \quad x^2-2x+1$$

is

$$(x-1)^2(x+1).$$

Tech. The l.c.m. of a set of quantities is a common multiple of the quantities which divides every common multiple of them. Also called *lowest common multiple.*

multiple correlation. See CORRELATION.

multiple integral. See INTEGRAL.

multiple point or tangent. See POINT—multiple (or k-tuple) point.

multiple root of an equation. In an algebraic equation, $f(x)=0$, a root a such that $(x-a)$ is contained in $f(x)$ two or more times. A root which is not a multiple root is said to be a **simple root.** If $(x-a)^n$ is the highest power of $(x-a)$ which is a factor of $f(x)$, a is called a *double* or *triple* root when n is 2 or 3, respectively, and an *n-tuple root* in general. A multiple root is a root of $f(x)=0$ and $f'(x)=0$, where $f'(x)$ is the derivative of $f(x)$. In general, a root is of order n, or an n-tuple root, if and only if it is a common root of $f(x)=0$, $f'(x)=0$, $f''(x)=0, \cdots$, to $f^{[n-1]}(x)=0$, but not of $f^{[n]}(x)=0$. Analogously, when $f(x)$ is not a polynomial, a root is said to be of order n if the nth derivative is the lowest order derivative of which it is not a root; the order of multiplicity of the root is then said to be n. *Syn.* Repeated root.

multiple-valued function. A function having more than one value for some values of the argument (independent variable); $y^2=x$ defines a *double-valued* function of x, since for each x, $x\neq 0$, there correspond two values of y, namely $\pm\sqrt{x}$; $y=\sin^{-1}x$ has infinitely many values of y for each value of x, for if $y_1=\sin^{-1}x$ then

$$n\pi+(-1)^n y_1=\sin^{-1}x,$$

where n is any integer. *Syn.* Many-valued function.

multiple-valued function of a complex variable. If the Riemann surface of a monogenic analytic function $f(z)$ of the complex variable z covers any part of the z-plane more than once, the function is called multiple-valued; *i.e.*, the function is multiple-valued if to any value z there corresponds more than one value of $f(z)$. A multiple-valued function may be considered as a single-valued function for z in a sub-domain lying on a single sheet of its Riemann surface of existence.

MUL'TI-PLI-CAND', *n.* The number to be multiplied by another number, called the **multiplier.**

MUL'TI-PLI-CA'TION, *n.* See various headings below and under PRODUCT.

abridged multiplication. The process of multiplying and dropping, after each multiplication by a digit of the multiplier, those digits which do not affect the degree of accuracy desired. If, in the product 235×7.1624, two-decimal place accuracy is desired (which requires the retention of only the third place throughout the multiplication), the abridged multiplication would be performed as follows:

$$\begin{aligned} &235\times 7.1624\\ &\quad =5+7.1624+30\times 7.1624+200\times 7.1624\\ &\quad =35.8120+214.872+1432.480\\ &\quad =1683.164=1683.16. \end{aligned}$$

multiplication of determinants. The product of determinants (or of a determinant and a scalar) is equal to the product of the values of the determinants (or of the value of the determinant and the scalar). The **multiplication of a determinant by a scalar** can be accomplished by multiplying each element of any one row (or any one column) by the scalar. The product of **two determinants of the same order** is another determinant of the same order, in which an element in the ith row and jth column is

the sum of the products of the elements in the ith row of the first determinant by the corresponding elements of the jth column of the second determinant (or vice versa). For 2nd order determinants,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} aA+bC & aB+bD \\ cA+dC & cB+dD \end{vmatrix}.$$

See PRODUCT—product of matrices.

multiplication of polynomials. See DISTRIBUTIVE—distributive law of arithmetic and algebra.

multiplication of the roots of an equation. The process of deriving an equation whose roots are each the same multiple of a corresponding root of the given equation. This is effected by the substitution (transformation) $x=x'/k$. The roots of the equation in x' are each k times one of the roots of the equation in x.

multiplication of vectors. (1) **Multiplication by a scalar.** The product of a scalar a and a vector **v** is the vector having the same direction as **v** and of length equal to the product of a and the length of **v**; *i.e.*, the vector obtained by multiplying each component of **v** by a. (2) **Scalar multiplication of two vectors.** The scalar product of two vectors is the *scalar* which is the product of the lengths of the vectors and the cosine of the angle between them. This is frequently called the **dot product**, denoted by $\mathbf{A}\cdot\mathbf{B}$, or the **inner product.** It is equal to the sum of the products of corresponding components of the vectors. See VECTOR—vector space. (3) **Vector multiplication of two vectors.** The vector product of two vectors **A** and **B** is the vector **C** whose length is the product of the lengths of **A** and **B** and the sine of the angle between them (the angle from the first to the second), and which is perpendicular to the plane of the given vectors and directed so that the three vectors in order **A**, **B**, **C** form a *right-handed trihedral* (see TRIHEDRAL—directed trihedral). The product is also called the **cross product.** If the product of **A** by **B** is **C**, one writes $A\times B=C$. (If the right forearm is placed along A with the fingers of the half closed right hand pointing in the direction A would be rotated to coincide with B, the erect thumb will point in the direction of C.) Scalar multiplication is commutative, but vector multiplication is not, for $B\times A=-A\times B$. The scalar product of $2i+3j+5k$ and $3i-4j+6k$ is $2\cdot3-3\cdot4+5\cdot6$ or 24, whereas the vector product is $38i+3j-17k$ if the vectors are multiplied in the given order, and the negative of this if they are multiplied in the reverse order.

MUL'TI-PLIC'I-TY, *n.* **multiplicity of a root of an equation.** See MULTIPLE—multiple root of an equation.

MUL'TI-PLI'ER, *n.* (1) The number which is to multiply another number, called the **multiplicand.** (2) In a computing machine, any arithmetic component that performs the operation of multiplication.

Lagrange's method of multipliers. See LAGRANGE.

MUL'TI-PLY, *adv.* **multiply connected region.** See CONNECTED.

MUL'TI-PLY, *v.* To perform the process of *multiplication.*

MUTATIS MUTANDIS. Necessary changes being made.

MU'TU-AL, *adj.* **mutual fund method.** (*Insurance.*) A method of computing the present value (single premium) for a *whole life annuity immediate.* See COMMUTATION—commutation tables.

mutual insurance company. A cooperative association of policyholders who divide the profits of the company among themselves. The policies, which usually provide for such division, are called *participating policies.*

mutual savings bank. A bank whose capital is that of the depositors who own the bank.

MU'TU-AL-LY, *adv.* **mutually equiangular polygons.** Polygons whose corresponding angles are equal.

mutually equilateral polygons. Polygons whose corresponding sides are equal.

mutually exclusive events. See EVENT.

N

NA'DIR, *n.* The point in the celestial sphere diametrically opposite to the zenith; the point where a plumb line at the

observer's position on the earth, extended downward, would pierce the celestial sphere.

NAPIER. **Napier's analogies.** Formulas for use in solving a spherical triangle. They are as follows (a, b, c representing the sides of a spherical triangle and A, B, C the angles opposite a, b, c, respectively):

$$\frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}=\frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2}c}$$

$$\frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}=\frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}c}$$

$$\frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)}=\frac{\tan \frac{1}{2}(A-B)}{\cot \frac{1}{2}C}$$

$$\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}=\frac{\tan \frac{1}{2}(A+B)}{\cot \frac{1}{2}C}.$$

Napier's rules of circular parts. Two ingenious rules by which one can write out the ten formulas needed in the solution of right spherical triangles. Omitting the right angle, one can think of the complements of the other two angles, the complement of the hypotenuse, and the other two sides as arranged on a circle in the same order as on the triangle. Any one of these is a middle point in the sense that there are two on either side of it. The two points nearest to a given point are called *adjacent parts*, and the two farthest are called *opposite parts*. Napier's rules are then stated as follows: I. The sine of any part is equal to the product of the tangents of the adjacent parts. II. The sine of any part is equal to the product of the cosines of the opposite parts.

NA-PIER'I-AN, or NA-PE'RI-AN, *adj.* **Napierian logarithms.** A name commonly used for *natural logarithms*, but Napier did not originate them. See LOGARITHM.

NAPPE, *n.* **nappe of a cone.** One of the two parts of a conical surface into which the surface is divided by the vertex. Better called *nappe* of a *conical surface*.

NAT'U-RAL, *adj.* **natural equations of a space curve.** See INTRINSIC—intrinsic equations of a space curve.

natural logarithms. Logarithms using the base e (2.71828183+). See e, and LOGARITHM.

natural numbers. The numbers 1, 2, 3, 4, etc. The same as *positive integers.*

NAUGHT, *n.* Same as ZERO.

NAU'TI-CAL, *adj.* **nautical mile.** 6080 feet as used by the British admiralty; 6080.27 feet as used by the U. S. Hydrographic office.

NEC'ES-SARY, *adj.* **necessary condition.** See CONDITION.

necessary condition for convergence of an infinite series: That the terms approach zero as one goes farther out in the series; that the nth term approaches zero as n becomes infinite. This is not a sufficient condition for convergence; *e.g.*, the series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$$

is divergent, although $1/n$ approaches zero as n becomes infinite. See CAUCHY—Cauchy's condition for convergence of a series.

NE-GA'TION, *n.* **negation of a proposition.** The proposition formed from the given proposition by prefixing "It is false that," or simply "not." *E.g.*, "Today is Wednesday" has the negation "It is false that today is Wednesday." "All cows are brown" has the negation "It is false that all cows are brown," which might be written as "There is at least one cow which is not brown." The negation of a proposition p is frequently written as $\sim p$ and read "*not p*." The negation of a proposition is true if and only if the proposition is false. *Syn.* Denial.

NEG'A-TIVE, *adj.* **negative angle, correlation, exponent, number.** See ANGLE, CORRELATION, EXPONENT, and POSITIVE—positive number.

negative direction. The direction opposite the direction that has been chosen as positive.

negative sign. The mark, $-$, denoting the negative of a number. See POSITIVE—positive number.

NE-GO'TI-A-BLE, *adj.* **negotiable paper.** An evidence of debt which may be transferred by indorsement or delivery, so that the transferee or holder may sue on it in his own name with like effect as if it had been made to him originally; *e.g.*, bills of exchange, promissory notes, drafts, and checks payable to the order of a payee or to bearer.

NEIGH'BOR-HOOD, *n.* **neighborhood of a point.** The interior of some bounded geometric figure (such as a square or circle in the plane) which contains the point. A neighborhood of a point on a line, plane, or surface is usually taken as the set of points within a stated distance of the point (*e.g.*, an *open interval* on the line or the interior of a circle in the plane, with the point as center). One speaks of a property as holding **in the neighborhood of** a point if there exists a neighborhood of the point in which the property holds, or of a numerical quantity (*e.g.*, curvature) depending on the nature of a curve or surface **in the neighborhood of a point** if the value of the quantity can be determined from knowledge of the portion of the curve or surface in an arbitrarily small neighborhood of the point. See SMALL—in the small, and TOPOLOGICAL—topological space.

NERVE, *n.* **nerve of a family of sets.** Let $S_0, S_1, \cdots, S_n$ be a finite family of sets. To each set S_k assign a symbol p_k. A *nerve* of this system of sets is the *abstract simplicial complex* whose vertices are the symbols $p_0, p_1, \cdots, p_n$ and whose *abstract simplexes* are all the subsets $p_{i_0}, p_{i_1}, \cdots, p_{i_r}$ whose corresponding sets have a nonempty intersection. *E.g.*, if S_0, S_1, S_2, S_3 are the four faces of a tetrahedron, the nerve is the abstract simplicial complex with vertices p_0, p_1, p_2, p_3 whose abstract simplexes are all sets of three or less vertices—it has a geometric realization as a tetrahedron.

NEST, *n.* See NESTED—nested sets.

NEST'ED, *adj.* **nested intervals.** A sequence of intervals such that each is contained in the preceding. It is sometimes required that the lengths of the intervals approach zero as one goes out in the sequence. The **nested interval theorem** states that for any sequence of nested intervals, each of which is *bounded* and *closed*, there is at least one point which belongs to each of the intervals (if the lengths of the intervals approach zero, there is exactly one such point). This theorem is true for intervals in n-dimensional Euclidean space, as well as for intervals on the line. See INTERVAL and MONOTONE.

nested sets. A collection of sets is nested if, for any two members A and B of the collection, either A is contained in B or B is contained in A. A nested collection of sets is also called a **nest, tower,** or **chain.**

NET, *adj.* Clear of all deductions (such as charges, cost, loss).

net premium. See PREMIUM.

net proceeds. The amount remaining of the sum received from the sale of goods after all expenses except the original cost have been deducted.

net profit. See PROFIT.

NET, *n.* See MOORE-SMITH CONVERGENCE.

NEUMANN. Neumann function. A function of type

$$N_n(z) = \frac{1}{\sin n\pi}\,[\cos n\pi\, J_n(z) - J_{-n}(z)],$$

where J_n is a Bessel function. This function is a solution of Bessel's differential equation (if n is not an integer) and is also called a *Bessel function of the second kind.* See HANKEL—Hankel function.

Neumann's function. (*Potential Theory.*) For a region R with boundary surface S, and for a point Q interior to R, the Neumann's function $N(P, Q)$ is a function of the form

$$N(P, Q) = 1/(4\pi r) + V(P),$$

where r is the distance PQ, $V(P)$ is harmonic, $\partial N/\partial n$ is constant on S, and $\iint_S N\, d\sigma_P = 0$. The solution $U(Q)$ of the Neumann problem can be represented in the form

$$U(Q) = \iint_S f(P)\, N(P, Q)\, d\sigma_P.$$

See GREEN—Green's function, BOUNDARY—second boundary value problem of potential theory (the *Neumann problem*).

NEWTON, *n.* A unit of force. A force of one newton is a force which will give an acceleration of one meter per second per second to a mass of one kilogram.

NEWTON. Gregory-Newton interpolation formula. See GREGORY-NEWTON FORMULA.

Newton-Cotes integration formulas. The approximation formulas

$$\int_{x_0}^{x_0+h} y\,dx = \frac{h}{2}(y_0+y_1) - \frac{h^3}{12}y''(\xi),$$

$$\int_{x_0}^{x_0+2h} y\,dx = \frac{h}{3}(y_0+4y_1+y_2) - \frac{h^5}{90}y^{iv}(\xi),$$

$$\int_{x_0}^{x_0+3h} y\,dx = \frac{3h}{8}(y_0+3y_1+3y_2+y_3) - \frac{3h^5}{90}y^{iv}(\xi),$$

etc., where y_k is the value of y at $x=x_0+kh$ and, in each formula, ξ is an intermediate value of x. The correction term involves the sixth derivative in the next two formulas after those shown, etc. Since the foregoing formulas involve the values of y at the limits of integration, they are said to be of **closed type.** The Newton-Cotes formulas of **open type** are

$$\int_{x_0}^{x_0+3h} y\,dx = \frac{3h}{2}(y_1+y_2) + \frac{h^3}{4}y''(\xi),$$

etc. The open-type formulas are used particularly in the numerical solution of differential equations.

Newtonian potential. See POTENTIAL—gravitational potential.

Newton's identities. Certain relations between sums of powers of all the roots of a polynomial equation and its coefficients. For the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0,$$

with the roots $r_1, r_2, \cdots, r_n$, Newton's identities are:

$$s_k + a_1s_{k-1} + a_2s_{k-2} + \cdots + a_{k-1}s_1 + ka_k = 0,$$

for $k \leqq n-1$, and

$$s_k + a_1s_{k-1} + a_2s_{k-2} + \cdots + a_ns_{k-n} = 0,$$

for $k \geqq n$, where

$$s_k = r_1{}^k + r_2{}^k + \cdots + r_n{}^k.$$

Newton's inequality. The logarithmic-convexity inequality

$$p_{r-1}p_{r+1} \leqq p_r{}^2, \quad 1 \leqq r < n,$$

where $p_r = b_r \Big/ \binom{n}{r}$ is the average value of the $\binom{n}{r}$ terms comprising the rth *elementary symmetric function* b_r of a set of numbers $a_1, a_2, \cdots, a_n$.

Newton's laws of motion. *First Law:* Every particle continues in its state of rest or of uniform motion in a straight line except in so far as it is compelled by forces to change that state. *Second Law:* The time rate of change of momentum is proportional to the motive force and takes place in the direction of the straight line in which the force acts. *Third Law:* The interaction between two particles is represented by two forces equal in magnitude but oppositely directed along the line joining the particles.

Newton's method of approximation. A method of step-by-step approximation to the roots of an equation in one unknown. It is based on the fact that the tangent of an ordinary curve very nearly coincides with a small arc of the curve, *i.e.*, that the subtangent is approximately the same as the distance from the curve's x-intercept to the foot of the ordinate of the point whose abscissa is the last approximation to the root. Suppose the equation is $f(x)=0$, and a_1 is an approximation to one of the roots. The next approximation, a_2, is the abscissa of the point of intersection of the x-axis and the tangent to the curve $y=f(x)$ at the point whose abscissa is a_1, *i.e.*, $a_2=a_1-f(a_1)/f'(a_1)$, where $f'(a_1)$ is the derivative of $f(x)$ evaluated for $x=a_1$. This is equivalent to using the first two terms (dropping all higher-degree terms) in Taylor's expansion of $f(x)$ about the point whose abscissa is a_1 and assuming that $f(a_2)=0$.

Newton's three-eighths rule. An alternative to Simpson's rule for approximating the area bounded by a curve $y=f(x)$, the x-axis, and the ordinates $x=a$ and $x=b$. The interval (a, b) is divided into $3n$ equal parts, and the formula is

$$A = \frac{(b-a)}{8n}[y_a + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots + 3y_{3n-1} + y_b].$$

The name of the rule comes from the fact that the coefficient $(b-a)/(8n)$ is equal to $(3/8)h$, where $h=(b-a)/(3n)$ is the length of each of the equal subintervals. For an analogous reason, Simpson's rule is some-

times called Simpson's one-third rule. The error in Simpson's rule is $-nk^5/90$ times the fourth derivative at an intermediate point, where $k=(b-a)/(2n)$, while that in Newton's rule is $-3nh^5/80$ times the fourth derivative at an intermediate point. See SIMPSON'S RULE, TRAPEZOID—trapezoid rule, and WEDDLE'S RULE.

trident of Newton. See TRIDENT.

NICOMEDES. conchoid of Nicomedes. Same as CONCHOID.

NIL'PO-TENT, *adj.* Vanishing upon being raised to some power. The matrix $A=$

$$\begin{pmatrix} 2 & 0 & -4 \\ 3 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

is nilpotent, since $A^3=0$.

NIL-SEGMENT, *n.* See SEGMENT—segment of a line.

NIM, *n.* **game of nim.** A game in which two players draw articles from several piles, each player in turn taking as many as he pleases from any one pile, the player who draws the last article winning. To win, a player must, with each draw, make the sum of the corresponding digits of the numbers of articles, written in the binary scale, either 2 or 0. If the numbers of articles are 17, 6, 5, they are written 10001, 110, 101 in the binary scale, which we write with digits having the same place value in the same column as follows:

$$\begin{array}{r} 10001 \\ 110 \\ 101 \end{array}$$

To win, the first player, A, must take fourteen articles from the large pile leaving the columns

$$\begin{array}{r} 11 \\ 110 \\ 101 \end{array}$$

The other player, B, has no choice but to make the sum, in at least one column, odd. A then makes it even and they finally come to the situation where it is B's draw and the numbers are 1 and 1, so that A wins. The above strategy assures a win for the first player who can make the sum of the digits even in each column. There are other variations of *nim*. *E.g.*, with one pile, each player can be required to pick up at least one and less than k articles, the *loser* being the person picking up the last article. The player first able to leave $nk+1$ articles (for some n) can win by leaving $n(k-1)+1$ at the next play, etc.

NINE, *adj., n.* **casting out nines.** See CASTING.

nine-point circle. See CIRCLE—nine-point circle.

NOD'AL, *adj.* **nodal line.** A line in a configuration which remains fixed while the configuration is being rotated or deformed in a certain manner; a line in an elastic plate which remains fixed while the plate vibrates. See EULER—Euler's angles.

NODE, *n.* **node of a curve.** A point at which two parts of a curve cross and have different tangents (sometimes a double point which is an *acnode* is also called a node). A set of nodes of curves which belong to a given family of curves is said to be a **node-locus.** See CRUNODE, and DISCRIMINANT—discriminant of a differential equation.

NOM'I-NAL, *adj.* **nominal rate of interest.** See INTEREST.

NOM'O-GRAM, *n.* A graph consisting of three lines or possibly curves (usually parallel) graduated for different variables in such a way that a straight edge cutting the three lines gives the related values of the three variables. *E.g.*, when considering automobile tires, one line might be graduated with the price, another with the cost per mile, and the other the mileage life of the tire, in such a way that a straight edge through a certain price point and mileage life point would cross the other line at the cost per mile. *Syn.* Alignment chart.

NON, *prefix.* Negating prefix.

nondense set. See DENSE—dense set.

non-Euclidean geometry. A geometry whose marked characteristic is its rejection of Euclid's parallel postulate; any geometry not based on the postulates of Euclid.

nonlinear. Of higher degree than the first.

nonremovable discontinuity. See DISCONTINUITY.

nonsingular linear transformation. A linear transformation whose determinant is not zero. See LINEAR—linear transformation.

nonterminating. See TERMINATING.

NON'A-GON, *n.* A polygon having nine sides. It is a **regular nonagon** if its sides are all equal and its interior angles are all equal.

NONRESIDUE, *n.* See RESIDUE.

NORM, *n.* (1) Mean; average. (2) Customary degree or condition. (3) Established pattern or form. (4) See VECTOR—vector space.

norm of a matrix. The square root of the sum of the squares of the absolute values of the elements; the square root of the *trace* of A^*A, where A is the given matrix and A^* is the *Hermitian conjugate* of A (or the *transpose* if A is real). The norm of a matrix is unchanged by multiplication on either side by a *unitary* matrix, from which it follows that the norm of a *normal matrix* (or of a Hermitian symmetric matrix) is equal to the sum of the squares of the *eigenvalues* of the matrix.

norm of a functional, transformation, or vector. See CONJUGATE—conjugate space, LINEAR—linear transformation, VECTOR—vector space.

NOR'MAL, *adj., n.* (1) Perpendicular. (2) According to rule or pattern; usual or natural. (3) Possessing a property which is commonly designated by the word *normal* (*e.g.*, see below, normal number, normal transformation).

binormal. See OSCULATING—osculating plane.

mean normal curvature of a surface at a point. See CURVATURE—mean curvature of a surface.

normal acceleration. See ACCELERATION.

normal curvature of a surface. See CURVATURE.

normal derivative. The *directional derivative* of a function in the direction of the normal at the point where the derivative is taken; the rate of change of a function in the direction of the normal to a curve or surface. See DIRECTIONAL.

normal divisor of a group. See INVARIANT—invariant subgroup.

normal equations. A set of equations derived by the method of least squares to obtain estimates of the parameters a and b in the equation $y=a+bx$, where y is a random variable and x is a fixed variate. By minimizing $\sum_{i=1}^{n}[y_i-(a+bx_i)]^2$ with respect to a and to b, where n is the number of paired observations, the following normal equations are derived:

$$\Sigma y_i - an - b\Sigma x_i = 0,$$
$$\Sigma y_i x_i - a\Sigma x_i - b\Sigma x_i^2 = 0.$$

The process may be applied to functions with k parameters so that k normal equations are obtained which, when solved, yield the least-squares estimates of the k parameters. See METHOD—method of least squares.

normal family of analytic functions. A family of functions $\{f(z)\}$ of the complex variable z, all analytic in a common domain D, and such that every infinite sequence of functions of the family contains a subsequence which converges uniformly to an analytic function (which might be identically infinite) in every closed region in D.

normal form of an equation. See LINE—equation of a line, PLANE—equation of a plane.

normal frequency curve. See FREQUENCY—normal frequency curve.

normal (or normalized) functions. See ORTHOGONAL—orthogonal functions.

normal lines and planes. See PERPENDICULAR—perpendicular lines and planes, and below, normal to a curve or surface.

normal matrix. See MATRIX.

normal number. A number whose decimal expansion is such that all digits occur with equal frequency, and all blocks of the same length occur equally often. *Tech.* Let a real number X be written as an infinite decimal with base r (r not necessarily 10). Let $N(d, n)$ be the number of occurrences of the digit d among the first n digits of X. Then the number X is **simply normal** with respect to the base r if

$$\lim_{n\to\infty}\frac{N(d, n)}{n}=\frac{1}{r}$$

for each of the values $0, 1, \cdots, r-1$ of d.

Now let $N(D_k, n)$ be the number of occurrences of the block D_k of k successive integers in the first n digits of X. Then X is normal with respect to the base r if

$$\lim_{n\to\infty} \frac{N(D_k, n)}{n} = \frac{1}{r^k}$$

for any positive integer k and block D_k (actually, it is sufficient to use only an infinite number of values of k). A normal number is irrational (see IRRATIONAL—irrational number), but a simply normal number may be rational (*e.g.*, the *repeating* decimal .01234567890123456789 · · ·). It is not known whether $\sqrt{2}, \pi$, or e is normal to some base, but the set of all real numbers which *do not* have the property of being normal with respect to every base is of *measure zero*. An example of a normal number with respect to base 10 is given by writing the integers in succession:

$$0.123456789101112131415 \cdots.$$

normal order. See ORDER.

normal section of a surface. A plane section made by a plane containing a normal (perpendicular) to the surface. *Syn.* Perpendicular section of a surface.

normal space. See REGULAR—regular space.

normal stress. See STRESS.

normal to a curve or surface. A normal line to a curve at a point is a line perpendicular to the tangent line at the point (see TANGENT—tangent line). For a plane curve, its equation is

$$(y - y_1) = [-1/f'(x_1)](x - x_1),$$

where $f'(x_1)$ is the slope of the curve at the point (x_1, y_1) in which the normal cuts the curve. See DERIVATIVE, and PERPENDICULAR—perpendicular lines and planes. The normal plane to a space curve at a point P is the plane perpendicular to the tangent line at P; the normal lines at P are the lines through P which lie in the normal plane. The binormal to a space curve at a point P is the line passing through P and normal to the osculating plane of the curve at P. The positive direction of the binormal is chosen so that its direction cosines are $\rho(y'z'' - z'y''), \rho(z'x'' - x'z''), \rho(x'y'' - y'x'')$, the primes denoting differentiation with respect to the arc length. The principal normal to a space curve at a point P is the line perpendicular to the space curve at the point P and lying in the *osculating plane* at P. The positive direction of the principal normal at P is chosen so that the tangent, principal normal, and binormal at P have the same mutual orientation as the positive x, y, and z axes. The normal line to a surface at a point is the line perpendicular to the tangent plane at the point (see TANGENT—tangent plane). *Syn.* Perpendicular to a curve or surface.

normal transformation. A bounded linear transformation T is normal if it commutes with its *adjoint* ($TT^* = T^*T$). A normal transformation must commute with its adjoint; but if it is not bounded, other conditions are usually imposed (*e.g.*, that it be *closed*). A bounded linear transformation T is normal if and only if $T = A + iB$, where A and B are symmetric transformations for which $AB = BA$. See MATRIX—normal matrix, and SPECTRAL—spectral theorem.

polar normal. See POLAR—polar tangent.

principal normal sections of a surface at a point. Normal sections in the *principal directions* of the surface at the point.

NOR'MAL-IZED, *adj.* normalized variate. (*Statistics.*) See VARIATE.

NORMED, *adj.* normed linear (vector) space. See VECTOR—vector space.

NORTH, *adj.* north declination. See DECLINATION—declination of a celestial point.

NO-TA'TION, *n.* Symbols denoting quantities, operations, etc.

continuation notation. See CONTINUATION.

factorial notation. The notation used in writing factorials. See FACTORIAL.

functional notation. See FUNCTIONAL.

Plücker's abridged notation. See ABRIDGED.

NOTE, *n.* A signed promise to pay a specified sum of money at a given time, or in partial payments at specified times.

bank note. A note given by a bank and used for currency. Usually has the shape and general appearance of government paper money.

demand note. A note that must be paid when the payee demands payment.

interest-bearing note. A note containing the words "with interest at — per cent," meaning that interest must be paid when the note is paid or at other times agreed upon by the payee and payer.

noninterest-bearing note. (1) A note on which no interest is paid. (2) A note on which the interest has been paid in advance.

note receivable. *A promissory note*, payment of which is to be made *to* the person under consideration, as contrasted to *note payable* which is to be paid *by* this person. Terms used in bookkeeping and accounting.

promissory note. A note, usually given by an individual, promising to pay a given sum of money at a specified time (or on demand).

NOUGHT, *n.* Same as ZERO; more commonly spelled naught.

NU'CLE-US, *n.* **nucleus of an integral equation.** Same as the KERNEL.

NULL, *adj.* (1) Nonexistent; of no value or significance. (2) Quantitatively zero (as *null circle*—a circle of zero area). (3) Not any, empty (as *null set*).

null hypothesis. See HYPOTHESIS—null hypothesis.

null matrix. A matrix whose elements are all zero.

null sequence. A sequence whose limit is zero.

null set. The set which is empty—has no members.

NUM'BER, *n.* (1) The positive integers (see INTEGER). *Number* is still used quite commonly to denote integers, in such phrases as *The Theory of Numbers.* (2) The set of all **complex numbers** (see COMPLEX—complex numbers). The **real numbers** constitute part of the set of complex numbers (the complex numbers which are not real numbers are the numbers $a+bi$ for which a and b are real numbers and $b\neq 0$). The real numbers are of two types, **irrational numbers** and **rational numbers** (see DEDEKIND CUT, and RATIONAL). The irrational numbers are of two types, **algebraic irrational numbers** and **transcendental numbers** (see IRRATIONAL—irrational numbers). The rational numbers are of two types, the **integers** and the **fractions** of the form m/n, where m and n are integers and m is not divisible by n (see INTEGER). (3) See CARDINAL—cardinal number, ORDINAL—ordinal number.

absolute number. See ABSOLUTE.

abstract number. See ABSTRACT.

abundant number. See below, perfect number.

amicable numbers. See AMICABLE.

arithmetic numbers. Real positive numbers; positive integers, fractions, or radicals such as occur in arithmetic; numbers themselves rather than letters denoting numbers.

Bernoulli's numbers. See BERNOULLI.

cardinal number. See CARDINAL.

code number. (*Statistics.*) The number assigned to class intervals for purposes of computational simplifications; frequently successive even or odd integers.

concrete number. See CONCRETE.

defective number. See below, perfect number.

denominate number. A number whose unit represents a unit of measure. The **denomination** of a denominate number is the kind of unit to which the number refers, as pounds, feet, gallons, tenths, hundreds, thousands, etc.

extended real number system. See EXTENDED.

Fermat numbers. See FERMAT.

imaginary number. See COMPLEX—complex number.

Liouville number. See LIOUVILLE.

mixed number. See MIXED.

negative number. See POSITIVE—positive number.

normal number. See NORMAL.

number class modulo *n*. The totality of integers each of which is congruent to a given integer modulo n. Thus if $x\equiv 2 \pmod 3$, then x belongs to the class of integers each of which is congruent to 2 modulo 3. This is the totality of all numbers of type $3k+2$, where k represents an arbitrary integer.

number field. See FIELD.

number scale. See SCALE—number scale.

number sieve, photoelectric number sieve. A mechanical device for factoring large numbers. See ERATOSTHENES.

number theory. See THEORY—number theory.

ordinal number. See ORDINAL.

perfect number. An integer which is equal to the sum of all of its factors except itself; 28 is a perfect number, since $28 = 1+2+4+7+14$. If the sum of the factors of a number (except itself) is less than the number, the number is said to be **defective** (or **deficient**); if it is greater than the number, the number is said to be **abundant.**

positive number. See POSITIVE.

Pythagorean numbers. See PYTHAGOREAN.

random numbers. (*Statistics.*) See RANDOM—table of random numbers.

square numbers. The numbers $1, 4, 9, 16, 25, \cdots, n^2, \cdots$, which are squares of integers.

transfinite number. See TRANSFINITE.

triangular numbers. The numbers $1, 3, 6, 10, \cdots$. They are called triangular numbers because they are the number of dots employed in making successive triangular arrays of dots. The process is started with one dot, successive rows of dots being placed beneath the first dot, each row having one more dot than the preceding one. The number of dots in the nth array is $(1+2+3+\cdots+n) = \frac{1}{2}n(n+1)$. See ARITHMETIC—arithmetic series.

NU'MER-ALS, *n.* Symbols used to denote numbers, as Arabic numerals or Roman numerals.

Arabic numerals. The symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Roman numerals. See ROMAN—Roman numerals.

NU'MER-A'TION, *n.* The process of writing or stating numbers in order of their size; the process of numbering.

NU'MER-A'TOR, *n.* The expression above the line in a fraction, the expression which is to be divided by the other expression, called the *denominator.*

NU-MER'I-CAL, *adj.* Consisting of numbers, rather than letters; of the nature of numbers.

numerical determinant. A determinant whose elements are all numerical, rather than literal. See DETERMINANT.

numerical equation. An equation in which the coefficients and constants are numbers rather than letters. The equation $2x+3=5$ is a numerical equation, whereas $ax+b=c$ is a *literal* equation.

numerical value. (1) Same as ABSOLUTE VALUE. (2) Value given as a number, rather than by letters.

O

OB'LATE, *adj.* **oblate ellipsoid of revolution (oblate spheroid).** See ELLIPSOID.

OB-LIQUE', *adj.* Neither perpendicular nor horizontal; slanting; changed in direction (as: "having a direction *oblique* to that of a former motion"). An **oblique angle** is any angle not a multiple of 90° (*e.g.*, any angle between 0° and 180° which is not 0°, 90°, or 180°); an **oblique triangle** is any triangle (plane or spherical) which does not contain a right angle. **Oblique lines** are lines which are neither parallel nor perpendicular; **oblique coordinates** are coordinates determined from oblique axes (see CARTESIAN—Cartesian coordinates). A line is **oblique to a plane** if it is neither parallel nor perpendicular to the plane (an **oblique circular cone** is a circular cone whose axis is oblique to its base).

OB'SER-VA'TION, *n.* **observation equation.** (*Statistics.*) An equation relating to the coefficients of an equation which is assumed to represent certain data. It is obtained by substituting values of the variables (taken from the data) in the assumed literal equation.

OB-TUSE', *adj.* **obtuse angle.** An angle greater than a right angle and smaller than a straight angle.

OC'TA-GON, *n.* A polygon having eight sides.

regular octagon. An octagon whose angles and sides are all equal.

OC'TA-HE'DRON, *n.* A polyhedron having eight faces. A **regular octahedron** is an octahedron whose faces are all congruent regular equilateral triangles. See POLYHEDRON—regular polyhedron.

OC'TANT, *n.* See CARTESIAN—Cartesian coordinates.

OC-TIL'LION, *n.* (1) In the U.S. and France, the number represented by one followed by 27 zeros. (2) In England, the number represented by one followed by 48 zeros.

ODD, *adj.* **odd function.** See FUNCTION—odd function.

odd number. An integer that is not evenly divisible by 2; any number of the form $2n+1$, where n is an integer; 1, 3, 5, 7 are odd numbers.

ODDS, *n.* (1) *In betting,* the ratio of the wager of one party to that of the other, as to lay or give *odds,* say 2 to 1. (2) The probability or degree of probability in favor of some event on which bets are laid. (3) An equalizing allowance given to a weaker side or player by a stronger, as a piece at chess or points at tennis.

OFF'SET, *n.* (*Surveying.*) A shift from a *given direction* perpendicular to that direction in order to pass an obstruction and yet obtain the total distance through the obstruction in the given direction. The distance across a pond in a given direction can be obtained by adding all the line segments in this direction which appear in a set of offsets (steps) going around the pond.

O-GIVE', *n,* Same as CUMULATIVE FREQUENCY CURVE.

OHM, *n.* (*Electricity.*) A unit of electrical resistance. (1) The **absolute ohm** is defined as the resistance of a conductor which carries a steady current of one absolute ampere when a steady potential difference of one absolute volt is impressed across its terminals. This is equivalent to the statement that the conductor dissipates heat at the rate of one watt when it carries a steady current of one absolute ampere. The absolute ohm has been the legal standard of resistance since 1950. (2) The **international ohm,** the legal standard before 1950, is the resistance offered to a steady electric current by a column of mercury of 14.4521 gm mass, constant cross-sectional area, and a length of 106.300 cm, at 0°C.

1 int. ohm = 1.000495 abs. ohm.

OHM'S LAW. (*Electricity.*) Current is proportional to electromotive force divided by resistance. This law applies to a metallic circuit if the electromotive force and the current are constant. See HELMHOLTZ.

ONE, *adj.* The cardinal number denoting a single unit.

one-parameter family of curves or surfaces. See FAMILY.

one-to-one correspondence. See CORRESPONDENCE.

one-valued, *adj.* Same as SINGLE-VALUED.

ON'TO, *prep.* A mapping or transformation of a set X which transforms points of X into points of Y is said to be a mapping of X *into* Y; it is a mapping of X *onto* Y if each point of Y is the image of at least one point of X. *E.g.,* $y=3x+2$ is a mapping of the real numbers *onto* the real numbers; $y=x^2$ is a mapping of the real numbers *into* the real numbers, or *onto* the nonnegative real numbers.

O'PEN, *adj.* **open interval.** See INTERVAL.

open mapping. A mapping (correspondence, transformation, or function) which associates with each point of a space D a unique point of a space Y, is open if the image of each open set of D is open in R; it is **closed** if the image of each closed set is closed. An open mapping is not necessarily closed and a closed mapping is not necessarily open. Also, a continuous mapping need not be either open or closed. See CONTINUOUS—continuous correspondence of points. *Syn.* interior mapping.

open set of points. A set U such that each point of U has a neighborhood all the points of which are points of U; the complement of a closed set. The interior of a circle and the set of all points on one side of a straight line in a plane are open sets. See TOPOLOGICAL—topological space.

OP'ER-A'TION, *n.* The process of carrying out the rules of procedure like addition, subtraction, differentiation, taking logarithms, making substitutions or transformations.

four fundamental operations of arithmetic. See FUNDAMENTAL.

OP'ER-A'TOR, *n.* **differential operator.** See DIFFERENTIAL—differential operator.

OP'PO-SITE, *adj.* **opposite sides in a rectangle, quadrilateral, or any polygon having an even number of sides.** Two sides having the same number of sides between them whichever way one travels around the polygon from one of the sides to the other.

opposite vertices (angles) of a polygon. Two vertices (angles) having an equal number of vertices (or sides) between them whichever way one counts around the polygon.

OP'TI-CAL, *adj.* **optical property of conics.** See ELLIPSE—focal property of ellipse, HYPERBOLA—focal property of hyperbola, and PARABOLA—focal property of parabola.

OP'TI-MAL, *adj.* **optimal strategy.** See STRATEGY.

OP'TI-MAL'I-TY, *n.* **principle of optimality.** In dynamic programming, the principle that, regardless of the initial state of the process under consideration and regardless of the initial decision, the remaining decisions must form an optimal policy relative to the state resulting from the first decision. See PROGRAMMING—dynamic programming.

OP'TION, *n.* **option term insurance.** See INSURANCE—life insurance.

OP'TION-AL, *adj.* (*Finance.*) Same as CALLABLE. See BOND.

ORACLE. An automatic digital computing machine at the Oak Ridge National Laboratory. ORACLE is an acronym for *Oak Ridge Automatic Computer and Logical Engine.*

OR'DER, *adj., n.* **derivatives of higher order.** See DERIVATIVE.

differences of first, second, third order. See DIFFERENCE.

normal order. An established arrangement of numbers, letters, or objects which is called *normal* relative to all other arrangements. If a, b, c is defined as the normal order for these letters, then b, a, c is an *inversion.* See INVERSION—inversion in a sequence of objects.

of the order of. See INFINITE—order of infinities.

order of algebraic curve or surface. The degree of its equation; the greatest number of points (real or imaginary) in which any straight line can cut it.

order of an a-point of an analytic function. See ANALYTIC—a-point of an analytic function.

order of a branch-point of a Riemann surface. If $k+1$ sheets of a Riemann surface hang together at a point, $k \geqq 1$, then the point is said to be a branch-point of order k.

order of contact of two curves. A measure of how close the curves lie together in the neighborhood of a point at which they have a common tangent. *Tech.* The order of contact of two curves $y=f(x)$ and $y=g(x)$ is one less than the order of the infinitesimal difference of the distances from the two curves to their common tangent, measured along the same perpendicular, relative to the distance from the foot of this perpendicular to their point of contact with the tangent; the order of contact of two curves is n when the nth-order differential coefficients, from their equations, and all lower orders, are equal at the point of contact, but the $(n+1)$st-order differential coefficients are unequal. See INFINITESIMAL—order of an infinitesimal.

order of a differential equation. See DIFFERENTIAL—differential equation (ordinary).

order of an elliptic function. The sum of the orders of the poles of the function in a primitive period parallelogram. There are no elliptic functions of order 0 or 1. An elliptic function of order k takes on every complex value exactly k times in a primitive period parallelogram.

order of the fundamental operations of arithmetic. When several of the fundamental operations occur in succession, multiplications and divisions are performed before additions and subtractions, and in the order in which they occur; *e.g.*,

$$3+6-2\times 4-7=3+3\times 4-7$$
$$=3+12-7=8.$$

order of a group. See GROUP, and PERIOD—period of an element of a group.

order of an infinitesimal. See INFINITESIMAL.

order of infinity. See INFINITY—order of infinities.

order of a pole of an analytic function. See ISOLATED—isolated singular point of an analytic function.

order of a radical. Same as the INDEX OF THE RADICAL.

order relation. See various headings under ORDERED.

order relation between real numbers. The relation which makes it possible to tell whether one of two distinct real numbers precedes or follows the other one.

order of units. The place of a digit in a number. Units in unit's place are units of the *first order*, in ten's place, of the *second order*, etc.

order of a zero point of an analytic function. See ZERO—zero point of an analytic function of a complex variable.

OR'DERED, *adj.* **ordered field.** See FIELD.

partially ordered set. A set which has a relation $x<y$ defined for some elements x and y and satisfying the conditions: (1) If $x<y$, then $y<x$ is false and x and y are not the same element. (2) If $x<y$ and $y<z$, then $x<z$. Sets of points are partially ordered if one defines $U<V$ for sets U and V to mean that U is a proper subset of V. Positive integers are partially ordered if one defines $a<b$ to mean a is a factor of b and $a\neq b$.

simply ordered set. A set S which has a relation $(<)$ defined and satisfying the conditions: (1) For any two elements x and y exactly one of the relations $x<y$, $x=y$, $y<x$ is valid. (2) If $x<y$ and $y<z$, then $x<z$. The set of positive integers, ordered according to their size, is a simply ordered set. *Syn.* Linearly ordered set, serially ordered set, ordered set, chain.

well-ordered set. A *simply ordered* set such that every subset of S has a first element (one preceding all others of the subset). Zermelo proved that every set can be well ordered if it is assumed that in each subset T one element of T can be chosen (or designated) as a "special" element. This assumption is called the **axiom of choice** or **Zermelo's axiom.** See CHOICE, and ZORN—Zorn's lemma. *Syn.* Normally ordered set.

OR'DIN-AL, *adj.* **ordinal numbers.** Numbers that denote order of the members of a set as well as the cardinal number property of the set. Two simply ordered sets are said to be **similar** if they can be put into one-to-one correspondence which preserves the ordering. All simply ordered sets which are similar to one another are said to have the same **ordinal type** and to have the same **ordinal number.** The ordinal number of the integers $1, 2, \cdots n$ is denoted by n, of all positive integers by ω, of all negative integers by ω^*, of all integers by π, of the rational numbers with the usual ordering by η, and of the real numbers with the usual ordering by λ. If α and β are the ordinal numbers of simply ordered sets P and Q, then $\alpha+\beta$ is defined as the ordinal number of the set (P, Q) containing all elements of P and Q with the given ordering of P and Q and with the definition that each element of P precedes each element of Q. We have $\omega\neq\omega^*$, $\omega^*+\omega=\pi\neq\omega+\omega^*$. Two ordered sets having the same ordinal number have the same *cardinal number*, but two ordered sets having the same cardinal number do not necessarily have the same ordinal number unless they are finite (*e.g.*, $\omega\neq\omega^*$). Many authors allow only ordinal types which correspond to *well-ordered* sets. With this restriction, any set of ordinal numbers is well-ordered if one defines $\alpha\leqq\beta$ to mean that any set of ordinal type α can be put into a one-to-one, order-preserving correspondence with an initial segment of any set of ordinal type β.

OR'DI-NAR'Y, *adj.* **ordinary annuity and life insurance.** See ANNUITY, and INSURANCE—life insurance.

ordinary differential equation. See DIFFERENTIAL—differential equation (ordinary).

ordinary point of a curve. See POINT—ordinary point of a curve.

OR'DI-NATE, *n.* The coordinate of a point, in the Cartesian coordinates in the plane, which is the distance from the axis of abscissas (x-axis) to the point, measured along a line parallel to the axis of ordinates (y-axis).

average (mean) ordinate. See MEAN—mean value of a function.

double ordinate. A line segment between two points on a curve and parallel to the axis of ordinates (in Cartesian coordinates) used with reference to curves that are symmetrical with respect to the axis of abscissas, such as the parabola $y^2 = 2px$ or the ellipse $x^2/a^2 + y^2/b^2 = 1$.

ORDVAC. An automatic digital computing machine built at the University of Illinois and installed at the Aberdeen Proving Ground. ORDVAC is an acronym for *Ordnance Discrete Variable Automatic Computer.*

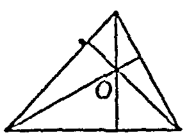
O'RI-EN-TA'TION, *n.* See COMPLEX—simplicial complex, MANIFOLD, SIMPLEX, and SURFACE.

OR'I-GIN, *n.* **origin of Cartesian coordinates.** The point of intersection of the axes. See CARTESIAN—Cartesian coordinates.

origin of a coordinate trihedral. The point of intersection of the planes. See COORDINATE—coordinate planes.

origin of a ray. See RAY.

OR'THO-CEN'TER, *n.* **orthocenter of a triangle.** The point of intersection of the three altitudes of the triangle.



OR-THOG'O-NAL, *adj.* Right-angled; pertaining to or depending upon the use of right angles.

orthogonal basis. See BASIS—basis of a vector space.

orthogonal complement. The orthogonal complement of a vector **v** (or of a subset S) of a vector space is the set of all vectors of the space which are orthogonal to **v** (or to each vector of S). The orthogonal complement of a vector in three dimensions is the set of all vectors perpendicular to the given vector, *i.e.*, the set of all linear combinations of any two linearly independent vectors perpendicular to the given vector. See VECTOR—vector space.

orthogonal functions. Real functions $f_1(x), f_2(x), \cdots$ are said to be *orthogonal* on the range (a, b) if

$$\int_a^b f_n(x) f_m(x)\, dx = 0,$$

for $m \neq n$, and to be normal, or normalized, or to be an orthonormal system if also

$$\int_a^b [f_n(x)]^2\, dx = 1$$

for all n. The integral $\int_a^b f_n(x) f_m(x)\, dx$ is called the inner product (f_n, f_m) of f_n and f_m. An orthonormal system of continuous functions $f_1, f_2, \cdots$ is said to be complete if $F(x) \equiv 0$ whenever $F(x)$ is continuous and

$$\int_a^b F(x) f_n(x)\, dx = 0$$

for all n. A necessary and sufficient condition for the completeness of an orthonormal system of continuous functions $f_1, f_2, \cdots$ is that, for any function F continuous on (a, b), $(F, F) = \sum_{n=1}^{\infty} (F, f_n)^2$, or that $\sum_{n=1}^{\infty} (F, f_n) f_n$ *converge in the mean* (of order 2) to $F(x)$. The above is valid for complex-valued functions if (F, G) is defined as $\int_a^b F(x)\overline{G(x)}\, dx$. It is also sufficient to assume the various functions are (Lebesgue) measurable and their squares are Lebesgue integrable, if $F(x) = 0$ is replaced by $F(x) = 0$ *almost everywhere*. Examples of complete orthonormal systems of continuous functions are: (1) The functions

$$\frac{1}{(2\pi)^{1/2}}, \frac{\cos nx}{\pi^{1/2}}, \frac{\sin nx}{\pi^{1/2}},$$

$n = 1, 2, 3, \cdots$, on the interval $(0, 2\pi)$; (2) the functions

$$e^{nix}/(2\pi)^{1/2}\ (n = 0, 1, \cdots)$$

on the interval $(0, 2\pi)$; (3) the functions $[\frac{1}{2}(2n+1)]^{1/2} P_n(x)\ (n = 0, 1, 2, \cdots)$ on the interval $(-1, 1)$, where $P_n(x)$ is the nth Legendre polynomial. See BESSEL—Bessel's inequality, and RIESZ-FISCHER THEOREM.

orthogonal matrix. See MATRIX—orthogonal matrix.

orthogonal projection. See PROJECTION—orthogonal projection.

orthogonal substitution or transformation. A substitution which transforms from one set of rectangular coordinates to another.

orthogonal system of curves on a surface. A system of two one-parameter families of curves on a surface S such that through any point of S there passes exactly one curve of each family, and such that at each point P of S the tangents to the two curves of the system through P are mutually orthogonal.

orthogonal trajectory of a family of curves. A curve which cuts all the curves of the family at right angles. Any line through the origin is an *orthogonal trajectory* of the family of circles which has the origin as a common center, and any one of these circles is an orthogonal trajectory of the family of lines passing through the origin. The equation of the orthogonal trajectories of a family of curves may be obtained from the differential equation of the family by replacing dy/dx in that equation by its negative reciprocal, $-dx/dy$, and solving the resulting differential equation.

orthogonal transformation. (1) A linear transformation of the form

$$y_i = \sum_{j=1}^{n} a_{ij}x_j \; (i=1, 2, \cdots, n),$$

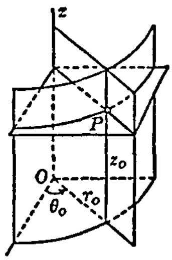
which leaves the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2$ invariant; a linear transformation whose matrix is an orthogonal matrix. (2) A transformation of the form $P^{-1}AP$ of a matrix A by an orthogonal matrix P. These two concepts are closely related. For let $A=(a_{ij})$ be the matrix of a linear transformation. Then since A^TA is the identity matrix (I) if A is orthogonal and A^T is the transpose of A,

$$\sum_{i=1}^{n} y_i^2 = (y)I\{y\} = (x)A^TA\{x\} = \sum_{i=1}^{n} x_i^2,$$

where (y) is the one-row matrix $(y_1, \cdots y_n)$, $\{x\}$ is the similar one-column matrix, and multiplication is matrix multiplication. A real orthogonal transformation is called **proper** or **improper** according as the determinant of A is 1 or -1. The rotation $x' = x\cos\phi + y\sin\phi$, $y' = -x\sin\phi + y\cos\phi$ is a proper orthogonal transformation. A proper orthogonal transformation is also called a **rotation,** being the usual rotation in two or three dimensions. If a matrix is symmetric, it can be reduced to diagonal form by an orthogonal transformation. Hence orthogonal transformations are often called *principal-axis transformations* and the eigenvectors of the matrix are called *normal coordinates.* See EQUIVALENT—equivalent matrices, TRANSFORMATION—congruent transformation, UNITARY—unitary transformation.

orthogonal vectors. Two vectors whose scalar product is zero. See MULTIPLICATION—multiplication of vectors, and VECTOR—vector space. For vectors represented by directed line segments in the plane or three-dimensional space, this is equivalent to the vectors (or lines) being perpendicular. See RECIPROCAL—reciprocal systems of vectors.

triply orthogonal system of surfaces. Three families of surfaces which are such that one member of each family passes through each point in space and each surface is orthogonal to every member of the other two families. In the figure the three



triply orthogonal systems of surfaces $x^2 + y^2 = r_0^2$, $y = x\tan\theta_0$ and $z = z_0$ intersect at right angles, for instance at the point P. See CONFOCAL—confocal quadrics, and CURVILINEAR—curvilinear coordinates of a point in space.

OR'THO-GRAPH'IC, *adj.* **orthographic projection.** Same as ORTHOGONAL PROJECTION. See AZIMUTHAL—azimuthal map, and ORTHOGONAL—orthogonal projection.

OR'THO-NOR'MAL, *adj.* **complete orthonormal system.** See ORTHOGONAL—orthogonal functions.

OS'CIL-LAT'ING, *adj.* **oscillating series.** See DIVERGENT—divergent series.

OS'CIL-LA'TION, *n.* A single swing from one extreme to another of an object

with an oscillating or vibrating motion. *Syn.* Vibration. Oscillations with variable amplitude decreasing toward zero as time increases are **damped oscillations**; **forced oscillations** are oscillations imparted to a body by an intermittent or oscillatory force, giving the motion of the body a different amplitude than it would have without such a force; oscillations which are not forced are **free** (a pendulum which has free oscillation very nearly describes simple harmonic motion if the oscillation is small); oscillations that tend toward fixed and well-defined limiting positions are said to be **stable oscillations** (an oscillation that is not stable is said to be **unstable**). Motion as described by the differential equation

$$\frac{d^2y}{dx^2}+A\frac{dy}{dx}+By=f(t)$$

has *free oscillations* if $A=0$, $B>0$, and $f(t)\equiv 0$; it then is *simple harmonic motion*. The motion has *damped oscillations* if $A>0$ [if $f(t)\equiv 0$, the motion is oscillatory if $A<\frac{1}{2}\sqrt{B}$]. If $f(t)$ represents an oscillatory force with

$$f(t)=k\sin(\lambda t+\theta),$$

then there are *forced oscillations*. If $\lambda^2=B$ and there is no damping ($A=0$), then the general solution is

$$y=-\frac{kt}{2\sqrt{B}}\cos(\sqrt{B}\,t+\theta)+C_1\sin(\sqrt{B}\,t+C_2)$$

and the motion becomes increasingly violent as t increases; this phenomenon is called *resonance* and results when the impressed force is of the same frequency as the free vibrations of the system.

oscillation of a function. The oscillation of a function on an interval I is the difference between the least upper bound and the greatest lower bound of values of the function at points of I. The oscillation of a function at a point P is the limit of the oscillation of the function on an interval which has P as an interior point as the length of the interval approaches zero. *Syn.* Saltus of a function. See VARIATION—variation of a function.

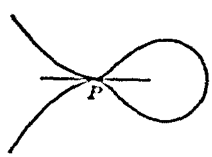
OS'CU-LAT'ING, *p.* **osculating circle of a curve at a point.** The circle in the limiting position, if this exists, of the circle tangent to the space curve C at the fixed point P on C, and through a variable point P' on C, as $P'\to P$ along C. The osculating circle has contact of at least the second order with C at P. The radius of the osculating circle is equal to the *radius of curvature* of C at P. Syn. Circle of curvature of a curve at a point.

osculating plane of a space curve at a point. The plane in the limiting position, if this exists, of the plane through the tangent to the space curve C at the fixed point P on C, and through a variable point P' on C, as $P'\to P$ along C. See NORMAL—normal to a curve or surface.

osculating sphere of a space curve at a point. The sphere through the osculating circle having contact of highest order (generally third) with the curve C at the point. Its center is on the polar line and its radius r is given by $r^2=\rho^2+\left(\tau\frac{d\rho}{ds}\right)^2$, where ρ and τ are the radii of curvature and torsion of C, respectively, and s is the arc length.

stationary osculating plane. The *osculating plane* of a space curve at a point where the rate of change of each of the direction cosines of the binormal to the curve vanishes.

OS'CU-LA'TION, *n.* **point of osculation.** A point on a curve at which two branches have a common tangent and each branch extends in both directions of the tangent. The curve $y^2=x^4(1-x^2)$ has a point of osculation at the origin, the double tangent there being the x-axis. Also called a **tacnode** and a **double cusp.**



POINT OF OSCULATION

OUT'PUT, *adj.* **output component.** In a computing machine, any component that is used in making available the results of the computation; *e.g.*, a typewriter, punched-card machine, or tape might be used for this purpose.

O'VAL, *n.* A curve shaped like a section of a football or of an egg; any curve that is

closed and always concave toward the center; a closed curve bounding a convex domain.

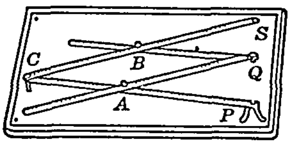
oval of Cassini. See CASSINI.

O'VER-HEAD', *adj.* **overhead expenses.** (*Finance.*) All expenses except labor and material.

P

PAIRED, *adj.* **paired observations.** (*Statistics.*) If the items in two different samples can be paired on the basis of some criterion, the joint observations are **paired observations.** Useful for comparing the means of two correlated samples. *E.g.*, heights of husbands and wives can be paired by pairing husband and wife.

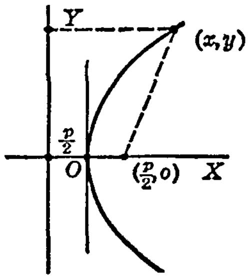
PAN'TO-GRAPH, *n.* A mechanical device for copying figures and at the same time changing the scale by which they are drawn, *i.e.*, for drawing figures similar to given figures. It consists of four graduated bars forming an adjustable parallelogram with the sides extended (see figure). The point P is fixed and, while the point Q traces out the figure, the point S traces out the copy (or vice versa). A, B, and C are free to move.



PAPPUS. theorems of Pappus. I. The area of a surface of revolution, formed by revolving a curve about a line in its plane not cutting the curve, is equal to the product of the length of the generating curve and the circumference of the circle described by the centroid of the curve. II. The volume of a solid of revolution, formed by revolving a plane area about a line in its plane not cutting the area, is equal to the product of the generating area and the circumference of the circle described by the centroid of the area.

PAR, *n.* (*Finance.*) (1) Value stated in a contract to pay, such as a bond or note. Also called **par value.** *At par*, *below par*, and *above par* refer to an amount equal to, less than, or greater than the face value (the amount stated in the contract). (2) The established value of the monetary unit of one country expressed in the monetary unit of another; called in full *par of exchange*, *mint par*, or *commercial par.*

PA-RAB'O-LA, *n.* The plane section of a conical surface by a plane parallel to an element; the locus of a point which moves so as to remain equidistant from a fixed point and a fixed line. Its **standard equation** in rectangular Cartesian coordinates is $y^2=2px$ (also written $y^2=4mx$), where the fixed point is on the positive x-axis at a distance $\frac{1}{2}p$ (or m) from the origin, and the given line is parallel to the y-axis at a distance $\frac{1}{2}p$ to the left of the origin. The



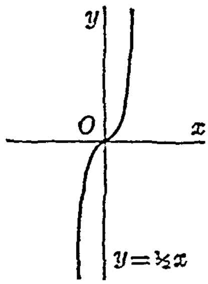
given point is called the focus of the parabola, and the given line is called the **directrix.** The axis of symmetry (the x-axis in the standard form given above) is called the **axis** of the parabola. The point where the axis cuts the parabola is called the **vertex**, and the chord through the focus and perpendicular to the axis is called the **latus rectum.** Important **parametric equations** of the parabola are those used, for instance, in determining the trajectory of a projectile. If v_0 is the initial velocity and β the angle the projectile makes with the horizontal plane when it starts, the equations of its path are

$$x=v_0t\cos\beta,\quad y=v_0t\sin\beta-\tfrac{1}{2}gt^2,$$

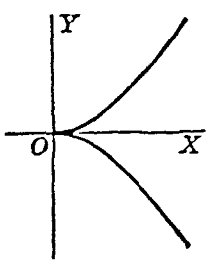
where the parameter t represents the time that has elapsed since the flight of the object started, and g is the acceleration of gravity. These equations represent a parabola. If $\beta=45°$ (the angle at which a projectile must be thrown to travel

farthest, neglecting air resistance), the equations reduce to $x=\frac{1}{2}\sqrt{2}v_0t$, $y=\frac{1}{2}\sqrt{2}v_0t-16t^2$, from which $y=x-32x^2/v_0^2$.

cubical parabola. The plane locus of an equation of the form $y=kx^3$. When k is positive the x-axis is an inflectional tangent, the curve passes through the origin and has infinite branches in the 1st and 3rd quadrants, and is concave up in the first and concave down in the 3rd quadrant.



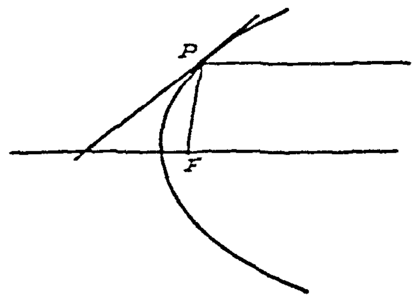
When k is negative the curve is the graph of $y=|k|x^3$ reflected in the y-axis. The semicubical parabola is the plane locus of the equation $y^2=kx^3$. It has a cusp of the first kind at the origin, the x-axis being the double tangent. It is the locus of the intersection of a variable chord, perpendicular to the axis of an ordinary parabola,



with a line drawn through the vertex of the parabola and perpendicular to the tangent at the end of the chord. The cubical and semicubical parabolas are *not* parabolas.

diameter of a parabola. The locus of the midpoints of a set of parallel chords. Any line parallel to the axis of the parabola is a diameter with reference to some set of chords. See DIAMETER—diameter of a conic.

focal property of the parabola. The focal radius to any point P on the parabola, and a line through P parallel to the axis of the parabola, make equal angles with the tangent to the parabola at P. If the parabola be constructed from a polished strip of metal, rays from a source of light at F (see figure) will be reflected from the parabola in rays parallel to the axis of the parabola. Likewise rays parallel to the axis of the parabola will be reflected and brought together at the focus. This property is referred to as the optical or reflection property of the parabola, and the corresponding property for sound is called the acoustical property of the parabola.



PAR'A-BOL'IC, *adj.* Of, relating to, resembling, or generated by a parabola.

parabolic cable. A cable suspended at both ends and supporting equal weights at equal distances apart horizontally. If the curve is an exact parabola, the weights must be uniformly and *continuously* distributed along the horizontal and the weight of the cable negligible. See CATENARY. A supporting cable of a suspension bridge hangs in a parabolic curve except for the slight modification of the curve due to the weight of the cable and the fact that the load is attached at intervals, not continuously.

parabolic curve. See CURVE—parabolic curve.

parabolic cylinder. See CYLINDER.

parabolic partial differential equation. A real second-order partial differential equation of the form

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + F\left(x_1, \cdots, x_n, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)=0$$

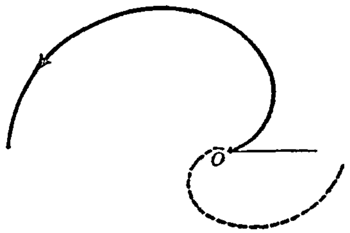
for which the determinant $|a_{ij}|$ vanishes. That is, the quadratic form $\sum_{i,j=1}^{n} a_{ij}y_iy_j$ is

singular (by means of a real linear transformation it can be reduced to the sum of fewer than n squares, not necessarily all of the same sign). A typical example is the *heat equation.* See INDEX—index of a quadratic form.

parabolic point on a surface. A point whose Dupin indicatrix is a pair of parallel straight lines (see DUPIN); a point at which the total curvature vanishes.

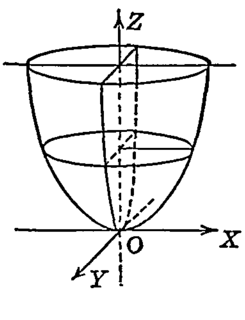
parabolic segment. A segment of a parabola which is subtended by a chord perpendicular to the axis of the parabola. Its area is $\frac{2}{3}cd$, where c (called the **base**) is the length of the chord, and d (called the **altitude**) is the distance from the vertex to the chord.

parabolic spiral. The spiral in which the square of the radius vector is proportional to the vectorial angle. Its equation in polar coordinates is $r^2 = a\theta$. Also called **Fermat's spiral.**



parabolic type. See TYPE—type of a Riemann surface.

PA-RAB'O-LOID, *n.* A term applied to the *elliptic* and *hyperbolic paraboloids.* The **elliptic paraboloid** is a surface which can be put in a position such that its sections parallel to one of the coordinate planes are ellipses, and parallel to the other coordinate planes are parabolas. When the surface is

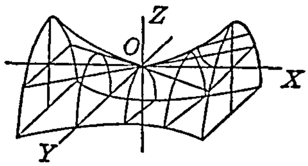


in the position illustrated, with its axis along the z-axis, its equation is

$$x^2/a^2 + y^2/b^2 = 2cz.$$

An elliptic paraboloid is a **paraboloid of revolution** if it is formed by revolving a parabola about its axis. This is the special case of the elliptic paraboloid in which the cross sections perpendicular to the axis are circles. The **hyperbolic paraboloid** is a surface which can be put in a position such that its sections parallel to one coordinate plane are hyperbolas, and parallel to the other coordinate planes are parabolas. In the position illustrated, its equation is

$$x^2/a^2 - y^2/b^2 = 2cz.$$



This is a ruled surface, the two families of rulings being

$$x/a - y/b = 1/p, \quad x/a + y/b = 2pcz,$$

and

$$x/a + y/b = 1/p, \quad x/a - y/b = 2pcz,$$

where p is an arbitrary parameter. These rulings are called RECTILINEAR GENERATORS, since either set may be used to generate the surface.

similar paraboloids. See SIMILAR.

PAR'A-COM-PACT', *adj.* **paracompact space.** A topological space T which is a *Hausdorff space* and has the property that, for any family F of open sets whose union contains T, there is a *locally finite* family F^* of open sets whose union contains T and which are such that each member of F^* is contained in a member of F. A paracompact space is *regular* and *normal.*

PAR'A-DOX, *n.* An argument in which it appears that an obvious untruth has been proved. See ZENO'S PARADOX, and RUSSELL'S PARADOX.

PAR'AL-LAC'TIC, *adj.* **parallactic angle of a star.** The angle between the arcs of great circles joining the star and the zenith,

and the star and the pole. See HOUR—hour angle and hour circle.

PAR′AL-LAX, *n.* geodesic parallax of a star. The plane angle subtended at the star by the radius of the earth.

PAR′AL-LEL, *adj., n.* Equidistant apart. See below, parallel curves, parallel lines and planes.

Euclid's postulate of parallels. If two lines are cut by a transversal and the sum of the interior angles on one side of the transversal is less than a straight angle, the two lines will meet if produced and will meet on that side of the transversal. If suitable other axioms are used, this is logically equivalent to: Only one line can be drawn parallel to a given line through a given point not on this line.

geodesic parallels on a surface. See GEODESIC.

parallel circle. See SURFACE—surface of revolution.

parallel curves. See CURVE—parallel curves (in a plane).

parallel displacement of a vector along a curve. If C is an arbitrarily given curve with parametric equations

$$x^i = f^i(t) \quad (t_0 \leqq t \leqq t_1),$$

and if ξ is any given contravariant vector at the point $x^i(t_0)$ on the curve C, then, under suitable restrictions on the metric tensor g_{ij} and on the curve C, the system of differential equations

$$\frac{d\xi^i(t)}{dt} + \Gamma^i_{\alpha\beta}(x^1(t), \cdots, x^n(t))\xi^\alpha(t)\frac{dx^\beta(t)}{dt} = 0,$$

subject to the initial conditions $\xi^i(r_0) = \xi_0{}^i$, will define a unique contravariant vector $\xi^i(t)$ at each point $x^i(t)$ of the given curve C. The vector $\xi^i(t)$ at the point $x^i(t)$ of the curve C is said to be parallel to the given vector $\xi_0{}^i$ with respect to the given curve C, and the vector $\xi^i(t)$ is said to be obtained from the given vector $\xi_0{}^i$ by a parallel displacement. The set of vectors $\xi^i(t)$, as the point $x^i(t)$ on C varies, is called a *parallel (contravariant) vector field with respect to the given curve C*. *E.g.*, the tangent vector field $dx^i(s)/ds$ to a geodesic forms a parallel (contravariant) field with respect to the geodesic.

parallel lines and planes. Two lines are parallel if there is a plane in which they both lie and they do not meet however far they are produced. The analytic condition that two noncoincident lines in a plane be parallel is that the coefficients of the corresponding variables in their rectangular Cartesian equations be proportional, that their slopes be equal, or that the determinant of the coefficients of the variables in their equations be zero. The condition that two noncoincident lines in space be parallel is that they have the same direction cosines (or direction cosines of opposite signs) or that their direction numbers be proportional. Two planes are parallel if they do not meet however far produced. The analytic condition that two noncoincident planes be parallel is that the direction numbers of their normals be proportional or that the coefficients of like variables in the Cartesian equations of the planes be proportional. A line and plane which do not meet however far produced are said to be parallel. A line is parallel to a plane if, and only if, it is perpendicular to the normal to the plane (see PERPENDICULAR—perpendicular lines and planes). A necessary and sufficient condition that three lines all be parallel to some plane is the vanishing of the third-order determinant whose rows are the direction numbers of the three lines, taken in a fixed order.

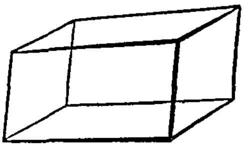
parallel sailing. See SAILING.

parallel surfaces. Surfaces having common normals. The only surfaces parallel to the surface S: $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, are the surfaces whose coordinates are $(x + aX, y + aY, z + aZ)$, where X, Y, Z are the direction cosines of the normal to S, and a is constant.

parallels of latitude. Circles on the earth's surface whose planes are parallel to the plane of the equator.

PAR′AL-LEL′E-PI′PED, *n.* A prism whose bases are parallelograms; a polyhedron, all of whose faces are parallelograms. The faces other than the bases are called lateral faces; their area, the lateral area of the prism; and their intersections, the lateral edges. A diagonal of a parallelepiped is a line segment joining two vertices which are not in the same face. There are four of these. They are usually called the

principal diagonals, the other diagonals being the diagonals of the faces. An **altitude** of a parallelepiped is the perpendicular distance from one face (called the

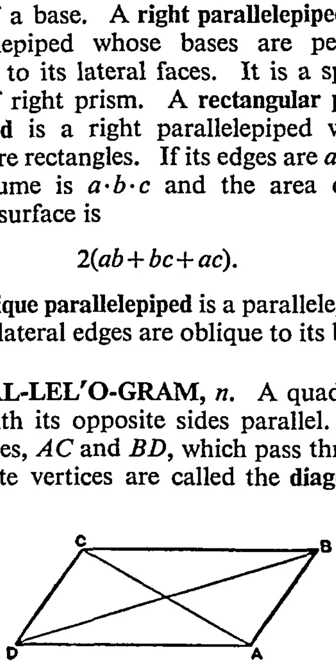


base) to the opposite face; the volume is equal to the product of an altitude and the area of a base. A **right parallelepiped** is a parallelepiped whose bases are perpendicular to its lateral faces. It is a special type of right prism. A **rectangular parallelepiped** is a right parallelepiped whose bases are rectangles. If its edges are a, b, c, its volume is $a \cdot b \cdot c$ and the area of its lateral surface is

$$2(ab+bc+ac).$$

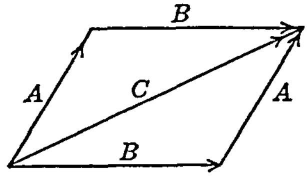
An **oblique parallelepiped** is a parallelepiped whose lateral edges are oblique to its bases.

PAR′AL-LEL′O-GRAM, *n.* A quadrilateral with its opposite sides parallel. The two lines, AC and BD, which pass through opposite vertices are called the **diagonals.**



The **altitude** of a parallelogram is the perpendicular distance between two of its parallel sides. The side to which the altitude is drawn is called the **base.** The **area** of a parallelogram is the product of an altitude by the area of the corresponding base, *i.e.*, the product of the area of any side (taken as a base) by the perpendicular distance from that side to the opposite side.

parallelogram law of forces. A law of vector addition of forces, established by Stevinus in 1586, which states that the resultant of two concurrent forces can be represented, in both magnitude and direction, by the diagonal of the parallelogram constructed with the concurrent forces as sides. This law forms one of the postulates of mechanics. If one constructs a parallelogram in which adjacent sides represent the given forces both in magnitude and direction, the diagonal of it, properly directed, represents the resultant of the two forces. In the figure, the vectors A and B represent forces, the diagonal C is the vector sum, or resultant, of A and B. The figure also shows that the vector addition is commutative, *i.e.*, $A+B=B+A$.



parallelogram of periods of a doubly periodic function of a complex variable. See PERIOD.

parallelogram of velocities or accelerations. Same as PARALLELOGRAM OF FORCES, with *velocities* or *accelerations* substituted for *forces.*

PAR′AL-LEL′O-TOPE, *n.* A parallelepiped the lengths of whose sides are proportional to $1, \frac{1}{2}, \frac{1}{4}$.

Hilbert parallelotope. The set of all points $x=(x_1, x_2, \cdots)$ of Hilbert space for which $|x_n| \leqq (\frac{1}{2})^n$ for each n. Any compact metric space is homeomorphic to a subset of the Hilbert parallelotope. The Hilbert parallelotope is homeomorphic to the set of all x for which $|x_n| \leqq 1/n$ for each n, this set sometimes being called a Hilbert parallelotope. Each of these sets is sometimes called a **Hilbert cube.**

PA-RAM′E-TER, *n.* An arbitrary constant or a variable in a mathematical expression, which distinguishes various specific cases. Thus in $y=a+bx$, a and b are parameters which specify the particular straight line represented by the equation. Also, the term is used in speaking of any letter, variable, or constant, other than the coordinate variables; *e.g.*, in the parametric equations of a line, $x=at+x_0$, $y=bt+y_0$, $z=ct+z_0$, the variable t is a parameter whose value determines a point on the line; a, b, c are parameters whose values determine a particular line.

conformal parameters. See CONFORMAL—conformal map.

differential parameters. See DIFFERENTIAL.

one-parameter family. See FAMILY.

parameter of distribution of a ruled surface. For a fixed ruling L on a ruled surface S, the value of the parameter of distribution b, to within algebraic sign, is the limit of the ratio, as the variable ruling L' of $S \to L$, of the minimum distance between L and L' to the angle between L and L'. The value of b is positive or negative according as the motion of the tangent plane is left-handed or right-handed as the point of tangency moves along the ruling L in the positive direction.

variation of parameters. See DIFFERENTIAL—linear differential equations.

PAR-A-MET'RIC, *adj.* **differentiation of parametric equations.** Finding the derivative from parametric equations. If the parametric equations are $y=h(t)$, $x=g(t)$, the derivative is given by

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt},$$

provided dx/dt is not zero (in case dx/dt is zero there may either be no derivative or it may be possible to use some other parametric equation to find it). *E.g.*, if $x=\sin t$ and $y=\cos^2 t$, then

$$\frac{dx}{dt}=\cos t,\quad \frac{dy}{dt}=-2\sin t\cos t,$$

and

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=-2\sin t.$$

equidistant system of parametric curves on a surface. A system of parametric curves for the surface such that the first fundamental quadratic form reduces to $ds^2=du^2+2F\,du\,dv+dv^2$. See SURFACE—fundamental quadratic form of a surface. *Syn.* Tchebychef net of parametric curves on a surface.

parametric curves on a surface. The curves of the families $u=$const. and $v=$const. on a surface S: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$; the former are called v-curves, the latter u-curves.

parametric equations. Equations in which coordinates are each expressed in terms of quantities called *parameters.* For a curve, parametric equations express each coordinate (two in the plane and three in space) in terms of a single parameter. The curve may be plotted point by point by giving this parameter values, each value of the parameter determining a point on the curve. Any equation can be written in an unlimited number of parametric forms, since the parameter can be replaced by an unlimited number of functions of the parameter. However, the term *parametric equations* sometimes refers to a specific parameter intrinsically related to the curve, as in the parametric equations of the circle: $x=r\cos\theta$, $y=r\sin\theta$, where θ is the central angle. See PARABOLA, ELLIPSE, and LINE—equation of a straight line, for specific parametric equations. The parametric equations of a surface are three equations (usually in Cartesian coordinates) giving x, y, z each as a function of two other variables, the parameters. Elimination of the parameters between the three equations results in the Cartesian equation of the surface. The points determined when one parameter is fixed and the other varies form a curve on the surface called a parametric curve. The parameters are called curvilinear coordinates, since a point on the surface is uniquely determined by the intersection of two parametric curves.

PA-REN'THE-SES, *n.* The symbols ( ). They enclose sums or products to show that they are to be taken collectively; *e.g.*, $2(3+5-4)=2\times 4=8$. See AGGREGATION, and DISTRIBUTIVE—distributive law.

PAR'I-TY, *n.* If two integers are both odd or both even they are said to have the same parity; if one is odd and the other even they are said to have different parity.

PARSEVAL. **Parseval's theorem.** If

$$a_k=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx$$

and

$$b_k=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$$

(for $n=0, 1, 2, \cdots$) and the numbers A_k, B_k are defined similarly for $F(x)$, then

$$\int_0^{2\pi} f(x)F(x)\,dx$$

$$=\pi\left[\tfrac{1}{2}a_0A_0+\sum_{n=1}^{\infty}(a_nA_n+b_nB_n)\right].$$

It is necessary to make restrictions on f and F, such that the function $f(x)$ is bounded in $(0, 2\pi)$ and that $\int_0^{2\pi} f(x)\,dx$ exists (and similarly for F), or that f and F are (Lebesgue) measurable and their squares are Lebesgue integrable on $(0, 2\pi)$. For a *complete orthonormal system* of vectors $x_1, x_2, \cdots$ in an infinite-dimensional vector space with an inner product (x, y) defined (such as Hilbert space), the theorem takes the form

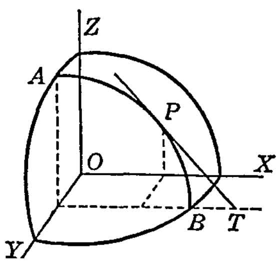
$$(u, v) = \sum_{k=1}^{\infty} (u, x_k)\overline{(v, x_k)},$$

where, if $u = v$, Parseval's theorem is Bessel's inequality with $\leqq$ replaced by $=$. See BESSEL—Bessel's inequality, and VECTOR—vector space.

PAR'TIAL, *adj.* **chain rule for partial differentiation.** See CHAIN—chain rule.

partial correlation. See CORRELATION.

partial derivative of a function of two or more variables. The ordinary derivative of the function with respect to one of the variables, considering the others as constants. If the variables are x and y, the partial derivatives of $f(x, y)$ are written $\partial f(x, y)/\partial x$ and $\partial f(x, y)/\partial y$, $D_x f(x, y)$ and $D_x f(x, y)$, $f_x(x, y)$ and $f_y(x, y)$, $f_1(x, y)$ and $f_2(x, y)$, or (x, y) may be deleted from any of these, leaving $\partial f/\partial x$, f_1, etc. The partial derivative of $x^2 + y$ with respect to x is $2x$; with respect to y it is 1. Geometrically, the partial derivative of a function of two variables, $f(x, y)$, with respect to one of the variables, is the slope of the tangent to the curve which is the intersection of the surface $z = f(x, y)$ and the plane whose equation is the other variable set equal to the constant which is its value at the point where the derivative is being evaluated. In the figure, the partial derivative with respect to x, evaluated at the point P, is the slope of the line PT which is tangent to the curve AB.



partial derivatives of the second and higher orders. Partial derivatives of partial derivatives. If the repeated partials are taken with respect to the same variable, the process is the same as fixing the other variables and taking the ordinary higher derivatives of a function of one variable. However, after having taken a partial derivative with respect to one variable it may be desirable to take the partial derivative of the resulting function with respect to another variable. One may desire the partial derivative with respect to y, of the partial derivative of a function $f(x, y)$ with respect to x, in which case the question arises as to when $f_{xy}(x, y) = f_{yx}(x, y)$. Such is the case when f_{xy} and f_{yx} are continuous at and in the neighborhood of the point at which the partial derivatives are taken. This result can be extended to partial derivatives of any order by successive applications of the second order case.

partial differential. See DIFFERENTIAL.

partial differential equations. Equations involving more than one independent variable and partial derivatives with respect to these variables. Such an equation is **linear** if it is of the first degree in the dependent variables and their partial derivatives, *i.e.*, if each term either consists of the product of a known function of the independent variables, and a dependent variable or one of its partial derivatives, or is itself a known function of the independent variables. The **order** of a partial differential equation is the order of the partial derivative of highest order which occurs in the equation.

elliptic, hyperbolic, and parabolic partial differential equations. See ELLIPTIC, HYPERBOLIC, and PARABOLIC.

partial fractions. A set of fractions whose algebraic sum is a given fraction. The term *method of partial fractions* is applied to the study of methods of finding these fractions and using them, particularly in integrating certain rational fractions. Undetermined coefficients are generally used to find partial fractions. *E.g.*, it is known that $1/(x^2-1) = A/(x-1) + B/(x+1)$ for some values of A and B, from which $A = \frac{1}{2}$, $B = -\frac{1}{2}$ are obtained by clearing of

fractions and equating coefficients of like powers of x. Any proper rational fraction can be expressed as a sum of fractions of the type

$$\frac{A}{x-a}, \frac{B}{(x-a)^n}, \frac{Cx+D}{x^2+bx+c}, \frac{Ex+F}{x^2+bx+c},$$

where n is a positive integer. Indeed, *partial fractions* are usually understood to be partial fractions of these relatively simple types.

partial product. The product of the multiplicand and one digit of the multiplier, when the latter contains more than one digit.

partial quotient. See FRACTION—continued fraction.

partial remainders. The (detached) coefficients of the quotient in synthetic division. See SYNTHETIC—synthetic division.

partial sum of an infinite series. See SUM.

PAR-TIC'I-PAT'ING, *adj.* **participating insurance policy.** See INSURANCE.

PAR'TI-CLE, *n.* Physically, a minute bit of matter of which material body is composed. A mathematical idealization of a physical particle is obtained by disregarding its spatial extensions, representing it as a mathematical point, and endowing it with the property of inertia (mass).

PAR-TIC'U-LAR, *adj.* **particular solution (or integral) of a differential equation.** Any solution that does not involve arbitrary constants (constants of integration); a solution obtainable from the general solution by giving special values to the constants of integration. See DIFFERENTIAL—solution of a differential equation.

PAR-TI'TION, *n.* **partition of an integer.** The *number of partitions* $p(n)$ of a positive integer n is the number of ways n can be written as a sum of positive integers, $n=a_1+a_2+\cdots+a_k$, where k is any positive integer and $a_1 \geqq a_2 \geqq \cdots \geqq a_k$. If k is restricted so that $k \leqq s$, this is called the *number of partitions of n into at most s parts.* Various other types of partitions have been studied. *E.g.*, the number of partitions of n for which the *summands are all different* can be shown to be equal to the number of partitions of n for which the *summands are all odd* (but repetitions are allowed); 5 is equal to 5, 4+1, 3+2; it is also equal to 5, 3+1+1, 1+1+1+1+1.

PARTS, *n.* **integration by parts.** See INTEGRATION—integration by parts.

PASCAL. **limaçon of Pascal.** See LIMAÇON.

Pascal's theorem. If any simple hexagon is inscribed in a conic, the three pairs of opposite sides intersect in collinear points.

Pascal's triangle. A triangular array of numbers composed of the coefficients of the expansion of $(a+b)^n$, for $n=0, 1, 2, 3$, etc. Ones are written on a vertical leg and on the hypotenuse of an isosceles right triangle. Each other element filling up the triangle is the sum of the element directly above it and the element above and to the left. The numbers in the $(n+1)$st row are then the coefficients in the binomial expansion of $(x+y)^n$. The array giving the binomial coefficients of orders 0, 1, 2, 3, 4, and 5 is

$$\begin{array}{cccccc} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

principle of Pascal. The pressure in a fluid is transmitted undiminished in all directions. *E.g.*, if a pipe projects vertically above a closed tank and the tank and pipe are filled with water, the total force on every unit of the inside surface of the tank is equal to that due to the water in the tank *plus a constant due to the water in the pipe.* That constant is equal to the weight of a column of water the height of the pipe and with unit cross section, regardless of the diameter of the pipe.

PATCH, *n.* **surface patch.** See SURFACE.

PATH, *n.* **path curve of a continuous surface deformation.** The locus of a given point of the surface under the deformation.

path curves. A name used for curves whose equations are in parametric form, a point being said to trace out the curve when the parameter varies.

path of a projectile. See PARABOLA—parametric equations of a parabola.

PAY'EE', *n.* The person to whom a sum of money is to be paid. Most frequently used in connection with notes and other written promises to pay.

PAY'MENT, *n.* A sum of money used to discharge a financial obligation, either in part or in its entirety.

defaulted payments. See DEFAULTED.

equal-payment method. See INSTALLMENT —installment payments.

installment payments. See INSTALLMENT.

PAY'OFF, *adj.*, *n.* The amount received by one of the players in a play of a game. For a two-person zero-sum game, the **payoff function** M is the function for which $M(x, y)$ (positive or negative) is the amount paid by the *minimizing player* to the *maximizing player* in case the maximizing player uses pure strategy x and the minimizing player uses pure strategy y. A **payoff matrix** for a finite two-person zero-sum game is a matrix such that the element a_{ij} in the ith row and jth column represents the amount (positive or negative) paid by the *minimizing player* to the *maximizing player* in case the maximizing player uses his ith pure strategy and the minimizing player uses his jth pure strategy. See GAME and PLAYER.

PEANO. Peano's postulates. See INTEGER.

Peano space. A space (or set) S which is a *Hausdorff topological space* and is the image of the *closed unit interval* [0, 1] for a continuous mapping. A Hausdorff topological space is a Peano space if and only if it is *compact*, *connected*, *locally connected*, *metrizable*, and *nonempty* (a Peano space is also *arc-wise connected*). A Peano space is sometimes called a **Peano curve**, although Peano curve sometimes means a continuous curve (a curve whose parametric equations are of the form $x=f(t)$, $y=g(t)$, where f and g are continuous and $0 \leqq t \leqq 1$) which passes through each point of the unit square.

PEARL-REED CURVE. Same as LOGISTIC CURVE.

PEARSON, Pearson's coefficient. See COEFFICIENT—correlation coefficient.

Pearson distribution. Let

$$\frac{df(x)}{dx}=\frac{(x-a)f(x)}{b_0+b_1x+b_2x^2}.$$

Then the set of frequency functions defined by this equality give what is known as the **Pearson distributions.** The values of a and b_i are functions of the moments of the distributions. The first four moments determine uniquely the particular distribution. The several curves are distinguished by the roots of the denominator of

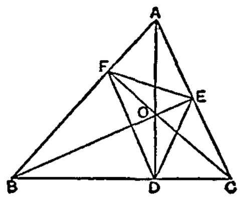
$$\frac{d(\log f)}{dx}=\frac{(x-a)}{B_0+B_1(x-a)+B_2(x-a)^2}.$$

If $a=B_1=B_2=0$, the normal curve results. A distribution function may be estimated from sample moments. The sampling variance of the moments is sometimes so large that more efficient alternative methods may be desirable.

PEAUCELLIER'S CELL. See INVERSOR, and INVERSION—inversion of a point with respect to a circle.

PED'AL, *adj.* **pedal curve.** The locus of the foot of the perpendicular from a fixed point to a variable tangent to a given curve. *E.g.*, if the given curve is a parabola and the fixed point the vertex, the *pedal curve* is a cissoid.

pedal triangle. The triangle formed within a given triangle by joining the feet of the perpendiculars from any given point to the sides. The triangle DEF is the pedal triangle formed within the triangle ABC by joining the feet of the altitudes.



The figure illustrates the fact that the altitudes of the given triangle bisect the angles of this pedal triangle.

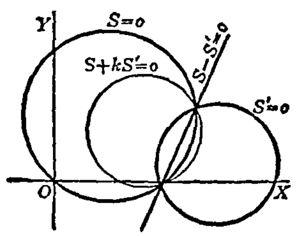
PELLIAN EQUATION. The Pellian equation is the special Diophantine equation, $x^2-Dy^2=1$, where D is a positive integer not a perfect square.

PEN'CIL, *n.* **axial pencil.** Same as pencil of planes given below.

pencil of circles. All the circles which lie in a given plane and pass through two fixed points. The equations of all members of the pencil can be obtained from the equations of any two circles passing through the two points by multiplying each equation by an arbitrary parameter and adding the results. The pencil of circles through the intersections of $x^2+y^2-4=0$ and $x^2+2x+y^2-4=0$ is given by

$$h(x^2+y^2-4)+k(x^2+2x+y^2-4)=0,$$

where h and k are arbitrary parameters not simultaneously zero. Frequently one of these parameters is taken as unity, but this excludes one of the circles from the pencil. In the figure, $S=0$ is the equation of one circle and $S'=0$ is the equation of the other. If the coefficients of x^2 and y^2 are unity in both equations then the equation $S-S'=0$ is the equation of the *radical axis* of any two members of the pencil.



pencil of families of curves on a surface. A one-parameter set of families of curves on a surface such that each two families of the set intersect under constant angle.

pencil of lines through a point. All the lines passing through the given point and lying in a given plane. The point is called the **vertex** of the pencil. The equations of the lines in the pencil can be obtained from the equations of any two lines of the pencil by multiplying each equation by an arbitrary parameter and adding the results. The pencil through the intersection of $2x+3y=0$ and $x+y-1=0$ is denoted by $h(2x+3y)+k(x+y-1)=0$ where h and k are not simultaneously zero. This equation contains only one essential parameter, because either h or k can be divided out since not both are zero. One parameter is frequently taken as unity, but that excludes one of the lines from the pencil; if $k=1$, for instance, there is no value of h which will reduce the equation of the pencil to $2x+3y=0$.

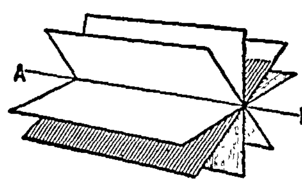
pencil of parallel lines. All the lines having a given direction; all the lines parallel to a given line. In projective geometry, a pencil of parallel lines (pencil of parallels) is included in the classification of *pencils of lines*; the vertex of the pencil, when the lines are parallel, being an *ideal point.* The notion of ideal point thus unifies the concepts of *pencil of lines* and *pencil of parallels.* The equations of the lines of a parallel pencil can be obtained by holding m (the slope) constant and varying b (the y-intercept) in the slope-intercept form $y=mx+b$ of the equation of a line, except when the pencil is perpendicular to the x-axis, in which case the equation $x=c$ suffices. Also, see above, pencil of lines through a point.

pencil of plane algebraic curves. All curves whose equations are given by assigning particular values to h and k, not both zero, in

$$hf_1(x, y)+kf_2(x, y)=0,$$

where $f_1=0$ and $f_2=0$ are of the same order (degree). If n is this order, the family passes through the n^2 points (with complex coordinates) common to $f_1=0$ and $f_2=0$. A pencil of conics consists of all conics passing through four fixed points, and a pencil of cubics consists of all the cubics passing through nine fixed points. A pencil of curves is often defined as above with either h or k taken equal to unity. See above, pencil of lines through a point.

pencil of planes. All the planes passing through a given line. The line is called the **axis** of the pencil (the line AB in the figure).



The equation of the pencil can be obtained by multiplying the equations of two of the planes by arbitrary parameters and adding the results. For any given values of the parameters, this is the equation of a plane passing through the line. In elementary

work, one of the parameters is usually taken as unity (see above, pencil of lines through a point).

pencil of spheres. All the spheres which pass through a given circle. The equations of all the members of the pencil can be determined by multiplying each of the equations of two spheres belonging to the pencil by an arbitrary parameter and adding the results; for particular values of these parameters this is the equation of a sphere of the pencil.

PEN'DU-LUM, *n.* **Foucault's pendulum.** A pendulum with a very long wire and a heavy bob, designed to exhibit the revolution of the earth about its axis. It is supported so as not to be restricted to remain in the same plane relative to the earth.

simple pendulum. A particle suspended by a weightless rod or cord; a body suspended by a cord whose weight is neglected, the body being treated as if it were concentrated at its center of gravity.

PEN'TA-DEC'A-GON, *n.* A polygon having fifteen sides. A **regular pentadecagon** is a pentadecagon having all of its sides and interior angles equal. Each interior angle is equal to 156°.

PEN'TA-GON, *n.* A polygon having five sides. A **regular pentagon** is a pentagon whose sides and interior angles are all equal. Each interior angle is equal to 108°.

PEN-TAG'O-NAL, *adj.* **pentagonal pyramid.** A pyramid whose base is a pentagon.

PEN'TA-GRAM (of Pythagoras). The five-pointed star formed by drawing all the diagonals of a regular pentagon and deleting the sides.

PE-NUM'BRA, *n.* See UMBRA.

PER CENT or PERCENT, *n.* Hundredths; denoted by %; 6% of a quantity is $\frac{6}{100}$ of it.

per cent error. See ERROR.

per cent profit on cost. The quotient of the selling price minus the cost, by the cost—all multiplied by 100. If an article costs 9 cents and sells for 10 cents, the *per cent gain* is $\frac{1}{9}\times 100$, or 11.11%.

per cent profit on selling price. The quotient of the selling price minus the cost, by the selling price—all multiplied by 100, or $100(s-c)/s$. The per cent gain on the cost price is always greater than the per cent gain on the selling price. If an article costs 9 cents and sells for 10 cents, the per cent gain on selling price is $\frac{1}{10}\times 100$, or 10%. Compare **per cent profit on cost.**

rate per cent. Rate in hundredths; same as YIELD.

PER-CENT'AGE, *n.* (1) The part of arithmetic dealing with per cent. (2) The result found by taking a certain per cent of the base. (3) Parts per hundred. One would say, "A percentage (or per cent) of the students are excellent"; but he would say, "Money is worth 6 per cent" (never 6 percentage).

PER-CEN'TILE, *n.* (*Statistics.*) A given percentile is the value which divides the range of a set of data into two parts such that a given percentage of the measures lies below this value.

PER'FECT, *adj.* **perfect cubes.** Numbers (quantities) which are exact third powers of some quantity, such as 8, 27, and $a^3+3a^2b+3ab^2+b^3$ which is equal to $(a+b)^3$.

perfect *n*th power. A number or quantity which is the exact nth power of some quantity. See above, perfect cubes.

perfect number. See NUMBER—perfect number.

perfect set. A set of points (or a set in a metric space) which is identical with its *derived set*; a set which is *closed* and *dense in itself.*

perfect square. A quantity which is the exact square of another quantity; 4 is a perfect square, as is also $a^2+2ab+b^2$, $(a+b)^2$.

perfect trinomial square. See SQUARE—perfect trinomial square.

PER'I-GON, *n.* An angle equal to the sum of two straight angles; an angle containing 360° or 2π radians.

PER-IM'E-TER, *n.* The length of a closed curve, as the perimeter of a circle, the perimeter of an ellipse, or the sum of the lengths of the sides of a polygon.

PE′RI-OD, *adj.*, *n.* **conversion period.** The time between two successive conversions of interest. *Syn.* Interest period.

parallelogram of periods. For a *doubly periodic function* $f(z)$ of the complex variable z, a *parallelogram of periods* is a parallelogram with vertices z_0, $z_0+\eta$, $z_0+\eta+\eta'$, $z_0+\eta'$, where η and η' are periods (with $\eta \neq k\eta'$ for any real k) but are not necessarily a *primitive period pair.* See below, primitive period parallelogram.

period in arithmetic. (1) The number of digits set off by a comma when writing a number. It is customary to set off periods of three digits, as 1,253,689. These periods are called units period, thousands period, millions period, etc. (2) When using certain methods for extracting roots, periods are set off equal to the index of the root to be extracted. (3) Period of a repeating decimal. The number of digits that repeat. See DECIMAL—repeating decimal.

period of an element of a group. The smallest power of the element which is the identity element. Sometimes called the **order** of the element. In the group whose elements are the roots of $x^6=1$ with multiplication as the group operation, $-\frac{1}{2}+\frac{1}{2}i\sqrt{3}$ is of period 3, since

$$(-\tfrac{1}{2}+\tfrac{1}{2}i\sqrt{3})^3=1.$$

See GROUP.

period of a function. See PERIODIC—periodic function of a real variable, and periodic function of a complex variable.

period region. For a periodic function of a complex variable, a *period region* is a *primitive period strip* or a *primitive period parallelogram*, according as the function is simply periodic or doubly periodic.

period of simple harmonic motion. See HARMONIC—simple harmonic motion.

primitive period pair. Two primitive periods ω and ω' of a *doubly periodic function* such that all periods of the function are of the form $n\omega+n'\omega'$, where n and n' are integers. See PERIODIC—periodic function of a complex variable. *Syn.* Fundamental period pair.

primitive period parallelogram. If $f(z)$ is a *doubly periodic function* of the complex variable z, if ω and ω' form a primitive period pair, and if z_0 is any point of the finite complex plane, then the parallelogram with vertices z_0, $z_0+\omega$, $z_0+\omega+\omega'$, $z_0+\omega'$ is called a primitive period parallelogram for the function. The vertex z_0 and the adjacent sides of the boundary, exclusive of their other end points, are considered as belonging to the parallelogram, the rest of the boundary being excluded. Thus each point of the finite plane belongs to exactly one parallelogram of a set of congruent primitive period parallelograms paving the entire finite plane. *Syn.* Fundamental period parallelogram.

primitive period strip. If $f(z)$ is a simply periodic function of the complex variable z in the domain D, and ω is a primitive period, then a region in D bounded by a line (or a suitable simple curve extending across D) C together with an image of C translated by an amount ω is called a **primitive period strip** for the function. *Syn.* Fundamental period strip.

select period of a select mortality table. See SELECT.

PE′RI-OD′IC, *adj.* **almost periodic function.** A continuous function $f(x)$ is **(uniformly) almost periodic** if, for any $\epsilon>0$, the set of all t for which the inequality

$$|f(x+t)-f(x)|<\epsilon$$

is satisfied for all x has the property that there is a number M such that any interval of length M contains at least one such t. *E.g.*, the function

$$f(x)=\sin 2\pi x+\sin 2\pi x\sqrt{2}$$

is uniformly almost periodic: $|f(x+t)-f(x)|$ is small if t is an integer and $\sqrt{2}t$ is near an integer. A function is uniformly almost periodic if and only if there is a sequence of finite trigonometric sums which converges uniformly to $f(x)$ (the terms in the sum are of type $a_r \cos rx$ and $b_r \sin rx$, where r need not be an integer). A number of generalized definitions of almost periodicity have been studied. *E.g.*, the expression $|f(x+t)-f(x)|$ in the above definition might be replaced by the least upper bound (for $-\infty<x<+\infty$) of

$$\left[\frac{1}{k}\int_x^{x+k} |f(x+t)-f(x)|^p\,dx\right]^{1/p},$$

for some specific value of k, or by the limit of this expression as $k\to\infty$.

periodic continued fraction. See FRACTION—continued fraction.

periodic curves. Curves whose ordinates repeat at equal distances on the axis of abscissas; the graph of a periodic function. The loci of $y=\sin x$ and $y=\cos x$ are periodic curves, repeating themselves in every successive interval of length 2π.

periodic function of a complex variable. The function $f(z)$, analytic in a domain D, is said to be **periodic** in D if $f(z)\neq$constant and if there is a complex number $\omega\neq 0$ such that if z is in D, then also $z+\omega$ is in D, and $f(z+\omega)\equiv f(z)$. The number ω is said to be a **period** of $f(z)$. If there is no period of the form $\alpha\omega$, where α is real and $|\alpha|<1$, then ω is said to be a **primitive period** or **fundamental period** of $f(z)$. A **simply periodic** (or **singly periodic**) **function** of a complex variable is a function $f(z)$ of the complex variable z having a primitive period ω but having no periods other than $\pm\omega, \pm 2\omega, \cdots$. A **doubly periodic function** of a complex variable is a periodic function of a complex variable which is not *simply periodic*. It can be shown that if a periodic function is not simply periodic then there exist two *primitive periods* ω and ω' such that all periods are of the form $n\omega+n'\omega'$, where n and n' are integers but not both zero. This is **Jacobi's theorem.** See ELLIPTIC—elliptic function.

periodic function of a real variable. A function $f(x)$ such that the range of the independent variable can be divided into equal subintervals such that the graph of the function is the same in each subinterval. The length of the smallest such equal subintervals is called the **period** of the function. *Tech.* If there is a smallest positive number p for which $f(x+p)=f(x)$ for all x (or $f(x)$ and $f(x+p)$ are both undefined), then p is the **period** of the function. Sin x has the period 2π (radians), since

$$\sin(x+2\pi)\equiv\sin x.$$

The **frequency** of a periodic function in a given interval is the quotient of the length of the interval and the period of the function (*i.e.*, the number of times the function repeats itself in the given interval). If an interval of length 2π is being considered, the frequency of $\sin x$ is 1; of $\sin 2x$, 2, and of $\sin 3x$, 3.

periodic motion. Motion which repeats itself, occurs in cycles. See HARMONIC—simple harmonic motion.

PE'RI-O-DIC'I-TY, *adj.* **periodicity of a function (or curve).** The property of having periods or being periodic.

PE-RIPH'ER-Y, *n.* The boundary line or *circumference* of any figure; the surface of a solid.

PER'MA-NENT-LY, *adj.* **permanently convergent series.** See CONVERGENT—permanently convergent series.

PER-MIS'SI-BLE, *adj.* **permissible values of a variable.** The values for which a function under consideration is defined and which lie on the interval or set on which the function is being considered. Zero is not a *permissible* value for x in the function $\log x$, and 4 is not a permissible value for x if $\log x$ is being considered on the interval (1, 2). *Permissible* is also used of *any* values for which a function is defined.

PER'MU-TA'TION, *n.* (1) An ordered arrangement or sequence of all or part of a set of things. All possible permutations of the letters a, b, and c are: a, b, c, ab, ac, ba, bc, ca, cb, abc, acb, bac, bca, cab, and cba. A **permutation of n things taken all at a time** is an ordered arrangement of all the members of the set. If n is the number of members of the set, the total possible numbers of such permutations is $n!$, for any one of the set can be put in the first place, any one of the remaining $n-1$ things in the second place, etc., until n places are filled. When **some members of the set are alike** (two permutations obtainable from each other by interchanging like objects are the same permutation) the number of such permutations is the number of permutations of n different things taken all at a time divided by the product of the factorials of the numbers representing the number of repetitions of the various things. The letters a, a, a, b, b, c can be arranged in $6!/(3!2!)$ or 60 different ways (permutations). A **permutation of n things taken r at a time** is a permutation containing only r members of the set. The number of such permutations is denoted by ${}_nP_r$ and is equal to

$$n(n-1)(n-2)\cdots(n-r+1),$$

or $n!/(n-r)!$, for any one can be put first,

any one of the remaining $n-1$ second, etc., until r places have been filled. A **permutation of n things taken r at a time with repetitions** is an arrangement obtained by putting any member of the set in the first position, any member of the set, including a repetition of the one just used, in the second, and continuing this until there are r positions filled. The total number of such permutations is $n \cdot n \cdot n \cdots$ to r factors, *i.e.*, n^r. The ways in which a, b, c can be arranged two at a time are aa, ab, ac, ba, bb, bc, ca, cb, cc. A **circular permutation** is an arrangement of objects around a circle. The total number of circular permutations of n different things taken n at a time is equal to the number of permutations of n things n at a time, divided by n because each arrangement will be exactly like $n-1$ others except for a shift of the places around the circle. (2) An **operation** which replaces each of a set of objects by itself or another object in the set in a one-to-one manner. The permutation which replaces x_1 by x_2, x_2 by x_1, x_3 by x_4, and x_4 by x_3, is denoted by

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

or (12)(34). A **cyclic** (or **circular**) **permutation** (or simply a **cycle**) is the advancing of each member of a set of ordered objects one position, the last member taking the position of the first. If the objects are thought of as arranged in order around a circle, a cyclic permutation is effected by rotating the circle; *cab* is a cyclic permutation of *abc*, denoted by $\begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}$ or (acb). The **degree** of a cyclic permutation is the number of objects in the set. A cyclic permutation of degree two is called a **transposition.** Every permutation can be factored into a product of transpositions. *E.g.*, $(abc)=(ab)(ac)$ in the sense that the permutation (abc) has the same effect as the permutation (ab) followed by the permutation (ac). A permutation is **even** or **odd** according to whether it can be written as the product of an even or odd number of *transpositions.* Let $x_1, x_2, \cdots, x_n$ be n indeterminants and let D be the product $(x_1-x_2)(x_1-x_3)\cdots(x_{n-1}-x_n)$ of all differences x_i-x_j, where $i<j$. A permutation of the subscripts $1, 2, \cdots, n$ is even or odd according to whether it leaves the sign of D unchanged or changes the sign of D.

permutation group. A group whose elements are permutations, the product of two permutations being the permutation resulting from applying each in succession. Thus the product of the permutation $p_1=(abc)$, which takes a into b, b into c, and c into a, and the permutation $p_2=(bc)$, which takes b into c and c into b, is $p_1p_2=(abc)(bc)=(ac)$, which takes a into c and c into a. The group of all permutations on n letters is a group of order $n!$, called a **symmetric group.** The subgroup of this group (of order $n!/2$) which contains all *even permutations* is called an **alternating group.** A permutation group of order n on n letters is called **regular.** See PERMUTATION (2), and GROUP. *Syn.* Substitution group.

permutation matrix. If a permutation on $x_1, x_2, \cdots, x_n$ carries x_i into $x_{i'}$ for each i, then the *permutation matrix* corresponding to this permutation is the square matrix of order n in which the elements in the ith column (for each i) are all zero except the one in the i'th row, which is unity. Any permutation group is isomorphic with the group of corresponding permutation matrices. In general, a permutation matrix is any square matrix whose elements in any column (or any row) are all zero, except for one element equal to unity.

PER'PEN-DIC'U-LAR, *adj.*, *n.* **perpendicular lines and planes.** Two **straight lines** which intersect so as to form a pair of equal adjacent angles are said to be perpendicular (each line is said to be perpendicular to the other). The condition (in *analytic geometry*) that two lines be perpendicular is: (1) *In a plane*, that the slope of one of the lines be the negative reciprocal of the slope of the other; (2) *in space*, that the sum of the products of the corresponding direction numbers (or direction cosines) of the two lines be zero (two **lines in space** are perpendicular if there exist intersecting perpendicular lines, each of which is parallel to one of the given lines). A **common perpendicular** to two or more lines is a line which is perpendicular to each of them. In a plane, the only lines that can have a common perpendicular are parallel lines, and they have any number. In space, any

two lines have any number of common perpendiculars (only one of which intersects both lines, unless the lines are parallel). A **line perpendicular to a plane** is a line which is perpendicular to every line through its intersection with the plane. It is sufficient that it be perpendicular to two nonparallel lines in the plane. The condition (in *analytic geometry*) that a line be perpendicular to a plane is that its direction numbers be proportional to those of the normal to the plane; or, what amounts to the same thing, that its direction numbers be proportional to the coefficient of the corresponding variables in the equation of the plane. The **foot** of the perpendicular to a line (or plane) is the point of intersection of the perpendicular with the line (or plane). Two **perpendicular planes** are two planes such that a line in one, which is perpendicular to their line of intersection, is perpendicular to the other; *i.e.*, planes forming a *right dihedral angle*. The condition (in *analytic geometry*) that two planes be perpendicular is that their normals be perpendicular, or that the sum of the products of the coefficients of like variables in their two equations be zero. See NORMAL—normal lines and planes.

PER'PE-TU'I-TY, *n.* (*Math. of Finance.*) An annuity that continues forever. See CAPITALIZED—capitalized cost.

PER-SPEC'TIVE, *adj.* **perspective position.** A pencil of lines and a range of points are in *perspective position* if each line of the pencil goes through the point of the range which corresponds to it. Two pencils of lines are in perspective position if corresponding lines meet in points which lie on a line called the **axis of perspectivity.** Likewise two ranges of points are in *perspective position* provided lines through their corresponding points meet in a point called the **center of perspectivity.** A range of points and an axial pencil (pencil of planes) are in *perspective position* if each plane of the pencil goes through the point which corresponds to it; a pencil of lines and an axial pencil are in perspective position if each line of the pencil lies in the plane to which it corresponds; likewise two axial pencils are in perspective position if intersections of corresponding planes lie in a plane. Each of the above relationships is called a **perspectivity.** See PROJECTIVE—projective relation.

PERSPECTIVITY. See PERSPECTIVE.

PFAFF'I-AN, *n.* An equation of the form

$$u_1\,dx_1+u_2\,dx_2+u_3\,dx_3+\cdots+u_n\,dx_n,$$

where the coefficients $u_1, \cdots, u_n$ are functions of the variables $x_1, \cdots, x_n$.

PHASE, *n.* **phase of simple harmonic motion.** The angle $\phi+kt$ in the equation of simple harmonic motion, $x=a\cos(\phi+kt)$. See HARMONIC—simple harmonic motion.

initial phase. The phase when $t=0$, namely ϕ in $x=a\cos(\phi+kt)$.

PHI. phi coefficient. See COEFFICIENT.

phi function. See EULER—Euler's ϕ-function.

PHRAGMÉN-LINDELÖF FUNCTION. Relative to an entire function $f(z)$ of finite order ρ, the function

$$h(\theta)\equiv\limsup_{r\to\infty}\frac{\log|f(re^{i\theta})|}{r^\rho}.$$

The Phragmén-Lindelöf function $h(\theta)$ is a *subsine function* of order ρ. See ENTIRE—entire function.

PI, *n.* The name of the Greek letter π, Π, which corresponds to the Roman P. The symbol π denotes the ratio of the circumference of a circle to its diameter; $\pi=3.14159+$; π is a *transcendental irrational number.* Π denotes a product. See INFINITE—infinite product.

Wallis' product for π. See WALLIS.

PI'CA, *n*, (*Printing.*) A measure of type body, equal to 12 points in U. S. scale. See POINT (4).

PICARD. Picard's method. An iterative method for solving differential equations. For the differential equation $dy/dx=f(x, y)$, the solution that passes through the point (x_0, y_0) satisfies the equation

$$y(x)=y_0+\int_{x_0}^{x} f[t, y(t)]\,dt.$$

Starting with an initial function, say the constant y_0, the method consists of making successive substitutions in accordance with the formula

$$y_n(x)=y_0+\int_{x_0}^{x} f[t, y_{n-1}(t)]\,dt.$$

The method extends to the solution of systems of linear differential equations and to the solution of higher-order linear differential equations and systems of equations.

Picard's theorems. *Picard's first theorem* states that if $f(z)$ is an entire function, and $f(z)\not\equiv\text{const.}$, then $f(z)$ takes on every finite complex value with at most one exception. *E.g.*, $f(z)=e^z$ takes on all values except 0. For *Picard's second theorem*, see SINGULAR—isolated singular point of an analytic function.

PIC'TO-GRAM, *n.* Any figure showing numerical relations, as *bar graphs*, *broken-line graphs*. See GRAPHING—statistical graphing.

PIECE'WISE, *adj.* **piecewise continuous function.** A function is piecewise continuous on R if it is defined on R and R can be divided into a finite number of pieces such that the function is continuous on the interior of each piece and such that the function approaches a finite limit as a point moves in the interior of a piece and approaches a boundary point in any way. It is necessary to restrict the nature of the pieces, *e.g.*, that their boundaries be simple closed curves if they are plane regions, or two points if on a straight line. If R is bounded, an equivalent definition is that it be possible to divide R into a finite number of pieces such that R is uniformly continuous on the interior of each piece.

PIERC'ING, *adj.* **piercing point of a line in space.** See POINT—piercing point of a line in space.

PITCH, *n.* **pitch of a roof.** The quotient of the rise (height from the level of plates to the ridge) by the **span** (length of the plates); one-half of the *slope* of the roof.

PLACE, *n.* **decimal place.** See DECIMAL.

place value. The value given to a digit by virtue of the place it occupies in the number relative to the units place. In 423.7, 3 denotes merely 3 units, 2 denotes 20 units, 4 denotes 400 units, and 7 denotes $\frac{7}{10}$ of a unit; 3 is in unit's place, 2 in ten's place, 4 in hundred's place, etc. *Syn.* Local value.

PLA'NAR, *adj.* **planar point of a surface.** A point of the surface at which $D=D'=D''=0$. See SURFACE—fundamental coefficients of a surface. At a planar point, every direction on the surface is an *asymptotic direction.* A surface is a plane if, and only if, all its points are planar points.

PLANE, *adj.*, *n.* A surface such that a straight line joining any two of its points lies entirely in the surface. *Syn.* Plane surface.

collinear planes. See COLLINEAR.

complex plane. See COMPLEX.

coordinate planes. See CARTESIAN—Cartesian coordinates.

diametral plane. See DIAMETRAL.

equation of a plane. In three-dimensional Cartesian coordinates, a polynomial equation which is of the first degree. The equation $Ax+By+Cz+D=0$ with A, B and C not all zero is called the **general form** of the equation of a plane. Several types of the general form are: (1) **Intercept form.** The equation $x/a+y/b+z/c=1$, where a, b and c are the x, y, z intercepts, respectively. (2) **Three-point form.** The equation of the plane expressed in terms of three points on the plane. The simplest form is obtained by equating to zero the determinant whose rows are $x, y, z, 1$; $x_1, y_1, z_1, 1$; $x_2, y_2, z_2, 1$; and $x_3, y_3, z_3, 1$, where the subscript letters are the coordinates of the three given points. (3) **Normal form.** The equation

$$lx+my+nz-p=0,$$

where l, m, and n are the direction cosines of the normal to the plane, directed from the origin to the plane, and p is the length of the normal from the origin to the plane. If l, m, n and the coordinates of a point in the plane, say x_1, y_1, z_1, are given, then $p=lx_1+my_1+nz_1$ and the equation of the plane can be written

$$l(x-x_1)+m(y-y_1)+n(z-z_1)=0.$$

The left member of this equation is the *scalar product* of the vectors (l, m, n) and $(x-x_1, y-y_1, z-z_1)$. Therefore the left member of either of the above equations is the distance from the point (x, y, z) to the plane. An equation, $Ax+By+Cz+D=0$, of a plane can be reduced to normal form by dividing by $\pm(A^2+B^2+C^2)^{1/2}$, the sign being opposite that of the constant term D.

normal (perpendicular) lines or planes to lines, curves, planes, and surfaces. See NORMAL—normal to a curve or surface, PERPENDICULAR—perpendicular lines and planes.

parallel lines and planes. See PARALLEL.

pencil of planes. See PENCIL.

plane angle of a dihedral angle. The angle formed by two intersecting lines, one of which lies in each face and both of which are perpendicular to the edge of the dihedral angle; the plane angle between the intersections of the faces of the dihedral angle with a third plane which is perpendicular to the edge of the dihedral angle. Such a plane angle is said to **measure** its dihedral angle. When the plane angle is *acute*, *right*, *obtuse*, etc., its *dihedral* angle is said to be *acute*, *right*, *obtuse*, etc.

plane curve. A curve all of whose points lie in a single plane.

plane figure. A figure lying entirely in a plane.

plane geometry. See GEOMETRY.

plane sailing. See SAILING.

plane section. The intersection of a plane and a surface or a solid.

principal plane of a quadric surface. See PRINCIPAL—principal plane of a quadric surface.

projection plane. The plane upon which a figure is projected; a plane section of the projection rays of a projection. See PROJECTION.

projective plane. See PROJECTIVE—projective plane.

sheaf of planes. See SHEAF.

shrinking of the plane. See SIMILITUDE—transformation of similitude, and STRAIN—one-dimensional strains.

PLA-NIM′E-TER, *n.* A mechanical device for measuring plane areas. Merely requires moving a pointer on the *planimeter* around the bounding curve. A common type is the *polar planimeter*. See INTEGRATOR.

PLAS-TIC′I-TY, *n.* **theory of plasticity.** The theory of behavior of substances beyond their elastic range.

PLATE, *n.* **plate of a building.** A horizontal timber (beam) that supports the lower end of the rafters.

PLATEAU PROBLEM. The problem of determining the existence of a *minimal surface* with a given twisted curve as its boundary. It might or might not be required that the minimal surface have minimum area. For several different contours the problem was solved by the physicist Plateau in soap-film experiments.

PLATYKURTIC, *adj.* **platykurtic distribution.** See KURTOSIS.

PLAY, *n.* **play of a game.** Any particular performance, from beginning to end, involved in a game. See GAME and MOVE.

PLAY′ER, *n.* An individual, or group of individuals acting as one, involved in the play of a game. In a two-person zero-sum game, the **maximizing player** is the player to whom all payments are considered as being made by the other player (a payment made to him is considered as a positive payment, while a payment made by him is considered as a negative payment); a **minimizing player** is the player to whom all payments are considered as being made (a payment made by him is considered as a positive payment, while a payment made to him is considered as a negative payment). See GAME and PAYOFF.

PLOT, *v.* **plot a point.** To locate the point geometrically, either in the plane or space, when its coordinates are given in some coordinate system. In Cartesian coordinates, a point is plotted by locating it on cross-section paper or by drawing lines on plain paper parallel to indicated axes of coordinates and at a distance from them equal to the proper coordinate of the point. See COORDINATE.

point-by-point plotting (graphing) of a curve. Finding an ordered set of points which lie on a curve and drawing through these points a curve which is assumed to resemble the required curve.

PLÜCKER'S ABRIDGED NOTATION. See ABRIDGED.

PLUMB, *adj., n.* A weight attached to a cord.

plumb line. See LINE—plumb line.

PLUS, *adj., n.* Denoted by +. (1) Indicates addition, as $2+3$ (3 added to 2). (2) Property of being positive. (3) A little more or in addition to, as 2.35+.

plus sign. The sign +. See PLUS.

POINCARÉ. Poincaré-Birkhoff fixed-point theorem. Let a continuous one-to-one transformation map the ring R formed by two concentric circles in such a way that one circle moves in the positive sense and the other in the negative sense and areas are preserved. Then the transformation has at least two fixed points. This theorem was conjectured by Poincaré and proved by G. D. Birkhoff.

Poincaré duality theorem. See DUALITY.

Poincaré recurrence theorem. Let X be a bounded open region of n-dimensional Euclidean space and let T be a homeomorphism of X onto itself that preserves volume, that is, any open set and its transform by T have the same volume (or *measure*). Poincaré proved that there is a set S of *measure zero* in X such that, if x is not in S and U is any open set in X which contains x, then an infinite number of the points $x, T(x), T^2(x), T^3(x), \cdots$ belong to U, where $T^n(x)$ is the result of applying T to x successively n times. The theorem is still true if S is required to be of *first category* as well as of measure zero. Numerous generalizations and modifications of Poincaré's theorem are known. See ERGODIC—ergodic theory.

POINT, *adj., n.* (1) An element of geometry which has position but no extension. (2) An element of geometry defined by its coordinates, such as the point (1, 3). (3) An element which satisfies the postulates of a certain space. See POSTULATE—Euclid's postulates, METRIC—metric space. (4) A unit used in measuring bodies of type, leads, etc. It is equal to .0138 inches or .0351 centimeters, in the U. S. system.

accumulation (or **cluster** or **limit**) **point.** See ACCUMULATION.

collinear points. See COLLINEAR.

condensation point. See CONDENSATION.

conjugate points relative to a conic. See CONJUGATE—conjugate points relative to a conic.

decimal point. See DECIMAL.

double point. See below, multiple point.

homologous point. See HOMOLOGOUS.

isolated point. A point in whose neighborhood there is no other point of the set under consideration. The origin is such a point on the graph of a polynomial equation when the lowest degree homogeneous polynomial in the equation of the curve, referred to a system of Cartesian coordinates having its origin at the given point, vanishes for no values of x and y in a neighborhood of zero, except for both x and y equal to zero. The curve $x^2+y^2=x^3$ has an isolated point at the origin, since the equation $x^2+y^2=0$ is satisfied only by the point (0, 0). The lowest degree homogeneous polynomial which can satisfy the above conditions is a quadratic; hence isolated points are at least double points. *Syn.* Acnode.

material point. See MATERIAL.

multiple (or k**-tuple**) **point.** A point on a curve at which the curve crosses or touches itself at least k times. A point on a curve where there are k tangents, distinct or coincident. The equations of the tangents at a k-tuple point on an algebraic curve can be determined by equating to zero the lowest degree terms (in this case the terms of the kth degree) in the equation of the curve referred to a Cartesian coordinate system whose origin is at the multiple point. If a curve has several (k) coincident tangents at a point, it is said to have a **multiple** (k**-tuple**) **tangent** at the point. A **double point** is a point at which a curve crosses itself (a point at which there are two tangents, real and distinct, coincident, or imaginary). The equations of the tangents at a double point on an algebraic curve can be determined by equating to zero the quadratic terms in the equation of the curve referred to a rectangular Cartesian coordinate system whose origin is at the double point, the linear terms and constant term being zero in this case. This quadratic may be a perfect square, in which case the two tangents are coincident.

ordinary (or simple) point of a curve. (1) A point at which the curve possesses a smoothly turning tangent, does not have an isolated point, and does not cross itself. *Tech.* A point is an ordinary point if in the neighborhood of this point either the ordinate (y) can be represented as a continuously differentiable function of the abscissa (x), or x can be represented as a continuously differential function of y (in either case, the curve has a tangent which is a close approximation to the curve near the point). If the equation of the curve is $f(x, y)=0$ and $f(x, y)$ has continuous second partial derivatives, then at a point for which $f_x=f_y=0$ the curve has two distinct tangents, or no tangent at all, according as $f_{xx}f_{yy}-f_{xy}^2$ is less than, or greater than, zero (if this expression is zero, the curve may have a double tangent, as at a cusp). A point of a curve which is not an ordinary point is said to be a **singular point.** Cusps, crunodes, isolated points, and multiple points are singular points. (2) A point at which either the ordinate can be expressed as an analytic function of the abscissa in a neighborhood of the point or the abscissa as such a function of the ordinate.

piercing point of a line in space. Any one of the points where the line passes through one of the coordinate planes.

point charge. See CHARGE.

point circle (null circle) and **point ellipse.** See CIRCLE—null circle, and ELLIPSE.

point of contact. See TANGENCY—point of tangency.

point of discontinuity. A point at which a curve (or function) is not continuous. See CONTINUOUS and DISCONTINUITY.

point of division. The point which divides the line segment joining two given points in a given ratio. If the two given points have the Cartesian coordinates (x_1, y_1) and (x_2, y_2) and it is desired to find a point such that the distance from the first point to the new point, divided by the distance from the new point to the second point, is equal to r_1/r_2, the formulas giving the coordinates x and y of the desired point are

$$x=\frac{r_2x_1+r_1x_2}{r_1+r_2}, \quad y=\frac{r_2y_1+r_1y_2}{r_1+r_2}.$$

When r_1/r_2 is positive, the point of division lies between the two given points, and the division is said to be **internal**; the new point is said to divide the line segment **internally** in the ratio r_1/r_2. When this ratio is negative, the point of division must lie on the line segment extended, and it is then said to divide the line segment **externally** in the ratio $|r_1/r_2|$. When $r_1=r_2$, the new point bisects the line segment and the above formulas reduce to

$$x=\frac{x_1+x_2}{2}, \quad y=\frac{y_1+y_2}{2}.$$

For **points in space,** the situation is the same as in the plane except that the points now have three coordinates. The formulas for x and y are the same, and the formula for z is

$$z=\frac{r_2z_1+r_1z_2}{r_1+r_2}.$$

point at infinity. (1) See IDEAL—ideal point. (2) See INFINITY—point at infinity in the complex plane.

point of inflection. See INFLECTION.

point of osculation. A point at which two branches of a curve have a common tangent and lie on opposite sides of it. *E.g.*, $y^2=(x-1)^4$ has a point of osculation at $x=1$, the curve consisting of the two parabolas $y=(x-1)^2$ and $y=-(x-1)^2$, both tangent to the x-axis at the point $(1, 0)$ and extending upward and downward, respectively. See OSCULATION.

point-by-point plotting (graphing) of a function. See PLOT.

point-slope form of the equation of a straight line. See LINE—equation of a straight line.

point of tangency. See TANGENCY.

power of a point. See POWER—power of a point.

salient point. See SALIENT.

simple point. Same as ORDINARY POINT.

singular point. See above, ordinary point of a curve, and various headings under SINGULAR.

umbilical point on a surface. See UMBILICAL.

POISSON. Poisson distribution. A distribution whose frequency function is of the form $f(x)=\frac{m^xe^{-m}}{x!}$ for $x=0, 1, 2, \cdots$, where m is a parameter called the **mean** or

variance, the mean and variance of the Poisson distribution being equal. Often appears in observing events which are very improbable but which occur occasionally since so many trials occur, *e.g.*, traffic deaths, accidents, and radioactive emissions. Specifically, if $n \to \infty$ and $p \to 0$ in the *binomial distribution* in such a fashion that $np = m$, the binomial distribution becomes the Poisson distribution.

Poisson's differential equation. The partial differential equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=-u,$$

or $\nabla^2 v = -u$. See DIRICHLET—Dirichlet characteristic properties of the potential function.

Poisson's integral. The integral

$$\frac{1}{2\pi}\int_0^{2\pi} U(\phi)\,\frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}\,d\phi$$

or, for $\zeta = ae^{i\phi}$ and $z = re^{i\theta}$,

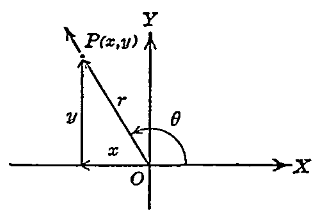
$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{\zeta+z}{\zeta-z}\right) U(\phi)\,d\phi,$$

which gives the value at the point $x = r\cos\theta$, $y = r\sin\theta$ of the function which is harmonic for $x^2+y^2<a^2$, continuous for $x^2+y^2 \leqq a^2$, and which coincides with the continuous boundary-value function $U(\phi)$ on $x^2+y^2=a^2$. More general boundary-value functions can be considered.

Poisson's ratio. The numerical value of the ratio of the strain in the transverse direction to the longitudinal strain. *E.g.*, a thin elastic rod, subjected to the action of a longitudinal stress T, undergoes a contraction e_1 in the linear dimensions of its cross section, and an extension e_2 in the longitudinal direction. The numerical value of the ratio $\sigma = \left|\frac{e_1}{e_2}\right|$ is **Poisson's ratio.** From Hooke's law, $T = Ee_2$, where E is *Young's modulus* in tension, so that $\sigma = -\frac{e_1 E}{T}$. For most structural materials Poisson's ratio has a value between $\frac{1}{4}$ and $\frac{1}{3}$.

PO'LAR, *adj., n.* Same as POLAR LINE (as a *noun*).

polar coordinates in the plane. The system of coordinates in which a point is located by its distance from a fixed point and the angle that the line from this point to the given point makes with a fixed line, called the **polar axis.** The fixed point, O in the figure, is called the **pole**; the distance, $OP = r$, from the pole to the given point, the **radius vector**; the angle θ (taken positive when counter-clockwise), the **polar angle** or **vectorial angle.** The polar coordinates of the point P are written (r, θ). The polar angle is sometimes called the **amplitude, anomaly** or **azimuth** of the point. From the figure it can be seen that the relations between rectangular Cartesian and polar coordinates are $x = r\cos\theta$, $y = r\sin\theta$. If r is positive (as in the figure), the amplitude, θ, of the point P is any angle (positive or negative) having OX as initial side and OP as terminal side. If r is negative, θ is any angle having OX as initial side and the extension of PO through O as terminal side. The point whose coordinates are $(1, 130°)$ or $(-1, -50°)$ is in the 2nd quadrant; the point whose coordinates are $(-1, 130°)$ or $(1, -50°)$ is in the 4th quadrant. See ANGLE.



polar coordinates in space. Same as SPHERICAL COORDINATES. See SPHERICAL.

polar distance. See CODECLINATION.

polar equation. An equation in polar coordinates. See CONIC, and LINE—equation of a straight line.

polar form of a complex number. The form a complex number takes when it is expressed in polar coordinates. This form is $r(\cos\theta + i\sin\theta)$, where r is the radius vector and θ the vectorial angle of the point represented by the complex number. The number r is called the **modulus** and the angle θ the **amplitude, argument,** or **phase.** *Syn.* Trigonometric form (representation) of a complex number. See COMPLEX—complex numbers, DE MOIVRE—De Moivre's theorem, and EULER—Euler's formula.

polar line or plane. See POLE—pole and polar of a conic, and pole and polar of a quadric surface.

polar line of a space curve. The line normal to the osculating plane of the curve at the center of curvature. *Syn.* Polar.

polar planimeter. See PLANIMETER.

polar of a quadratic form. The bilinear form obtained from a quadratic form $Q = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ $(a_{ij} = a_{ji})$ by the operator

$$\tfrac{1}{2}\sum_{i=1}^{n} y_i \frac{\partial}{\partial x_i},$$

i.e., the bilinear form

$$Q' = \sum_{i,j=1}^{n} a_{ij}y_ix_j.$$

If x and y are regarded as points in $n-1$ dimensions with homogeneous coordinates $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$, then $Q=0$ is the equation of a quadric and $Q'=0$ is that of the polar of y with respect to the quadric. See below, pole and polar of a conic.

polar reciprocal curves. See RECIPROCAL—polar reciprocal curves.

polar tangent. The segment of the tangent line to a curve cut off by the point of tangency and a line through the pole perpendicular to the radius vector. The projection of the polar tangent on this perpendicular is the **polar subtangent.** The segment of the normal between the point on the curve and this perpendicular is the **polar normal.** The projection of the polar normal on this perpendicular is the **polar subnormal.**

polar triangle of a spherical triangle. The spherical triangle whose vertices are poles of the sides of the given triangle, the poles being the ones nearest to the vertices opposite the sides of which they are poles. See POLE—pole of an arc of a circle on a sphere.

reciprocal polar figures. See RECIPROCAL.

PO'LAR-I-ZA'TION, *n.* **polarization of a complex of charges.** See POTENTIAL—concentration method for the potential of a complex.

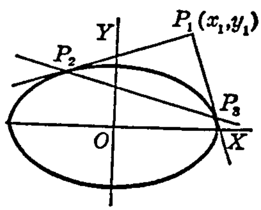
POLE, *n.* **pole of an analytic function.** See SINGULAR—isolated singular point of an analytic function.

pole of the celestial sphere. One of the two points where the earth's axis, produced, pierces the celestial sphere. They are called the *north* and *south celestial poles.* See HOUR—hour angle and hour circle.

pole of a circle on a sphere. A point of intersection of the sphere and the line through the center of the circle and perpendicular to the plane of the circle. The north and south poles are the poles of the equator. The poles of an **arc of a circle** on a sphere are the poles of the circle containing the arc.

pole of geodesic polar coordinates. See GEODESIC—geodesic polar coordinates.

pole and polar of a conic. A point and the line which is the locus of the harmonic conjugates of this point with respect to the two points in which a secant through the given point cuts the conic; a point and the line which is the locus of points conjugate (see CONJUGATE—conjugate points relative to a conic) to the given point. The point is said to be the **pole** of the line and the line the **polar** of the point. Analytically, the polar of a point is the locus of the equation obtained by replacing the coordinates of the point of contact in the equation of a general tangent to the conic by the coordinates of the given point. See CONIC—tangent to a general conic. *E.g.*, if a **circle** has the equation $x^2+y^2=a^2$, the equation of the *polar* of the point (x_1, y_1) is $x_1x+y_1y=a^2$. When a point lies so that two tangents can be drawn from it to the conic, the *polar* of the point is the secant through the points of contact of the tangents. The *polar line* of P_1 (in the figure), relative to the ellipse, is the line P_2P_3.



pole and polar of a quadric surface. A point (called the **pole** of the plane) and a plane (called the **polar** of the point) which is the locus of the harmonic conjugates of the point with respect to the two points in which a variable secant through the pole cuts the quadric. Analytically, the polar plane of a given point is the plane whose

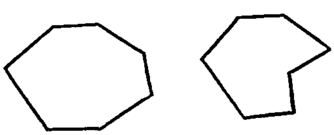
equation is that obtained by replacing the coordinates of the point of tangency in the general equation of a tangent plane by the coordinates of the given point. See TANGENT—tangent plane to a quadric surface. If, for instance, the quadric is an ellipsoid whose equation is $x^2/a^2+y^2/b^2+z^2/c^2=1$, the polar of the point (x_1, y_1, z_1) is the plane $x_1x/a^2+y_1y/b^2+z_1z/c^2=1$.

pole of stereographic projection. See PROJECTION—stereographic projection of a sphere on a plane.

pole of a system of coordinates. See POLAR—polar coordinates in the plane.

POL′I-CY, *n.* **annuity and insurance policies.** See ANNUITY and INSURANCE.

POL′Y-GON, *n.* A plane figure consisting of n points, $p_1, p_2, p_3, \cdots, p_n$, $n \geqq 3$, (called **vertices**) and of the line segments p_1p_2, $p_2p_3, \cdots, p_{n-1}p_n$, p_np_1 (called **sides**). In elementary geometry it is usually required that the sides have no common point except their end points. A polygon of 3 sides is a **triangle**; of 4 sides, a **quadrilateral**; of 5 sides, a **pentagon**; of 6 sides, a **hexagon**; of 7 sides, a **heptagon**; of 8 sides, an **octagon**; of 9 sides, a **nonagon**; of 10 sides, a **decagon**; of 12 sides, a **dodecagon**; of n sides, an **n-gon.** The plane area bounded by the sides of the polygon is the **interior** of the polygon. The **(interior) angles** of a polygon are the angles made by adjacent sides of the polygon and lying within the polygon. A polygon is **convex** if it lies on one side of any one of its sides extended; *i.e.*, if each interior angle is less than or equal to 180°.



CONVEX CONCAVE

A polygon is **concave** if it is not convex, *i.e.*, if at least one of its interior angles is larger than 180°. A polygon is concave if and only if there is a straight line which passes through the interior of the polygon and cuts the polygon in four or more points. A convex polygon always has an interior. A concave polygon has an interior if no side touches any other side, except at vertex, and no two vertices coincide (*i.e.*, if it is a *simple closed curve* or a *Jordan curve*). A polygon is **equiangular** if its interior angles are equal; it is **equilateral** if its sides are equal. A triangle is equiangular if and only if it is equilateral, but this is not true for polygons of more than three sides. A polygon is **regular** if its sides are equal and its interior angles are equal.

circumscribed and inscribed polygons. See CIRCUMSCRIBED.

diagonal of a polygon. A line segment joining any two nonadjacent vertices of the polygon.

frequency polygon. See FREQUENCY—frequency distribution.

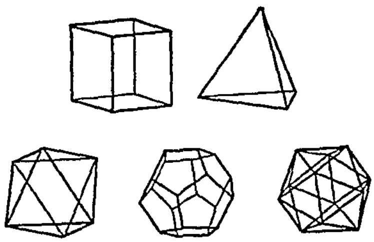
similar polygons. Polygons having their corresponding angles equal and their corresponding sides proportional. See SIMILAR.

spherical polygon. A portion of a sphere bounded by arcs of great circles.

POL′Y-HE′DRAL, *adj.* **polyhedral angle.** See ANGLE—polyhedral angle.

POL′Y-HE′DRON, *n.* A solid bounded by plane polygons. The bounding polygons are called the **faces**; the intersections of the faces, the **edges**; and the points where three or more edges intersect, the **vertices.** A polyhedron of four faces is a **tetrahedron**; one of six faces, a **hexahedron**; one of eight faces, an **octahedron**; one of twelve faces, a **dodecahedron**; and one of twenty faces, an **icosahedron.** A **convex polyhedron** is a polyhedron which lies entirely on one side of any plane containing one of its faces, *i.e.*, a polyhedron any plane section of which is a convex polygon. A polyhedron that is not convex is **concave.** For a concave polyhedron, there is at least one plane which contains one of its faces and is such that there is a part of the polyhedron on each side of the plane. A **simple polyhedron** is a polyhedron which is *topologically equivalent* to a sphere, a polyhedron with no "holes" in it. A **regular polyhedron** is a polyhedron whose faces are congruent regular polygons and whose polyhedral angles are congruent. There are only five regular polyhedrons: the regular **tetrahedron, hexahedron** (or cube), **octahedron, dodecahedron,** and **icosahedron.** These are shown in the figures. More generally, a

polyhedron may be an object which is homeomorphic to a set consisting of all the points which belong to simplexes of a *simplicial complex.*



circumscribed and inscribed polyhedrons. See CIRCUMSCRIBED.

diagonal of a polyhedron. See DIAGONAL—diagonal of a polyhedron.

Euler's theorem on polyhedrons. See EULER.

similar polyhedrons. Polyhedrons which can be made to correspond in such a way that corresponding faces are similar each to each and similarly placed and such that their corresponding polyhedral angles are congruent.

symmetric polyhedrons. Two polyhedrons each of which is congruent to the mirror image of the other.

POL'Y-NO'MI-AL, *adj., n.* A **polynomial in one variable** (usually called simply a **polynomial**) is a rational integral algebraic expression of the form $a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$, where a_i, $i=0, 1, 2, \cdots n$, are complex numbers (real or imaginary), and n is a positive integer. Sometimes n is only required to be nonnegative. A polynomial is said to be **linear, quadratic, cubic, quartic** (or **biquadratic**), etc., according as its degree is 1, 2, 3, 4, etc. A **polynomial in several variables** is an expression which is the sum of terms, each of which is the product of a constant and various nonnegative powers of the variables.

continuation of sign in a polynomial. See CONTINUATION.

degree of a polynomial. See DEGREE.

polynomial equation. See EQUATION—polynomial equation.

polynomials of Bernoulli, Hermite, Laguerre, Legendre, and Tchebycheff. See the respective names.

POOLED, *adj.* **pooled sum of squares.** (*Statistics.*) $S=\sum_{j=1}^{k}\sum_{i=1}^{n_j}(x_{ij}-\bar{x}_j)^2$, where $j=1, 2, \cdots, k$ over k samples and $i=1, 2, \cdots, n_j$, where n_j is the number of observations in the jth sample. Actually, the squared deviations around the individual sample means is the set of quantities that are summed, but standard statistical practice is to use the phrase *pooled sum of squares.*

POP'U-LA'TION, *n.* (*Statistics.*) The total set of items (actual or potential) defined by some characteristic of the items. Thus the totality of potential measurements of a rod of fixed length is a population; the totality of automobile tires produced under prescribed conditions is the population of that defined set. If the total set is infinitely large, it is called an **infinite population.**

POS'I-TIVE, *adj.* **positive angle.** See ANGLE.

positive correlation. See CORRELATION.

positive number. *Positive* and *negative numbers* are used to denote numbers of units taken in opposite directions or opposite senses. If a positive number denotes miles east, a negative number denotes miles west. *Tech.* If a is a positive number, then b is the negative number containing a units if $a+b=0$; positive a is written $+a$, or simply a, while b, the negative of a, is written $-a$. The set of real numbers is an *ordered field.* See FIELD—ordered field.

positive sign. Same as PLUS SIGN. See PLUS.

POS'TU-LATE, *n.* See AXIOM.

Euclid's postulates. (1) A straight line may be drawn between any two points. (2) Any terminated straight line may be produced indefinitely. (3) About any point as center a circle with any radius may be described. (4) All right angles are equal. (5) (The parallel postulate.) If two straight lines lying in a plane are met by another line, making the sum of the internal angles on one side less than two right angles, then those straight lines will meet, if sufficiently produced, on the side on which the sum of the angles is less than two right angles. (There is not complete agreement on how

many of Euclid's assumptions were designated as postulates, but these five are generally so recognized.)

PO'TEN-CY, *n.* **potency of a set.** See CARDINAL—cardinal number.

POTENTIAL, *adj., n.* The work done against a conservative field or its negative (depending on conventions) in bringing a unit of the proper sort from infinity to the point in question, or the value at the given point of a function whose directional derivative is equal in magnitude to the component of the field intensity in that direction at the given point. This concept is so extensive and well developed that it can only be described satisfactorily by enumerating its special instances. See various headings below.

concentration method for the potential of a complex. This method consists of selecting a point O inside the complex and expressing r_i in terms of quantities associated with O, namely, r (the distance from O to the field point), l_i (the distance from O to the charge e_i), and θ_i (the angle between r and l_i). Thus by the law of cosines and the binomial theorem we have

$$r_i^{-1}=(r^2+l_i^2-2rl_i\cos\theta_i)^{-1/2}=\frac{1}{r}+(l_i\cos\theta_i)/r^2+l_i(3\cos^2\theta_i-1)/(2r^3)+\cdots.$$

If we let λ_i be the vector from O to e_i, ρ_1 the unit vector pointing from O toward the field point, and μ_i the vector $e_i\lambda_i$, then $\mu_i\cdot\rho_1=e_il_i\cos\theta_i$ and therefore $\Sigma e_il_i\cos\theta_i=\Sigma\mu_i\cdot\rho_1=\mu\cdot\rho_1$, where $\mu=\Sigma\mu_i$. The quantity μ is called the **polarization of the complex.** If we multiply the foregoing equation for r_i^{-1} by e_i, sum on i, and denote the total charge Σe_i by e, we see that the potential assumes the form $e/r+(\mu\cdot\rho_1)/r^2+$ higher order terms. If the complex consists of but two charges equal in magnitude but opposite in sign and we call the negative charge e_1, then

$$\mu_1+\mu_2=e_2(-\lambda_1+\lambda_2)=\mu.$$

Now $-\lambda_1+\lambda_2$ is a vector from the negative charge e_1 to the positive charge e_2. Consequently, the polarization of this special complex (which in one sense is the electrical counterpart of a magnet) is a vector having the direction from the negative charge to the positive charge and having magnitude m equal to the magnitude of the positive charge times the distance between charges. A **doublet** or **dipole** is the abstraction obtained by allowing l_1 and l_2 to approach zero while e_2 $(e_2=-e_1)$ approaches infinity in such a way that μ remains constant. This limiting process eliminates the higher-order terms in the expansion of r_i^{-1}. Hence the potential of the doublet is given by the single term $(\mu\cdot\rho_1)r^{-2}$ or $(m\cos\theta)r^{-2}$, where θ is the angle between ρ_1 and μ. Returning to the more general complex of charges, we see that, except for terms involving the third and higher powers of $1/r$, its potential is that due to a single charge of magnitude e and a dipole of moment μ both located at O.

conductor potential. See CONDUCTOR.

Dirichlet characteristic properties of the potential function. See DIRICHLET.

electrostatic potential. See ELECTROSTATIC.

first, second, and third, boundary-value problems of potential theory. See BOUNDARY.

Gauss' mean value theorem for potential functions. See GAUSS—Gauss' mean value theorem.

gravitational potential of a complex of particles (Newtonian potential). The function obtained from $\Sigma e_i/r_i$ by replacing e_i with $-Gm_i$, where G denotes the gravitational constant and m_i the mass of the ith particle. Many writers omit the minus sign and compensate for it otherwise. When this is done it is the positive gradient of the potential that gives the field strength or force a unit mass would experience if placed at the point in question. If the minus sign is dropped and G is given the value unity, the *Newtonian potential function* for the set of point-masses is then $\Sigma m_i/r_i$.

kinetic potential. The difference between the kinetic energy and the potential energy. *Syn.* Lagrangian function.

potential energy. See ENERGY.

potential function for a double layer of distribution of dipoles on a surface. This potential function U is given by

$$U=\iint m\cos\theta r^{-2}\,dS.$$

Here m is the moment per unit area of the

dipole distribution and θ is the angle between the polarization vector and the vector to the field point (see concentration method for the potential of a complex). If m is not only continuous but is of class C and the polarization vector is normal to the surface, then

$$\lim \frac{\partial U}{\partial n}\bigg|_{\text{at } N} = \lim \frac{\partial U}{\partial n}\bigg|_{\text{at } M}$$

as M and N approach P. However, in this case U suffers a jump on passing through the surface; for, if m is continuous and M and N are points on the positive and negative sides of the normal through the surface point P, then $\lim U(M) = U(P)+2\pi m(P)$, while $\lim U(N) = U(P) - 2\pi m(P)$.

potential function for a surface distribution of charge or mass. The function U defined by $U = \iint \sigma/r\, dS$. Here σ is the surface density of charge or mass if σ is continuous. U is continuous but its normal derivative suffers a jump at the surface. More precisely, if we select a point P on the surface (but not on its edge if it is a surface patch), draw the normal through P, select two points M and N on the normal but on opposite sides of P, and compute the normal derivatives in the sense M to N at both M and N, then the limit as M and N approach P of

$$\partial U/\partial n|_{\text{at } N} - \partial U/\partial n|_{\text{at } M}$$

is $-4\pi\sigma$.

potential function relative to a given vector point function $\boldsymbol{\phi}$. A scalar point function S such that $\nabla S = \boldsymbol{\phi}$, or $-\nabla S = \boldsymbol{\phi}$, depending on the convention adopted. If $\boldsymbol{\phi}$ is the velocity, then S is called the **velocity potential.** See IRROTATIONAL—irrotational vector in a region.

potential function for a volume distribution of charge or mass. If we are given a continuous space distribution of charge or mass (*i.e.*, a density function instead of a discrete collection of points endowed with charge or mass), the potential function is $\iiint \rho/r\, dV$. Here ρ is a point function, say $\rho(X, Y, Z)$ in Cartesian coordinates, r is the distance from the charge point (X, Y, Z) to the field point (x, y, z), whereas the region of integration is the volume occupied by the charge. Thus in

$$\iiint \rho/r\, dX\, dY\, dZ,$$

the integration variables are the coordinates X, Y, Z, while the letters x, y, z appear as parameters. Consequently,

$$\iiint \rho/r\, dV$$

is a function of the field point variables x, y, z.

potential in magnetostatics. Work done by the magnetic field in repelling a unit positive pole from the given point to a point at infinity, or to a point which has been selected as a point of zero potential. The potential due to a distribution of magnetic material is essentially that of a similar dipole distribution.

potential theory. The theory of potential functions. From one point of view, it is the theory of Laplace's equation. Every harmonic function can be regarded as a potential function and the Newtonian potential functions are harmonic functions in free space.

spreading method for the potential of a complex. Instead of replacing the complex with a series of fictitious elements located at a single point, the spreading method replaces the set of point charges with a continuous distribution of charge characterized by a density function $\rho(x, y, z)$, or with both a density of charge and a density of polarization. If both charge and polarization are spread, simpler functions will suffice for a given degree of approximation than will if only charge is distributed. The potential in the two cases is taken to be $\iiint \rho/r\, dV$ if charge alone is spread, and otherwise

$$\iiint \rho/r\, dV + \iiint (m \cos\theta)/r^2\, dV.$$

Here m is the absolute value of the polarization per unit volume. If the polarization may be regarded as concentrated on a surface, the last integral should be replaced with the surface integral

$$\iint m \cos\theta r^{-2}\, dS,$$

in which m is the magnitude of the polarization per unit area.

vector potential relative to a given vector point function ϕ. A vector point function ψ such that $\nabla \times \psi = \phi$. See SOLENOIDAL—solenoidal vector in a region.

POUND, *n.* A unit of weight; the weight of one mass pound. See MASS and WEIGHT. Since weight varies slightly at different points on the earth, for extremely accurate work one pound of force is taken as the weight of a mass pound at sea level and 45° north latitude.

pound of mass. See MASS.

POUND'AL, *n.* A unit of force. See FORCE—unit of force.

POW'ER, *adj., n.* **Abel's theorem on power series.** See ABEL.

difference of like powers of two quantities. See DIFFERENTIAL.

differentiation of a power series. See SERIES—differentiation of an infinite series.

fitting by a power law. (*Statistical graphing.*) Determining the constants, a and n, in the equation $y = ax^n$ by means of a set of statistics which are known (or assumed) to approximately satisfy this type of equation. This problem is reduced to the problem of fitting a linear equation to the data, by taking the logarithm of both sides of this equation, which changes it to

$$\log y = n \log x + \log a \quad \text{or} \quad u = nv + c,$$

where $\log y = u$, $\log x = v$. and $\log a = c$.

integration of a power series. See SERIES—integration of an infinite series.

perfect ***n*****th power.** See PERFECT.

power. (*Physics.*) The rate at which work is done.

power of a number. See EXPONENT.

power of a point with reference to a circle or a sphere. *With reference to a circle,* the quantity obtained by substituting the coordinates of the point in the equation of the circle when written with the right-hand side equal to zero and the coefficients of the square terms equal to unity. This is equal to the algebraic product of the distances from the point to the points where any line through the given point cuts the circle (this product is the same for all such lines); it is equal to the square of the length of a tangent from the point to the circle when the point is external to the circle. *With reference to a sphere,* the power of a point is the power of the point with reference to any circle formed by a plane passing through the point and the center of the sphere. This is equal to the value obtained by substituting the coordinates of the point in the equation of the sphere when written with the right-hand side equal to zero and the coefficients of the square terms equal to unity. It is also the product of the distances from the point to the points where any line through the given point cuts the sphere, or the square of the length of a tangent from the point to the sphere if the point is external to the sphere.

power residue. See RESIDUE.

power series. See SERIES.

power of a set. See CARDINAL—cardinal number.

sums of like powers of two quantities. See SUM—sums of like powers of two quantities.

PRE-CI'SION, *n.* **double precision.** A term applied when, in a computation, two words (or storage positions) of a computing machine are used to denote a single number to more decimal places than would be possible with one storage position.

index of precision. See INDEX—index of precision.

PRE'MI-UM, *adj., n.* (1) The amount paid for the loan of money, in addition to normal interest. (2) The difference between the selling price and par value of stocks, bonds, notes, and shares, when the selling price is larger (*e.g.,* **premium bonds** are bonds which sell at a premium—for more than par value). Compare DISCOUNT. (3) One kind of currency is at a *premium* when it sells for more than its face value in terms of another; when one dollar in gold sells for $1.10 in paper money, gold is at a *premium* of 10 cents per dollar. (4) The amount paid for insurance. A **net premium** is a premium which does not include any of the company's operating expenses. Explicitly, **net annual premiums** are equal annual payments made at the beginning of each policy year to pay the cost of a policy figured under the following assumptions: all policyholders will die at a rate given by a standard (accepted) mortality table; the

insurance company's funds will draw interest at a certain given rate; every benefit will be paid at the close of the policy year in which it becomes due, and there will be no charge for carrying on the company's business; the **net single premium** is the present value of the contract benefits of the insurance policy. The **natural premium** is the net single premium for a one-year term insurance policy at a given age (this is the yearly sum required to meet the cost of insurance each year, not including the company's operating expenses). **Net level premiums** are fixed (equal) premiums (usually annual), which are equivalent over a period of years to the natural premiums over the same period. In the early years, the premiums are greater than the natural premium; in the later years, they are less. **Gross premium** (or **office premium**) is the premium paid to the insurance company; net premium, plus allowances for office expenses, medical examinations, agents' fees, etc., minus deductions due to income. **Installment premiums** are annual premiums payable in installments during the year. The single premium for an insurance policy is the amount which, if paid on the policy date, would meet all premiums on the policy. See RESERVE.

PRES'ENT, *adj.* **present value.** See VALUE—present value.

PRES'SURE, *n.* (*Physics.*) A force, per unit area, exerted over the surface of a body. See below, fluid pressure.

center of pressure. See CENTER—center of pressure.

fluid pressure. The force exerted per unit area by a fluid. The fluid pressure on a unit horizontal area (plate) at a depth h is equal to the product of the density of the fluid and h. The total force on a horizontal area at depth k is khA, where A is the area and k the density of the fluid. The total force on a nonhorizontal area is found by dividing the area into infinitesimal (differential) areas (horizontal strips, if the area is in a vertical plane) and summing the force on these strips by integration. See ELEMENT—element of integration. (The height of the fluid can be measured from any point in the element of area and the force on the element computed the same as if the element were in a horizontal plane at that depth. See DUHAMEL'S THEOREM.)

PRE-VAIL'ING, *adj.* **prevailing interest rate for a given investment.** The rate which is generally accepted for that particular type of investment at the time under consideration. *Syn.* Income rate, current rate, yield rate.

PRICE, *n.* The quoted sum for which merchandise or contracts (bonds, mortgages, stock, etc.) are offered for sale, or the price for which they are actually sold (the **selling price**). The price recorded in wholesale catalogues and other literature is the **list price** (it is usually subject to a discount to retail merchants). The **net price** is the price after all discounts and other reductions have been made. For bonds, the total payment made for a bond is the **flat price** or **purchase price**; the **quoted price** (or "**and interest price**") is the same as the *book value* of the bond (see VALUE). The theoretical value of the purchase price of a bond on a dividend date is the present value of the redemption price (usually face value) plus the present value of an annuity whose payments are equal to the dividends on the bond; between dividend dates, the purchase price is the sum of the price of the bond at the last interest date and the *accrued interest* (the proportionate part of the next coupon which is paid to the seller). The *flat price* is equal to the *quoted price* plus the *accrued interest*. The **redemption price** of a bond is the price that must be paid to redeem a bond. If a bond specifies that it may be redeemed at specified dates prior to maturity, the price at which it may be redeemed on such dates is the **call price.**

PRI'MA-RY, *adj.* **primary infinitesimal and infinite quantity.** See STANDARD—standard infinitesimal and infinite quantity.

PRIME, *adj., n.* **prime direction.** An initial directed line; a fixed line with reference to which directions (angles) are defined; usually the positive x-axis or the *polar axis.*

prime factors of a quantity. All the prime quantities (numbers, polynomials) that will exactly divide the given quantity. *E.g.,* (1) the numbers 2, 3 and 5 are the

prime factors of 30; (2) the quantities x, $(x+1)$ and $(x-1)$ are the prime factors of x^5-2x^3+x. See below, prime number, and prime polynomial.

prime meridian. See MERIDIAN.

prime number. An integer which has no integral factors except unity and itself, as 2, 3, 5, 7, or 11; 1 is usually excluded.

prime number theorem. Let $\pi(n)$ be the number of prime integers not greater than n. The prime number theorem states that the limit of the ratio of $\pi(n)$ and $n/\log_e n$, as n becomes infinite, is 1.

prime polynomial. A polynomial which has no polynomial factors except itself and constants. The polynomials $x-1$ and x^2+x+1 are prime. *Tech.* A polynomial which is *irreducible.* See IRREDUCIBLE—polynomials irreducible in a given field.

prime (or accent) as a symbol. The symbol (′) placed to the right and above a letter. (1) Used to denote the first derivative of a function: y' and $f'(x)$, called *y-prime* and *f-prime*, denote the first derivatives of y and $f(x)$. Similarly y'' and $f''(x)$, called *y double prime* and *f double prime*, denote second derivatives. In general, $y^{[n]}$ and $f^{[n]}(x)$ denote the nth derivatives. (2) Sometimes used on letters to denote constants, x' denoting a particular value of x, (x', y') denoting the particular point whose coordinates are x' and y', in distinction to the variable point (x, y). (3) Used to denote different variables with the same letters, as x, x', x'', etc. (4) Used to denote feet and inches, as 2′ 3″, read two feet and three inches. (5) Used to denote minutes and seconds in circular measurement of angles, as 3° 10′ 20″, read three degrees, ten minutes, and twenty seconds.

relatively prime quantities. Quantities which have no factor in common except unity. They are also said to be *prime to each other.*

PRIM′I-TIVE, *adj.* (1) A geometrical or analytic form from which another is derived; a quantity whose derivative is a function under consideration. (See DERIVATIVE.) (2) A function which satisfies a differential equation. (3) A curve of which another is the polar or reciprocal, etc.

primitive curve. A curve from which another curve is derived; a curve of which another is the polar, reciprocal, etc.; the graph of the primitive of a differential equation (a member of the family of curves which are graphs of the solutions of the differential equation). See INTEGRAL—integral curves.

primitive of a differential equation. See DIFFERENTIAL—solution of a differential equation.

primitive element of a monogenic analytic function. See MONOGENIC—monogenic analytic function.

primitive *n*th root of unity. See UNITY—root of unity.

primitive period of a periodic function of a complex variable. See PERIODIC—periodic function of a complex variable, and various headings under PERIOD.

PRIN′CI-PAL, *adj., n.* Most important or most significant. In *finance,* money put at interest, or otherwise invested.

principal diagonal. See DETERMINANT, MATRIX, and PARALLELEPIPED.

principal ideal. See IDEAL.

principal meridian. See MERIDIAN.

principal normal. See NORMAL—normal to a curve or surface.

principal part of the increment of a function. See INCREMENT—increment of a function.

principal parts of a triangle. The sides and interior angles. The other parts, such as the bisectors of the angles, the altitudes, the circumscribed and inscribed circles, are called **secondary parts.**

principal plane of a quadric surface. A plane of symmetry of the quadric surface.

principal radii of curvature at a point on a surface. The two radii of curvature of the normal sections whose curvatures are respectively the maximum and minimum of the curvatures of all normal sections at the point. These two sections are orthogonal at the point.

principal root of a number. The positive real root in the case of roots of positive numbers; the negative real root in the case of odd roots of negative numbers. Every number has two square roots, three cube roots, and in general n nth roots.

principal value of an inverse trigonometric function. See TRIGONOMETRIC—inverse trigonometric functions.

PRIN-CI′PI-A, *n.* One of the greatest scholarly works of all time, written by Sir

Isaac Newton, and first printed in London in 1687 under the title *Philosophiae Naturalis Principia Mathematica.* This work lies at the base of all present structure of mechanics of rigid and deformable bodies and mathematical astronomy.

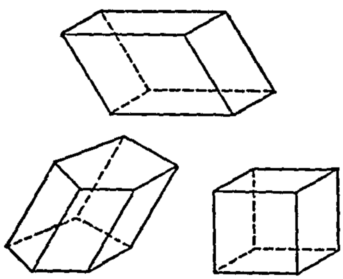
PRIN'CI-PLE, *n.* A general truth or law, either assumed or proved. See AXIOM—axiom of continuity, DUALITY, ENERGY—principle of energy, PROPORTIONAL—proportional parts, and SAINT-VENANTS' PRINCIPLE.

principle of the maximum. The principle that if $f(z)$ is a regular analytic function of the complex variable z in the domain D, and $f(z)$ is not a constant, then $f(z)$ does not attain a maximum absolute value at any interior point of D.

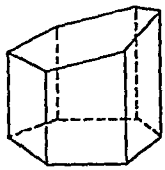
principle of the minimum. The principle that if $f(z)$ is a nonvanishing regular analytic function of the complex variable z in the domain D, and if $f(z)$ is not constant, then $|f(z)|$ does not take on a minimum value at any interior point of D. Note that if $f(z)=z$, $|f(z)|$ does take on a minimum at the origin.

PRINGSHEIM'S THEOREM on double series. See SERIES—double series.

PRISM, *n.* A polyhedron with two congruent and parallel faces, called the **bases,** and whose other faces, called **lateral faces,**



are parallelograms formed by joining corresponding vertices of the bases; the intersections of lateral faces are called **lateral edges.** A **diagonal** is any line segment joining two vertices that do not lie in the same face or base. The **altitude** is the perpendicular distance between the bases. The **lateral area** is the total area of the lateral faces (equal to an edge times the perimeter of a right section), and its volume is equal to the product of its base and its altitude. A prism with a triangle as base is called a **triangular prism**; one with a quadrilateral as base is called a **quadrangular prism**; etc. A **right prism** is a prism whose bases are perpendicular to the lateral edges; a right prism is a **regular prism** if the bases are regular polygons. A **truncated prism** is a portion of a prism lying between two nonparallel planes which cut the prism and have their line of intersection outside the prism. A **right truncated prism** is a truncated prism in which one of the cutting planes is perpendicular to a lateral edge.



circumscribed and inscribed prisms. See CIRCUMSCRIBED.

right section of a prism. A plane section perpendicular to the lateral faces of the prism.

PRIS-MAT'IC, *adj.* **prismatic surface.** A surface generated by a moving straight line which always intersects a broken line lying in a given plane and is always parallel to a given line not in the plane. When the broken line is closed, the surface is called a **closed prismatic surface.**

PRIS'MA-TOID, *adj., n.* A polyhedron whose vertices all lie in one or the other of two parallel planes. The faces which lie in the parallel planes are the **bases** of the prismatoid and the perpendicular distance between the bases is the **altitude.** See PRISMOIDAL—prismoidal formula.

prismatoid formula. Same as the PRISMOIDAL FORMULA.

PRIS'MOID, *n.* A prismatoid whose bases are polygons having the same number of sides, the other faces being trapezoids or parallelograms.

PRIS-MOI'DAL, *adj.* **prismoidal formula.** The volume of a prismatoid is equal to one-sixth of the altitude times the sum of the areas of the bases and four times the area of a plane section midway between the bases: $V=\frac{1}{6}h(B_1+4B_m+B_2)$. This formula also gives the volume of any solid having two parallel plane bases, whose cross-sectional area (by a plane parallel to the bases) is given by a linear, quadratic, or cubic function of the distance of the cross section from one of the bases (an elliptic cylinder, and quadratic cone, satisfy these conditions). The prismoidal formula is sometimes given as $V=\frac{1}{4}h(B_1+3S)$, where S is the area of a section parallel to the base and $\frac{2}{3}$ the distance from B_1 to B_2. This is equivalent to the preceding form. See SIMPSON'S RULE.

PROB'A-BIL'I-TY, *adj., n.* (1) Let n be the number of exhaustive, mutually exclusive, and equally likely cases of an event under a given set of conditions. If m of these cases are known as the event A, then the **(mathematical** or ***a priori*****) probability** of event A under the given set of conditions is m/n. *E.g.*, if one ball is to be drawn from a bag containing two white balls and three red balls, the probability of drawing a white ball is $\frac{2}{5}$ and the probability of drawing a red ball is $\frac{3}{5}$. This definition is circular in that *equally likely* means *equally probable*, but the intuitive meaning is useful. (2) If, in a *random sequence* of n trials of an event with m favorable events, the ratio m/n as n increases indefinitely has the limit P, then P is the **probability** of the event m. This is sometimes amended to state that, if it is practically certain that m/n is approximately equal to P when n is very large, P is the probability of the event m. (3) A third general type of definition is based on certain axiomatic statements and leaves the problem of applying these to empirical situations to the ingenuity of the practicing statistician. If, in a given set of N elements, N_1 are considered elements A, then the probability of an element of the set being A is N_1/N. All the above definitions have either logical or empirical difficulties. They usually give the same numerical values for the usual empirical problems. See below, empirical or *a posteriori* probability.

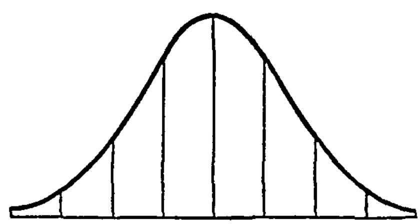
empirical or *a posteriori* probability. If in a number of trials an event has occurred n times and failed m times, the *probability* of its occurring in the next trial is $n/(n+m)$. It is assumed, in determining *empirical probability*, that there is no known information relative to the probability of the occurrence of the event other than the past trials. The probability of a man living through any one year, based upon past observations as recorded in a mortality table, is *empirical probability*. See BAYES' THEOREM.

inverse probability. See BAYES' THEOREM.

mathematical or *a priori* probability. See above, PROBABILITY (1).

probability convergence. Let $x_1, x_2, x_3, \cdots$ be a sequence of random variables (*e.g.*, the means of samples of size $1, 2, 3, \cdots$). Then x_n **converges in probability** to a constant k if, for any $\epsilon>0$, the probability of $|x_n-k|>\epsilon$ tends to zero as $n\to\infty$.

probability curve. A curve of the type shown in the figure; the locus of any curve



of the type $y=ke^{-h^2(x-A)^2}$ where A, h, and k are constants. *Syn.* Error curve, normal distribution curve, normal frequency curve. See FREQUENCY—normal frequency curve.

probability density function. Let the range of a continuous variable x be the interval (a, b). Let (c, d) be an interval in (a, b). Let $P(c, d)$ denote the probability that x has a value in the interval (c, d). Let it be assumed that (1) $P(c, d)\geqq 0$, (2) $P(a, b)=1$, (3) $P(c, d)=P(c, e)+P(e. d)$ if $c<e<d$, (4) when (c, d) is infinitesimal, $P(c, d)$ is also infinitesimal (of the same or higher order). Then $P(c, c)=0$ and $P(c, d)$ is a continuous function of c and d. Let $\lim_{\Delta x\to 0}\frac{P(x, x+\Delta x)}{\Delta x}=p(x)$; then $p(x)$ is called the **probability density** of x, or the **probability function.** Also,

$$\int_c^d p(x)\,dx=P(c, d)$$

and $\int_a^b p(x)\,dx = 1$. If $p(x)=0$ for those values of x which are not possible, then

$$\int_{-\infty}^{\infty} p(x)\,dx = 1.$$

The probability of $x \leqq x_0$ is the **relative distribution function,** often called a **distribution function.** It is equal to

$$\int_{-\infty}^{x_0} p(x)\,dx.$$

The probability density function is often called the **relative frequency function,** or merely the **frequency function.** Usually from the context it is apparent whether the relative or absolute frequency function is meant. The above concepts can be generalized, a **probability measure** for a set S being a nonnegative real measure for which $m(S)=1$. The above measure $P(c, d)$ of intervals (c,d) can be extended to be a measure for the σ-ring generated by the intervals. Then $m(R)=\int_R p(x)dx$. See MEASURE—measure of a set.

probability limit. T is the *probability limit* of the statistic t_n, derived from a random sample of n observations, if the probability of $|t_n - T| < \epsilon$ approaches 1 as a limit as $n \to \infty$, for any $\epsilon > 0$. See above, probability convergence.

probability of the occurrence of an event in a number of repeated trials. (1) The probability that an event will happen *exactly r times* in n trials, for which p is the probability of its happening and q of its failing in any given trial, is given by the formula $n!p^r q^{n-r}/[r!(n-r)!]$, which is the $(n-r+1)$th term in the expansion of $(p+q)^n$. The probability of throwing exactly two aces in five throws of a die is

$$5!(\tfrac{1}{6})^2(\tfrac{5}{6})^3/(2!3!) = .16+.$$

(2) The probability that an event will happen *at least r times* in n throws is the probability that it will happen every time plus the probability that it will happen exactly $n-1$ times, $n-2$ times, etc., to exactly r times. This probability is given by the sum of the first $n-r+1$ terms of the expansion of $(p+q)^n$.

probability paper. Graph paper, one axis of which is scaled so that the graph of the *cumulative frequency* of the *normal distribution function* forms a straight line.

PROB'A-BLE, *adj.* Likely to be true or to happen.

probable deviation, or **probable error.** See DEVIATION.

PROB'LEM, *n.* A question proposed for solution; a matter for examination; a proposition requiring an operation to be performed or a construction to be made, as to bisect an angle or find a cube root of 2. See ACCUMULATION—accumulation problem, DIDO'S PROBLEM, DISCOUNT—discount problem under compound interest, FOUR—four color problem, and APOLLONIUS.

problem formulation. In numerical analysis, problem formulation is the process of deciding what information the customer really wanted, or should have wanted, and then of writing this in mathematical terms preparatory to programming the problem for machine solution. See PROGRAMMING—programming for a computing machine.

PRO'CEEDS, *n.* (1) The sum of money obtained from a business transaction or enterprise. The proceeds of a farm for a year is the sum of all the money taken in during the year; the proceeds of a sale of goods is the money received in return for the goods. The amount of money left after deducting all discounts and expenses from the proceeds of a transaction is called the **net proceeds.** (2) The difference between the face of a note, or other contract to pay, and the discount; the balance after interest, in advance, has been deducted from the face of the note.

PRO-DUCE', *v.* **produce a line.** To continue the line. *Syn.* Prolong, extend.

PROD'UCT, *adj., n.* The **product** of two or more objects is the object which is determined from these objects by a given operation called **multiplication.** See various headings below (particularly, *product of real numbers*) and under MULTIPLICATION. Also see COMPLEX—complex numbers, and SERIES—multiplication of series.

Cartesian product. The Cartesian product of two sets A and B is the set (denoted by $A \times B$) of all pairs (x, y) such that x is a member of A and y is a member of B. If multiplication, addition, or multiplication by scalars is defined for each of the sets A

and B, then the same operation can be defined for $A \times B$ by

$$(x_1, y_1)\cdot(x_2, y_2) = (x_1\cdot x_2, y_1\cdot y_2),$$
$$(x_1, y_1)+(x_2, y_2) = (x_1+x_2, y_1+y_2),$$
$$a(x, y) = (ax, ay).$$

If A and B are **groups**, their Cartesian product is a group. A matrix representation of the Cartesian product of two groups is given by the *direct product* of corresponding matrices in representations of the two groups. If a group G has subgroups H_1 and H_2 such that H_1 and H_2 have only the identity in common, each element of G is a product of an element of H_1 and an element of H_2, and each element of H_1 commutes with each element of H_2, then G is isomorphic with the Cartesian product $H_1 \times H_2$. If A and B are **rings**, then $A \times B$ is a ring. If A and B are **vector spaces** (with the same scalar multipliers), then $A \times B$ is a vector space. If A and B are **topological spaces**, then $A \times B$ is a topological space if a set in $A \times B$ is defined to be open if it is a Cartesian product $U \times V$, where U and V are open sets in A and B. If A and B are **topological groups** (or **topological vector spaces**), then $A \times B$ is a topological group (or a topological vector space). If A and B are **metric spaces**, the usual distance relation for $A \times B$ is

$$d[(x_1, y_1), (x_2, y_2)] = [d(x_1, x_2)^2 + d(y_1, y_2)^2]^{1/2}.$$

With this definition, the Cartesian product $R \times R$, where R is the space of real numbers, is the two-dimensional space of all points (x, y) with the usual distance of plane geometry. If A and B are **normed vector spaces**, $A \times B$ is a normed vector space if the norm is defined by

$$\|(x, y)\| = [\|x\|^2 + \|y\|^2]^{1/2}.$$

Many other definitions are used, such as $\|(x, y)\| = \|x\| + \|y\|$. If A and B are **Hilbert spaces**, then $A \times B$ is a Hilbert space if the norm is defined as above or, equivalently, if the *inner product* of (x_1, y_1) and (x_2, y_2) is defined to be the sum of the inner product of x_1 and x_2 and the inner product of y_1 and y_2. The above definitions can be extended in natural ways to the product of any finite number of spaces. The Cartesian product of sets X_a, where a is a member of an index set A, is the set of all functions x defined on A for which $x(a)$ is a member of X_a for each a of A. This means that a point of the product space is a set consisting of a point chosen from each of the sets X_a, the point $x(a)$ being the *a-th coordinate* of the point x of the product. If each of the sets X_a is a **topological space**, then their Cartesian product is a topological space if an open set is defined to be any set which is a union of sets which are Cartesian products of sets Y_a, where $Y_a = X_a$ for all but a finite number of members of A and Y_a is an open set of X_a for the other members of A. For a Cartesian product of a finite number of topological spaces $X_1, X_2, \cdots, X_n$, a set is open in the product if and only if it is a product of sets $U_1, U_2, \cdots, U_n$, where U_k is open in X_k for each k. With this topology for the Cartesian product, it can be shown that the Cartesian product is compact (*i.e.*, bicompact) if and only if each X_a is compact (this is called the **Tychonoff theorem**). Sometimes the elements of the Cartesian product of a nonfinite number of spaces are restricted by some convergence requirement. *E.g.*, the Cartesian product of Hilbert spaces $H_1, H_2, \cdots$ is the set of all sequences $h = (h_1, h_2, \cdots)$ for which h_n belongs to H_n for each n and $\|h\|$ is finite, where

$$\|h\| = [\|h_1\|^2 + \|h_2\|^2 + \cdots]^{1/2}.$$

A Cartesian product is sometimes called a **direct product** or a **direct sum.**

continued product. See CONTINUED.

direct product of matrices. The direct product of square matrices A and B (not necessarily of the same order) is the matrix whose elements are the products $a_{ij}b_{mn}$ of elements of A and B, where i, m are row indices and j, n are column indices; the row containing $a_{ij}b_{mn}$ precedes that containing $a_{i'j'}b_{m'n'}$, if $i < i'$, or if $i = i'$ and $m < m'$, and similarly for columns. Other ordering conventions are sometimes used.

infinite product. A product which contains an unlimited number of factors. An infinite product is denoted by a capital pi, Π; *e.g.*, $\Pi[n/(n+1)] = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdots$ is an infinite product. An infinite product $u_1 \cdot u_2 \cdot \cdots u_n \cdots$ is said to **converge** if it is possible to choose k so that the sequence

$$u_k,\ u_k\cdot u_{k+1},\ u_k\cdot u_{k+1}\cdot u_{k+2}, \cdots$$

converges to some limit which is not zero.

When the product becomes infinite, or if the above sequence approaches zero for all k, it is said to diverge. When there is a k such that the sequence above neither approaches a limit nor becomes infinite, it is said to oscillate. Because of certain relations to infinite series, infinite products are frequently written in the form $\Pi(1+a_n)$. A necessary and sufficient condition for the convergence of $\Pi(1+a_n)$ and $\Pi(1-a_n)$, if each $a_n>0$, is the convergence of Σa_n. If the series Σa_n^2 is convergent, the infinite product converges if, and only if, Σa_n converges. An infinite product, $\Pi(1+a_n)$, is said to **converge absolutely** if $\Sigma|a_n|$ is absolutely convergent. An absolutely convergent infinite product is convergent. The factors of a convergent infinite product can be rearranged in any way whatever without changing the limit of the product if, and only if, this product converges absolutely.

limit of a product. See LIMIT—fundamental theorems on limits.

partial product. See PARTIAL—partial product.

product of determinants, polynomials, and vectors. See the corresponding headings under MULTIPLICATION.

product formulas. See TRIGONOMETRY—identities of plane trigonometry.

product of matrices. The product AB of matrices A and B is the matrix whose elements are determined by the rule that the element c_{rs} in row r and column s is the sum over i of the product of the element a_{ri} in row r and column i of A by the element b_{is} in row i and column s of B:

$$c_{rs}=\sum_{i=1}^{n} a_{ri}b_{is}.$$

This product is defined only if the number n of columns in A is equal to the number of rows in B. Matrix multiplication is associative, but not commutative. The product of a **scalar c and a matrix** A is the matrix whose elements are the products of c and the corresponding elements of A. The determinant of cA (when A is a square matrix of order n) is equal to the product of c^n and the determinant of A.

product moment. See MOMENT—product moment.

product moment correlation coefficient. See CORRELATION—normal correlation.

product of real numbers. Positive integers (and zero) can be thought of as symbols used to describe the "manyness" of sets of objects (also see PEANO—Peano's postulates). Then the **product of two integers** A and B (denoted by $A\times B$, $A\cdot B$ or AB) is the integer which describes the "manyness" of the set of objects obtained by combining A sets, each of which contains B objects (or by combining B sets, each of which contains A objects: $AB=BA$). *E.g.*,

$$3\cdot 4=4+4+4=3+3+3+3=12$$

(see SUM—sum of real numbers). Also, $0\cdot 3=3\cdot 0=0+0+0=0$. The **product of two fractions** $\frac{a}{b}$ and $\frac{c}{d}$ $\left(\text{an integer } n \text{ may be regarded as the fraction } \frac{n}{1}\right)$ is defined by

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

The same rule applies if some of a, b, c, d are fractions. *E.g.*,

$$\frac{3}{5}\cdot\frac{1}{2}=\frac{3}{10};\quad \frac{\frac{2}{3}}{\frac{1}{5}}\cdot\frac{3}{\frac{1}{2}}=\frac{6/3}{1/10}=20$$

[the last equality results from multiplying numerator and denominator by 10 and then dividing by 3, which is valid since $\frac{a}{b}=\frac{ak}{bk}$ for any numbers a, b, k (b and k not zero)]. Multiplication by a fraction a/b can be interpreted as dividing the other factor into b equal parts and taking a of these ($\frac{1}{2}$ is equal to the sum of 5 addends, each equal to $\frac{1}{10}$, so $\frac{3}{5}\cdot\frac{1}{2}$ is $\frac{1}{10}+\frac{1}{10}+\frac{1}{10}=\frac{3}{10}$). The **product of mixed numbers** can be obtained by multiplying each term of one number by each term of the other, or by reducing each mixed number to a fraction. *E.g.*,

$$(2\tfrac{1}{2})(3\tfrac{2}{3})=(2+\tfrac{1}{2})(3+\tfrac{2}{3})=6+\tfrac{4}{3}+\tfrac{3}{2}+\tfrac{2}{6}=9\tfrac{1}{6}$$
$$=(\tfrac{5}{2})(\tfrac{11}{3})=\tfrac{55}{6}.$$

Two **decimals** can be multiplied by reducing them to fractions, or by ignoring the decimal point and multiplying as if the decimals were whole numbers and then pointing off as many decimal places in this product as there are in both multiplicand and multiplier together. The meaning and application of this rule are shown by the example

$$2.3\times .02=\tfrac{23}{10}\times\tfrac{2}{100}=\tfrac{46}{1000}=.046.$$

When the numbers being multiplied have been given signs, the multiplication is done by multiplying the numerical values of the numbers and making the result positive, if both numbers are positive or both are negative, and making it negative if the numbers have different signs (this is sometimes called **algebraic multiplication**). The rule of signs is: *Like signs give plus, unlike give minus. E.g.*, $2\times(-3)=-6$, $-2\times 3=-6$, $-2\times(-3)=6$. An explanation of this rule is that $a(-b)$ is the number which added to ab will give zero (*i.e.*, the *negative* or *additive inverse* of ab), since

$$ab+a(-b)=a[b+(-b)]=a\cdot 0=0.$$

Likewise, $(-a)(-b)$ is the number which added to $a(-b)$ will give zero, *i.e.*,

$$(-a)(-b)=ab.$$

The **product of irrational numbers** may be left in indicated form after similar terms have been combined, until some specific application indicates the degree of accuracy desired. Such a product as $(\sqrt{2}+\sqrt{3})(2\sqrt{2}-\sqrt{3})$ would be left as $1+\sqrt{6}$. A product such as $\pi\sqrt{2}$ can be approximated as

$$(3.1416)(1.4142)=4.443.$$

It is necessary to have a specific definition of irrational numbers before one can specifically define the product of numbers one or more of which is irrational. See DEDEKIND CUT.

product of sets and spaces. See above, Cartesian product, and INTERSECTION.

product of the sum and difference of two quantities. Such products as $(x+y)(x-y)$ used in factoring, since this product is equal to x^2-y^2.

products of inertia. See MOMENT—moment of inertia.

scalar and vector products. See MULTIPLICATION—multiplication of vectors.

PRO'FILE, *adj.* **profile map.** A vertical section of a surface, showing the relative altitudes of the points which lie in the section.

PROF'IT, *n.* The difference between the price received and the sum of the original cost and the selling expenses, when the price received is the larger. The selling expenses include storage, depreciation, labor, and sometimes accumulations in a reserve fund. This is sometimes called the **net profit**; the **gross profit** is the difference between the selling price and the original cost. Also see headings under PER CENT.

PRO'GRAM-MING, *n.* **convex programming.** The particular case of nonlinear programming in which the function to be extremized, and also the constraints, are appropriately convex or concave functions of the x's. See below, linear programming, quadratic programming.

dynamic programming. The mathematical theory of multistage decision processes.

linear programming. The mathematical theory of the minimization or maximization of a linear function subject to linear constraints. As often formulated, it is the problem of minimizing the linear form $\sum_{i=1}^{n} a_i x_i$, $x_i \geqq 0$, subject to the linear constraints $\sum_{i=1}^{n} b_{ij}x_i=c_j$, $j=1,2,\cdots,m$. See TRANSPORTATION—Hitchcock transportation problem. The analogous mathematical theory for which the function to be extremized and the constraints are not all linear is called **nonlinear programming.** A **solution** of a linear programming problem is any set of values x_i that satisfy the m linear constraints; a solution consisting of nonnegative numbers is called a **feasible solution**; a solution consisting of m x's for which the matrix of coefficients in the constraints is not singular, and otherwise consisting of zeros, is a **basic solution**; a feasible solution that minimizes the linear form is called an **optimal solution.** See below, quadratic programming, and SIMPLEX—simplex method.

programming for a computing machine. In preparing a problem for machine solution, programming is the process of planning the logical sequence of steps to be taken by the machine. It usually, but not necessarily, involves the preparation of flow charts. This process usually follows problem formulation and precedes coding. See CODING, CHART—flow chart, and PROBLEM—problem formulation.

quadratic programming. The particular case of nonlinear programming in which

the function to be extremized, and also the constraints, are quadratic functions of the x's wherein the second-degree terms constitute appropriately semidefinite quadratic forms. See above, convex programming.

PRO-GRES′SION, *n.* **arithmetic progression.** A sequence of terms each of which is equal to the preceding plus a fixed constant. See ARITHMETIC—arithmetic progression.

geometric progression. A sequence of terms such that the ratio of each term to the immediately preceding one is the same throughout the sequence. The general form is usually written: $a, ar, ar^2, \cdots ar^{n-1}$, where a is the **first term,** r the **common ratio** (or simply **ratio**), and ar^{n-1} the last or nth term. The indicated sum of such terms is sometimes called a *geometric progression,* although *geometric series* is more common. See SERIES—geometric series.

harmonic progression. See HARMONIC.

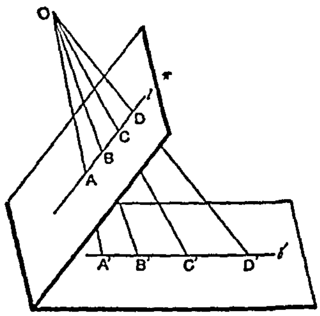
PRO-JEC′TILE, *n.* **path of a projectile.** See PARABOLA—parametric equations of a parabola.

PRO-JECT′ING, *adj.* **projecting cylinder.** A cylinder whose elements pass through a given curve and are perpendicular to one of the coordinate planes. There are three such cylinders for any given curve, unless the curve lies in a plane perpendicular to a coordinate plane, and their equations in rectangular Cartesian coordinates can each be obtained by eliminating the proper one of the variables x, y, and z between the two equations which define the curve. The space curve, a circle, which is the intersection of the sphere $x^2+y^2+z^2=1$ and the plane $x+y+z=0$ has the three projecting cylinders whose equations are $x^2+y^2+xy=\frac{1}{2}$, $x^2+z^2+xz=\frac{1}{2}$, and $y^2+z^2+yz=\frac{1}{2}$. These are elliptic cylinders.

projecting plane of a line in space. A plane containing the line and perpendicular to one of the coordinate planes. There are three projecting planes for every line in space, unless the line is perpendicular to a coordinate axis. The equation of each projecting plane contains only two variables, the missing variable being the one whose axis is parallel to the plane. These equations can be derived as the three equations given by the double equality of the symmetric space equation of the straight line. See LINE—equation of a straight line in space.

PRO-JEC′TION, *n.* **center of projection.** See below, central projection.

central projection. A projection of one configuration (A, B, C, D, in figure) on a given plane (called the **plane of projection**) in which the projection in this plane (A', B', C', D') is formed by the intersections, with this plane, of all lines passing through a fixed point (not in the plane) and the various points in the configuration. The image on a photographic film is a projection of the image being photographed, if the lens is considered as a point. The point is called the **center of projection** and the lines, or rays, are called the **projectors.** When the center of projection is a point at infinity (when the rays are parallel), the projection is called a **parallel projection.** See below, orthogonal projection.

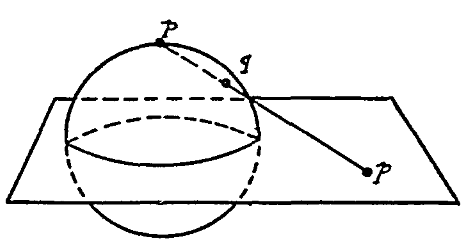


orthogonal projection. A projection of one configuration into a given line or plane for which the resulting configuration is formed by the intersections with the line or plane of lines perpendicular to the line or plane and passing through points of the given configuration. The projection of a **point** is the foot of the perpendicular from the point to the line or plane; the projection of a **line segment** is the line segment which joins the projections of the end points of the given line segment; the projection of a **vector (force, velocity, etc.)** is the vector whose initial and terminal points are the projections of the initial and terminal points, respectively, of the given vector; the projection of a **directed broken line** into a

line is the directed line segment whose initial and terminal points are, respectively, the projections of the initial point of the first segment of the directed broken line and the terminal point of the last segment (sometimes one speaks of the signed length of this projection as the *projection* of the directed broken line). The projection of a **curve** (or **polygon**), **surface**, or **solid** is the configuration consisting of the projections of the points of the curve (or polygon), surface, or solid.

projection of a vector space. A transformation P of the vector space into itself which is *linear* (*i.e.*, *additive* and *homogeneous*) and *idempotent* ($P \cdot P \equiv P$). If P is a projection of the vector space T, then there are vector spaces M and N contained in T such that each element of T is uniquely representable as the sum of an element of M and an element of N. Explicitly, M is the *range* of P and N is the *null space* of P (the space of all vectors x such that $P(x)=0$). It is said that P **projects** T **onto** M along N. If T is a **Banach space**, then P is *continuous* if and only if there is a positive number ϵ such that $\|x-y\| \geqq \epsilon$ if x and y are vectors of unit norm (length) belonging to M and N, respectively, or (equivalently) if there is a constant M such that $\|P(x)\| \leqq M\|x\|$ for each x. If T is a Hilbert space, then P is said to be an **orthogonal projection** (or sometimes simply a *projection*) if $\|P(x)\| \leqq \|x\|$ for each x, or (equivalently) if M and N are orthogonal.

stereographic projection of a sphere on a plane. For a given point P, called the **pole,** on the surface of a sphere S, and for a given plane π not passing through P, and perpendicular to a diameter through P, the line joining P with a variable point p on π intersects S in a second point q.



This mapping of the points q of the sphere S on the points p of π is called a **stereographic projection** of S on π. If an ideal "point at infinity" is adjoined to the plane π, to correspond to P, then the correspondence between the points of S and those of π is one-to-one. The map is a conformal one, much used in the theory of functions of a complex variable. The plane π is often taken as the equatorial plane of S relative to P, or as the tangent plane to S diametrically opposite P.

PRO-JEC'TIVE, *adj.* **projective geometry.** The study of those properties of geometric configurations which are invariant under projection.

projective plane. The set of all number triples (x_1, x_2, x_3), except $(0, 0, 0)$, with the convention that $(x_1, x_2, x_3)=(y_1, y_2, y_3)$ if there are two nonzero numbers a and b such that $ax_i=by_i$ for $i=1, 2, 3$. Points with $x_3 \neq 0$ can be regarded as points of the *Euclidean plane* with abscissa x_1/x_3 and ordinate x_2/x_3; points with $x_3=0$ are called *points at infinity* or *ideal points* (see IDEAL—ideal point). Each "direction" in the Euclidean plane determines a single point at infinity and the projective plane is topologically equivalent to a disc (a circle and its interior) with the two end points of each diameter identified. It is also topologically equivalent to a *sphere with one cross-cap.*

projective relation. Two fundamental forms are in *projection relation* and are said to form a *projectivity* if a one-to-one correspondence exists between the elements of the two such that each four harmonic elements of one correspond to four harmonic elements of the other.

PROJECTIVITY. See PROJECTIVE—projective relation.

PRO-JEC'TORS, *n.* See PROJECTION—central projection.

PRO'LATE, *adj.* **prolate cycloid.** A *trochoid* which has loops.

prolate ellipsoid of revolution. See ELLIPSOID.

PRO-LONG', *v.* **prolong a line.** To continue the line. *Syn.* Produce.

PROM'IS-SO'RY, *adj.* **promissory note.** A written promise to pay a certain sum on

a certain future date, the sum usually drawing interest until paid. The person who signs the note (the one who first guarantees its payment) is called the **maker** of the note (also see INDORSE).

PROOF, *n.* (1) The logical argument which establishes the truth of a statement. (2) The process of showing by means of an assumed logical process that what is to be proved follows from certain previously proved or axiomatically accepted propositions. See ANALYTIC—analytic proof, DEDUCTIVE—deductive method or proof, INDIRECT—indirect proof, INDUCTION—mathematical induction, INDUCTIVE—inductive methods, REDUCTIO AD ABSURDUM PROOF, and SYNTHETIC—synthetic method of proof.

PROP'ER, *adj.* **proper fraction.** See FRACTION.

proper subset. See SUBSET.

PROP'ER-LY, *adv.* **contained properly.** See SUBSET.

properly divergent series. See DIVERGENT—divergent series.

PROP'ER-TY, *n.* **evaluation of mining property.** Evaluating it as a depreciating asset. A redemption fund is usually built up to equal the cost of the mine when the ore is all gone.

property of finite character. See CHARACTER.

PRO-POR'TION, *n.* The statement of equality of two ratios; an equation whose members are ratios. Four numbers, a, b, c, d, are in proportion when the ratio of the first pair equals the ratio of the second pair. This is denoted by $a:b=c:d$, or better by $a/b=c/d$; a notation becoming obsolete is $a:b::c:d$. The letters a and d are called the **extremes,** b and c the **means,** of the proportion. A **continued proportion** is an ordered set of three or more quantities such that the ratio between any two successive ones is the same. This is equivalent to saying that any one of the quantities except the first and last is the *geometric mean* between the previous and succeeding ones, or that the quantities form a *geometric progression*; 1, 2, 4, 8, 16 form a continued proportion, written

$$1:2:4:8:16 \quad\text{or}\quad \tfrac{1}{2}=\tfrac{2}{4}=\tfrac{4}{8}=\tfrac{8}{16}.$$

If four numbers are in proportion, then various other proportions can be derived from this proportion. If $\frac{a}{b}=\frac{c}{d}$, then

$$\frac{a+b}{b}=\frac{c+d}{d};$$

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}, \quad \text{if} \quad a\neq b;$$

$$\frac{a}{c}=\frac{b}{d};$$

$$\frac{b}{a}=\frac{d}{c}, \quad \text{if} \quad a\neq 0;$$

$$\frac{a-b}{b}=\frac{c-d}{d}.$$

These five proportions are said to be derived from the given proportion by **addition, addition and subtraction, alternation, inversion,** and **subtraction,** respectively.

PRO-POR'TION-AL, *adj., n.* As a noun, one of the terms in a proportion. A **fourth proportional** for numbers a, b and c is a number x such that $a/b=c/x$; a **third proportional** for numbers a and b is a number x such that $a/b=b/x$. A **mean proportional** between two numbers a and b is a number x such that $a/x=x/b$. *E.g.*, 10 is a fourth proportional to 1, 2 and 5, since $\frac{1}{2}=\frac{5}{10}$; 4 is a third proportional to 1 and 2, since $\frac{1}{2}=\frac{2}{4}$ (also, 2 is a mean proportional between 1 and 4). See PROPORTION.

directly proportional quantities. Same as PROPORTIONAL QUANTITIES.

inversely proportional quantities. Two quantities whose product is constant; two quantities such that one is *proportional* to the reciprocal of the other.

proportional parts. Parts which are in the same proportion as a given set of numbers. The parts of 12 proportional to 1, 2, and 3 are 2, 4, and 6. The **principle of proportional parts** is the assumption that a function of a variable varies linearly with the independent variable for small values of the difference between the values of the variable (in other words, short arcs of the graph of the function are very nearly straight line segments). This principle is

used mostly in interpolation. See LOGARITHM—proportional parts in a table of logarithms.

proportional quantities. Two variable quantities having fixed (constant) ratio.

proportional sets of numbers. Two sets of numbers such that the ratios of corresponding numbers are equal, except for ones that are both zero; *i.e.*, two sets of numbers for which there exist two numbers m and n, not both zero, such that m times any number of the first set is equal to n times the corresponding number of the second set. The two sets of numbers are also said to be **linearly dependent,** although the concept of linear dependence is not limited to two sets. The sets 1, 2, 3, 7, and 4, 8, 12, 28 are proportional. The numbers $m=4$ and $n=1$ suffice for these sets. This definition is more general than if corresponding numbers must have equal quotients, for the sets 1, 5, 0, 9, 0 and 0, 0, 0, 0, 0 are in proportion, where m is zero and n is any number not zero; but some corresponding numbers do not have quotients, because of the impossibility of dividing by zero.

PRO-POR'TION-AL'I-TY, *n.* The state of being in proportion.

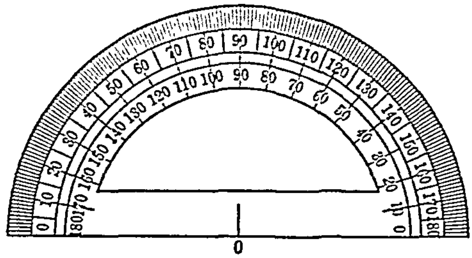
factor of proportionality. See FACTOR—factor of proportionality.

PROP'O-SI'TION, *n.* (1) A theorem or problem. (2) A theorem or problem with its proof or solution. (3) Any statement which makes an assertion which is either true or false, or which has been designated as true or false.

PROP'O-SI'TION-AL, *adj.* **propositional function.** An expression which becomes a proposition when suitable values are given to certain symbols in the expression. *E.g.*, "$x<3$" is a propositional function which is a *true* proposition if $x=2$, a *false* proposition if $x=4$. *Syn.* Sentential function. See QUANTIFIER.

PRO-SPEC'TIVE, *adj.* **prospective method of computing reserves.** See RESERVE.

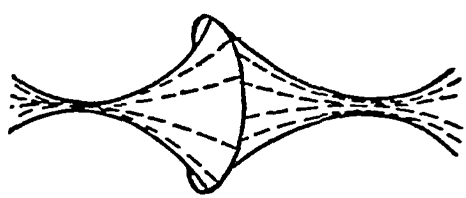
PRO-TRAC'TOR, *n.* A semicircular plate graduated, usually in degrees, from one extremity of the diameter to the other and used to measure angles.



PROVE, *v.* To establish by evidence or demonstration; show the truth of; find a proof of. See PROOF.

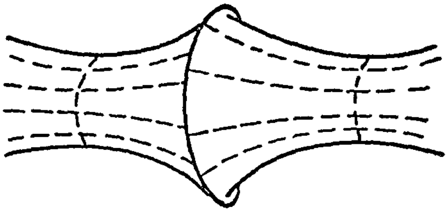
PSEU'DO-SPHERE, *n.* The surface of revolution of a tractrix about its asymptote; a pseudospherical surface of revolution of parabolic type.

PSEU-DO-SPHER'I-CAL, *adj.* **pseudospherical surface.** A surface whose total curvature K has the same negative value at all its points. See SPHERICAL—spherical surface. A pseudospherical surface of **elliptic type** is a pseudospherical surface whose linear element is reducible to the form $ds^2=du^2+a^2\sinh^2(u/a)\,dv^2$; the coordinate system is a geodesic polar one. A pseudospherical surface of revolution of

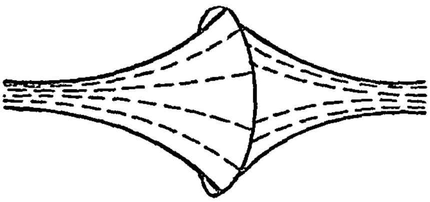


elliptic type consists of a succession of hour-glass-shaped zones with cusps at the maximum parallels. A pseudospherical surface of **hyperbolic type** is a pseudospherical surface whose linear element is reducible to the form $ds^2=du^2+\cosh^2(u/a)\,dv^2$; the coordinate system is a geodesic one, with the coordinate geodesics orthogonal to a geodesic, $u=0$. A *pseudospherical surface of revolution of hyperbolic type* consists of a succession of congruent spool-shaped zones with cusps at the maximum parallels. A pseudospherical surface of **parabolic type** is a pseudospherical surface

whose linear element is reducible to the form $ds^2 = du^2 + e^{2u/a}\,dv^2$; the coordinate system is a geodesic one, with the coordinate geodesics orthogonal to a curve of



constant geodesic curvature. The only pseudospherical surface of revolution of parabolic type is the pseudosphere, or surface of revolution of a tractrix about its asymptote.



PTOLEMY'S THEOREM. A necessary and sufficient condition that a convex quadrilateral be inscribable in a circle is that the sum of the products of the two pairs of opposite sides be equal to the product of the diagonals.

PUR'CHASE, *adj., n.* **purchase price of a bond.** See PRICE.

PURE, *adj.* **pure geometry.** See SYNTHETIC—synthetic geometry.

pure imaginary number. See COMPLEX—complex number.

pure mathematics. See MATHEMATICS.

pure projective geometry. Projective geometry employing only geometric methods and only bringing up properties other than projective in a subordinate way. See GEOMETRY—modern analytic geometry.

pure surd. See SURD.

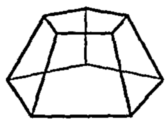
PYR'A-MID, *n.* A polyhedron with one face a polygon and the other faces triangles with a common vertex. The polygon is called the **base** of the pyramid and the triangles are called the **lateral faces.** The common vertex of the lateral faces is called the **vertex** of the pyramid and the intersections of pairs of lateral faces are called **lateral edges.** The **altitude** is the perpendicular distance from the vertex to the base. The **lateral area** is the total area of the lateral faces, and the **volume** is equal to $\frac{1}{3}bh$, where b is the area of the base and h is the altitude. A **regular pyramid** is a pyramid whose base is a regular polygon and whose lateral faces make equal angles with the base (a pyramid with a regular polygon for base and with the foot of its altitude at the center of the base). Its lateral surface area is $\frac{1}{2}SP$, where S is the **slant height** (common altitude of its faces) and P the perimeter of the base.

circumscribed and inscribed pyramids. See CIRCUMSCRIBED.

frustum of a pyramid. The section of a pyramid between the base and a plane parallel to the base. The **bases** of the frustum are the base of the pyramid and the intersection of the pyramid with this plane parallel to the base. The **altitude** is the perpendicular distance between the base and the plane. The volume of a frustum of a pyramid is equal to

$$\tfrac{1}{3}h(A+B+\sqrt{AB}),$$

where A and B are the areas of the bases and h is the altitude. If the pyramid is regular, the lateral area of a frustum is $\frac{1}{2}S(P_1+P_2)$, where S is the slant height (altitude of a face) and P_1 and P_2 are the perimeters of the bases.



spherical pyramid. A figure formed by a spherical polygon and planes passing through the sides of the polygon and the center of the sphere. Its volume is

$$\frac{\pi r^3 E}{540},$$

where r is the radius of the sphere and E the *spherical excess* of the base of the pyramid. The polyhedral angle, at the center of the sphere, made by the plane faces of the pyramid is said to **correspond** to the

spherical polygon which forms the base of the pyramid.

truncated pyramid. The part of a pyramid between the base and a plane oblique to the base. The bases are the base of the pyramid and the intersection of the plane and the pyramid.

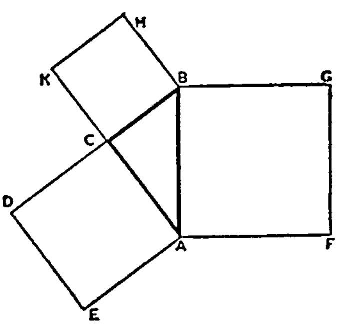
PY-RAM′I-DAL, *adj.* **pyramidal surface.** A surface generated by a line passing through a fixed point and moving along the sides of a polygon whose plane does not contain the fixed point.

PY-THAG′O-RE′AN, *adj.* **Pythagorean identities.** See TRIGONOMETRY—identities of plane trigonometry.

Pythagorean numbers. Any set of integers satisfying the equation $x^2+y^2=z^2$; *e.g.*, 3, 4, 5. Such numbers are given by m^2-n^2, $2mn$, m^2+n^2, where m and n are arbitrary integers.

Pythagorean relation between direction cosines. The sum of the squares of the direction cosines of a line is equal to unity.

Pythagorean theorem. The sum of the squares of the lengths of the legs of a right triangle is equal to the square of the hypotenuse. The right triangle whose legs are 3 and 4, and hypotenuse 5, has been used for ages to square corners. Geometrically this theorem states that the area of $ABGF$ (in the figure) is equal to the sum of the areas of $ACDE$ and $BCKH$.



pentagram of Pythagoras. See PENTAGRAM.

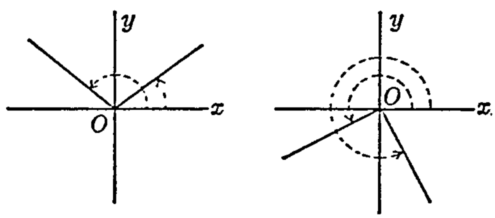
Q

QUAD′RAN′GLE, *n.* A **simple quadrangle** is a plane geometric figure consisting of four points, no three of which are collinear, and the four lines connecting them in a given order. A **complete quadrangle** consists of four points, no three of which are collinear, and the six lines determined by the points in pairs.

QUAD-RAN′GU-LAR, *adj.* **quadrangular prism.** A prism whose bases are quadrilaterals.

QUAD-RANT, *adj., n.* **laws of quadrants** for a right spherical triangle. (1) Any angle and the side opposite it are in the same quadrant; (2) when two of the sides are in the same quadrant the third is in the first quadrant and when two are in different quadrants the third is in the second. (The *first*, *second*, *third*, and *fourth* quadrants mean angles from 0° to 90°, 90° to 180°, 180° to 270°, and 270° to 360°, respectively.)

quadrant angles. Angles are designated as first, second, third, or fourth quadrant angles when the initial side coincides with the positive abscissa axis in a system of rectangular coordinates and the terminal side lies in the first, second, third, or fourth quadrants, respectively. The angles in the first figures are in the first and second quadrants; those in the second are in the third and fourth quadrants.

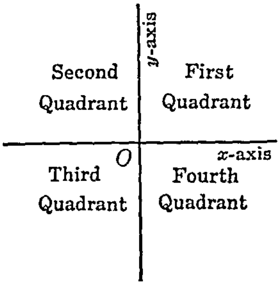


quadrant of a circle. (1) One-half of a semicircle; one-fourth of a circumference. (2) The plane area bounded by two perpendicular radii and the arc they subtend.

quadrant of a great circle on a sphere. One-fourth of the great circle; the arc of the great circle subtended by a right angle at the center of the sphere.

quadrant in a system of plane rectangular coordinates. One of the four compartments into which the plane is divided by the axes of reference in a Cartesian system of coordinates. They are called first, second, third, and fourth quadrants as counted counterclockwise beginning with the quadrant in which both coordinates are positive.

See CARTESIAN—Cartesian coordinates in the plane.



QUAD-RAN'TAL, *adj.* **quadrantal angles.** The angles 0°, 90°, 180°, 270° or in radians 0, $\pi/2$, π, $3\pi/2$, and all angles having the same terminal sides as any one of these, as 2π, $5\pi/2$, 3π, $7\pi/2$, $-\pi/2$, $-\pi$, $\cdots$.

quadrantal spherical triangle. See SPHERICAL.

QUAD-RAT'IC, *adj.* Of the second degree.

discriminant of a quadratic. See DISCRIMINANT—discriminant of a polynomial equation.

quadratic equation. An equation of the second degree. The **general form** (sometimes called an **affected quadratic**) is $ax^2+bx+c=0$. The **reduced form (p-form)** is

$$x^2+px+q=0.$$

A **pure quadratic** equation is an equation of the form $ax^2+b=0$.

quadratic form. See FORM.

quadratic formula. A formula for computing the roots of a quadratic equation. If the equation is in the form $ax^2+bx+c=0$, the formula is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

See DISCRIMINANT—discriminant of a polynomial equation.

quadratic reciprocity law. If p and q are distinct odd primes, then

$$(q|p)(p|q)=(-1)^{1/4(q-1)(p-1)}.$$

See LEGENDRE—Legendre's symbol.

QUAD'RA-TURE, *n.* The process of finding a square equal in area to the area of a given surface.

quadrature of a circle. Finding (constructing) a square which has the same area as a given circle; usually called **squaring the circle.** It is impossible to do this with straight-edge and compass alone, since a line segment of length π cannot be so constructed from one of unit length.

QUAD'RE-FOIL, *n.* See MULTIFOIL.

QUAD'RIC, *adj., n.* (1) Of the second degree; quadratic. (2) An expression of the second degree in all its terms; a homogeneous expression of the second degree. A **quadric curve (quadric surface)** is a curve (surface) whose equation in Cartesian coordinates is algebraic and of the second degree (see CONIC). See CENTRAL—central quadrics, and CONICAL—quadric conical surfaces.

confocal quadrics. See CONFOCAL.

quadric quantic. See QUANTIC.

QUAD'RI-LAT'ER-AL, *n.* A polygon having four sides. See PARALLELOGRAM, RECTANGLE, RHOMBUS, TRAPEZOID.

complete quadrilateral. A figure consisting of four lines and their six points of intersection.

quadrilateral inscribable in a circle. See PTOLEMY'S THEOREM.

regular quadrilateral. A quadrilateral whose sides and interior angles are all equal; a square.

simple quadrilateral. A figure consisting of four lines and their four successive intersections in pairs. Simple as distinguished from a *complete* quadrilateral.

QUAD-RIL'LION, *n.* (1) In the U. S. and France, the number represented by one followed by 15 zeros. (2) In England, the number represented by one followed by 24 zeros.

QUAN'TIC, *n.* A rational integral homogeneous function of two or more variables; a homogeneous algebraic polynomial in two or more variables. Quantics are classified as *quadric, cubic, quartic,* etc., according as they are of the second, third, fourth, etc., degrees. They are classified as *binary, ternary, quarternary,* etc., according as they contain two, three, four, etc., variables.

QUAN'TI-FI'ER, *n.* Phrases such as *for any* $x, y, z, \cdots$, and *there are* $x, y, z, \cdots$

for which. The first type is a **universal quantifier,** the latter an **existential quantifier.** Quantifiers precede a *propositional function* and may be represented symbolically; *e.g.*, "*for any* x, $p(x)$" might be written as $\forall_x[p(x)]$, $A_x[p(x)]$, or $(x)[p(x)]$; "*there is an* x *for which* $p(x)$" as $\exists_x[p(x)]$, $E_x[p(x)]$, or $(\exists x)p(x)$. Such a statement as "there is a man who is disliked by all men" could be written as

$$\exists_x[\forall_y\ (x \text{ is disliked by } y)].$$

QUAN'TI-TY, *n.* Any arithmetic, algebraic, or analytic expression which is concerned with value rather than relations between such expressions.

QUAR'TER, *n.* One fourth-part.

QUAR'TIC, *adj., n.* Of the fourth degree; of the fourth order. A **quartic curve** is an algebraic curve of the fourth order (the graph of a fourth degree equation). A **quartic equation** is a polynomial equation of the fourth degree.

quartic symmetry. Symmetry like that of a regular octagon, that is, symmetry of a plane figure with respect to four lines through a point, neighboring pairs intersecting at 45°.

solution of the quartic. See FERRARI'S solution of the quartic.

QUAR'TILE, *n.* (*Statistics.*) The twenty-fifth, fiftieth, and seventy-fifth *percentiles* are the 1st, 2nd, and 3rd *quartiles.* See PERCENTILE.

quartile deviation. See DEVIATION—quartile deviation.

quartile magnitude or measurement. A magnitude corresponding to a *quartile division* of certain measurements or magnitudes. *Syn.* Quartile.

QUA'SI. quasi-analytic function. See ANALYTIC—quasi-analytic function.

QUA-TER'NA-RY, *adj.* Consisting of four; containing four.

quarternary quantic. See QUANTIC.

QUA-TER'NI-ON, *n.* A symbol of type $x=x_0+x_1i+x_2j+x_3k$, where x_0 and the coefficients of i, j, k are real numbers. *Scalar multiplication* is defined by

$$cx=cx_0+cx_1i+cx_2j+cx_3k;$$

the *sum* of x and $y=y_0+y_1i+y_2j+y_3k$ is

$$x+y=(x_0+y_0)+(x_1+y_1)i+(x_2+y_2)j+(x_3+y_3)k;$$

the *product* xy is computed by formally multiplying x and y by use of the distributive law and the conventions

$$i^2=j^2=k^2=-1,$$
$$ij=-ji=k,\quad jk=-kj=i,\quad ki=-ik=j.$$

The quaternions satisfy all the axioms for a *field* except the commutative law of multiplication.

QUIN'TIC, *adj., n.* (1) Of the fifth degree. (2) An algebraic function of the fifth degree. A **quintic curve** is an algebraic curve of the fifth order (the graph of a fifth degree equation). A **quintic equation** is a polynomial equation of the fifth degree.

quintic quantic. See QUANTIC.

QUIN-TIL'LION, *n.* (1) In the U. S. and France, the number represented by one followed by 18 zeros. (2) In England, the number represented by one followed by 30 zeros.

QUO'TIENT, *adj., n.* The quantity resulting from the division of one quantity by another. The division may have been actually performed or merely indicated; *e.g.*, 2 is the quotient of 6 divided by 3, as is also 6/3. In case the division is not exact one speaks of the *quotient* and the *remainder*, or simply the *quotient* (meaning the integer obtained plus the indicated division of the remainder), *e.g.*, $7\div 2$ gives the quotient 3 and the remainder 1, or the *quotient* $3\frac{1}{2}$.

derivative of a quotient. See DIFFERENTIATION FORMULAS in the appendix.

difference quotient. See DIFFERENCE.

quotient space or factor space. Let T be a set for which an *equivalence relation* is defined and let T be divided into *equivalence classes* (see EQUIVALENCE). If certain operations, distance, etc., are defined for elements of T, then it may be possible to define these operations, distance, etc., for the equivalence classes in such a way that the set of equivalence classes is a space of

the same type as T. In this case, the set of equivalence classes is said to be a *quotient space* or *factor space* of T. *E.g.*, the quotient space of the set C of complex numbers, *modulo* the set R of real numbers, is the set C/R of equivalence classes defined by the equivalence relation $x \equiv y$ if and only if $x-y$ is a real number. The elements of C/R are the sets of numbers represented by the horizontal lines in the complex plane, and the sum of two "lines" is the "line" which contains the sum of two numbers, one on each of the given lines (the elements of C/R are also called *residue classes* (modulo R)). The **quotient group** of a group G by an *invariant subgroup* H is the group (denoted by G/H) whose elements are the cosets of H (these cosets are also equivalence classes if one defines x and y to be equivalent if xy^{-1} belongs to H). The unit element of G/H is H and the product of two cosets is the coset containing a product of an element of one coset by an element of the other, the multiplication being in the same order as that of the corresponding cosets. The uniqueness of the product and the group properties of G/H are a consequence of H being an invariant subgroup of G. If G is also a **topological group** and H is a closed set as well as being an invariant subgroup, then G/H is a topological group if one defines a set U^* of elements to be open if and only if U is open in G, where U is the set of all elements of G which belong to a coset of H which is a member of U^*. If G is a metric space, then there is a metric for G which is equivalent to the metric of G and which is right-invariant (*i.e.*, distance as given by the metric satisfies $d(xa, ya) = d(x, y)$ for any elements a, x, y). Then G/H is a metric space if the distance between cosets H_1 and H_2 is defined to be

$$\bar{d}(H_1, H_2) = \text{g.l.b. } d(x_1, x_2)$$

for x_1 in H_1 and x_2 in H_2. For the above example of the quotient space C/R of the set of complex numbers modulo the set of real numbers, an open set in C/R is a set of horizontal "lines" in the complex plane whose points form an open set in the plane, while the distance between two elements of C/R is the distance between the corresponding "lines" in the plane. The **quotient ring** of a ring R by an *ideal* I is the ring (denoted by R/I) whose elements are the cosets of I. These cosets are equivalence classes if one defines x and y to be equivalent if $x-y$ belongs to I (they are also called residue classes and R/I a **residue class ring**). The zero element of R/I is I and the sum (product) of two cosets is the coset containing a sum (product) of an element of one coset and an element of the other (the multiplication being in the same order as that of the corresponding cosets). The uniqueness of the sum and product and the ring properties of R/I are consequences of I being an ideal. Also, if R is a ring with unit element, or a commutative ring, or an integral domain, then R/I is a set of the same type. Let V be a **vector space** and L be a subset of V which is also a vector space. Let V/L be the set of equivalence classes (or residue classes) defined by the equivalence relation $f \equiv g$ if and only if $f-g$ belong to L. Then V/L is a vector space if the sum of two equivalence classes F and G is the equivalence class which contains the sum of an element of F and an element of G, and the product of a scalar α and an equivalence class F is the equivalence class which contains the product of α and an element of F. If B is a **Banach space** and L is a subset of B which is also a Banach space, then B/L can be defined in the same way as for vector spaces. If one also defines $\|F\|$ for an equivalence class F to be the greatest lower bound of $\|f\|$ for f belonging to F, then B/L is a Banach space. If H is a **Hilbert space,** then H/L can be defined in the same way as for Banach spaces and is isometric with the orthogonal complement of L in H.

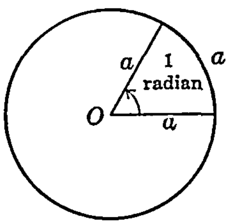
R

RAABE. Raabe's ratio test. See RATIO—ratio test.

RA'DI-AL-LY, *adv.* **radially related figures.** Figures which are central projections of each other; figures such that a line drawn from some fixed point to a point of one of them passes through a point of the other, such that the ratio of the distances from the fixed point to the two points is always the same (two similar figures can always be so placed). The fixed point is

called the **homothetic center,** the **center of similitude,** or **ray center.** The ratio of the two line segments is called the **ray ratio, ratio of similitude,** or **homothetic ratio.** Two radially related figures are similar. They are also called **homothetic figures.**

RA'DI-AN, *n.* A central angle subtended in a circle by an arc whose length is equal to the radius of the circle. Thus the **radian measure** of an angle is the ratio of the arc it subtends to the radius of the circle in which it is the central angle (a constant ratio for all such circles); also called circular measure, π measure (rare), natural measure (rare); 2π *radians* $=360°$, π *radians* $=180°$ or 1 *radian* $=(180/\pi)°$; $\frac{1}{4}\pi$ *radians* $=45°$, $\frac{1}{3}\pi$ *radians* $=60°$, $\frac{1}{2}\pi$ *radians* $=90°$. See SEXAGESIMAL and MIL.



RA'DI-ATE, *v.* **radiate from a point.** To be a *ray* with the point as origin.

RA'DI-A'TION, *adj., n.* **radiation phenomena.** Wave phenomena in which a disturbance at a single point at time $t=0$ spreads out with the passage of time. The region into which the disturbance spreads is called the **range of influence.** See DEPENDENCE—domain of dependence.

RAD'I-CAL, *adj., n.* (1) The indicated root of a quantity, as $\sqrt{2}$, $\sqrt{x}$. (2) The sign indicating a root to be taken, a **radical sign,** the sign $\sqrt{\ }$ (a modified form of the letter *r*, the initial of the Latin *radix*, meaning root), placed before a quantity to denote that its root is to be extracted. To distinguish the particular root, a number (the **index**) is written over the sign; thus, $\sqrt[2]{\ }$, $\sqrt[3]{\ }$, $\sqrt[n]{\ }$, etc., denote respectively the square root, cube root, *n*th root, etc. In the case of the square root, the index is omitted, $\sqrt{\ }$ instead of $\sqrt[2]{\ }$ being written. The *radical sign* is frequently said to include a bar above the radicand as well as the above sign. This combination is written $\sqrt{\quad}$. See SIMPLIFICATION—simplification of radicals.

radical axis. The radical axis of **two circles** is the locus of the equation resulting from eliminating the square terms between the equations of the circles. When the circles intersect, the radical axis passes through their two points of intersection. The radical axis of two circles is also the line consisting of those points whose *powers* with respect to the two circles are equal (see POWER—power of a point with reference to a circle or a sphere). The radical axis of **three spheres** is the line of intersection of the three radical planes taken with respect to the three possible pairs of spheres. The line is finite if, and only if, the centers of the spheres do not lie on a straight line.

radical center. The radical center of **three circles** is the point in which the three radical axes of the circles, taken in pairs, intersect. This point is finite if, and only if, the centers of the circles do not lie on a line. The radical center of **four spheres** is the point of intersection of the six radical planes formed with respect to the six possible pairs of spheres made up from the given four. The point is finite if, and only if, the centers of the four spheres are not coplanar.

radical plane of two spheres. The locus of the equation resulting from eliminating the square terms between the equations of the two spheres. When the spheres intersect, the radical plane is the plane of their circle of intersection.

RAD'I-CAND', *n.* The quantity under a radical sign; as 2 in $\sqrt{2}$, or $a+b$ in $\sqrt{a+b}$.

RA'DI-US, *n.* [*pl.* **radii**]. **focal radius.** See FOCAL.

long radius of a regular polygon. The distance from the center to a vertex; the radius of the circumscribed circle.

radius of a circle. The distance from the center to the circumference.

radius of convergence of a power series. The radius of the circle of convergence. See CONVERGENCE—circle of convergence.

radius of curvature. See CURVATURE—curvature of a plane curve, curvature of a

space curve. Also see various headings under CURVATURE and GEODESIC, and below, radius of total curvature, radius of torsion of a space curve.

radius of geodesic torsion. The reciprocal of the *geodesic torsion.* See GEODESIC—geodesic torsion of a surface at a point in a given direction.

radius of gyration. The distance from a fixed line (point, or plane) to a point in, or near, a body where all the mass of the body could be concentrated without altering the *moment of inertia* of the body about the line (point, or plane); the square root of the quotient of the moment of inertia by the mass.

radius of normal curvature. See CURVATURE—normal curvature of a surface.

radius of total curvature of a surface at a point. The quantity ρ defined by $K=-\frac{1}{\rho^2}$, where K is the *total curvature* of the surface at the point. If K is negative, then ρ is real. If the asymptotic lines are taken as parametric curves, so that we have $D=D''=0$, then $\frac{1}{\rho}=\frac{D'}{H}$, where $H=\sqrt{EG-F^2}$.

radius of torsion of a space curve. See TORSION—torsion of a space curve at a point. *Syn.* Radius of second curvature.

radius vector [*pl.* **radii vectores; radius vectors**]. See POLAR—polar coordinates, and SPHERICAL—spherical coordinates.

short radius of a regular polygon. The perpendicular distance from the center to a side; the radius of the inscribed circle. *Syn.* Apothem.

RA'DIX, *n.* [*pl.* **radices**]. (1) A root. (2) Any number which is made the fundamental number or base of any system of numbers; thus, 10 is the radix of the decimal system of numeration. (3) A name sometimes given to the base of a system of logarithms. In the common system of logarithms the radix is 10; in the natural system it is 2.7182818284 $\cdots$, denoted by e. See BASE—base of a system of numbers.

radix fraction. An indicated sum of fractions, of the form $a/r+b/r^2+c/r^3+d/r^4\cdots$, where the letters $a, b, \cdots$ are all integers less than r (which is also an integer). This is a generalization of decimals; when r is 10 it reduces to decimal fraction.

radix of a mortality table. See MORTALITY—mortality table.

RAN'DOM, *adj.* **random device.** A device for the determination of numbers from a prescribed frequency distribution.

random sample. (*Statistics.*) A sample obtained by a selection of items from the population is a **random sample** if each item in the population has an equal chance of being drawn. *Random* describes a method of drawing a sample, rather than some resulting property of the sample discoverable after the observance of the sample. See SAMPLE—stratified random sample.

random sequence. (*Statistics.*) A sequence of values that is irregular, nonrepetitive, or haphazard. A **sequence of random digits** is a random sampling from the population consisting of the ten digits $0, 1, \cdots, 9$. A random sample of k objects from n objects can be obtained by numbering the objects from 0 to n and choosing a sequence of numbers (each with the same number of digits as n), disregarding those larger than n, until k numbers have been chosen. A completely satisfactory definition of random sequence is yet to be discovered. However, tests of randomness can be made; *e.g.*, by subdividing the sequence into blocks and using the chi-square test to analyze the frequencies of occurrence of specified individual integers or of runs consisting of specified digits in a specified order. A table of one million random digits has been published.

random variable. Same as CHANCE VARIABLE. See CHANCE.

random walks. A succession of "walks" along line segments in which the direction (and possibly the length) of each walk is determined in a random way. Random walks are used to obtain probabilistic solutions to mathematical or physical problems. *E.g.*, if a person makes a step of length h every r seconds and each step is equally likely to be to the right or to the left, then the probability at time t of being at a distance of x from where he was at time $t=0$ can be shown to be given by a function $U(x, t)$ which satisfies the difference equation

$$U(x, t+r)=\tfrac{1}{2}U(x+h, t)+\tfrac{1}{2}U(x-h, t).$$

The function U can be approximately evaluated by letting a computing machine "make" a large number of random walks by reference to a sequence of random numbers. If $h^2=r$ and $h\to 0$, the limit of $U(x, t)$ satisfies the heat equation

$$\frac{\partial^2 u}{\partial x^2}=2\frac{\partial u}{\partial t}$$

with the boundary conditions $U(x, 0)=0$ if $x\neq 0$ and $\int_{-\infty}^{\infty} U(x, t)dx=1$. A random walk method is a type of Monte Carlo method.

randomized blocks. See BLOCK.

stratified random sample. See SAMPLE.

systematic random sample. Let a population have nk elements, the population being divided into n sub-populations of k elements each. Select a number from 1 to k at random and then sample every kth consecutive element, where $1/k$ is the ratio of sample to population. This is a special kind of *random sample* and is in some populations more efficient than simple random sampling.

table of random numbers. A set of numbers arranged such that a random succession of numbers may be selected according to any procedure, subject to the sole restriction that the selection of a number from the set be influenced only by its location in the table. Designed to permit the drawing of random samples. After numbering the items in a population, one may select those items whose numbers are obtained from the table of random numbers.

RANGE, *n.* (*Statistics.*) (1) The most general measure of dispersion; the difference between the greatest and the least of a set of quantities, or the interval between the greatest and the least. (2) The **range of a function or transformation** is the set of values the function or transformation may take on. The range of the function $f(x)=x^2$ is the set of all nonnegative real numbers, if the domain of the function is the set of all real numbers. The range of a mapping or transformation is the set which includes each point which is the map or transform of some point by means of the mapping or transformation. (3) The **range of a variable** is the set of values the variable may take on.

RANK, *n.* **rank of a matrix.** See MATRIX.

RATE, *n.*, *v.* (1) Reckoning by comparative values or relations. (2) Relative amount, quantity, or degree; as, the *rate* of interest is 6% (*i.e.*, \$6 for every \$100 for every year); the *rate* per mile of railroad charges; a rapid *rate* of growth. See CORRESPONDING—corresponding rates, DIVIDEND—dividend rate, DEATH—central death rate, INTEREST, MORTALITY—rate of mortality, SPEED, VELOCITY, and YIELD.

rate of change of a function at a point. The limit of the ratio of an infinitesimal increment of the function at the point to that of the independent variable; the limit of the average rate of change over an interval including the point as the length of the interval approaches zero. This is sometimes called the **instantaneous rate of change** since the rates of change at neighboring points are in general different. The rate of change of a function at a point is the slope of the tangent to the graph of the function, the **derivative at the point.**

RA'TIO, *adj.*, *n.* The quotient of two numbers (or quantities); the relative sizes of two numbers (or quantities). One also speaks of the ratio of two numbers when the second is zero; if $a\neq 0$, then a and 0 have a ratio, denoted by $a{:}0$, and $a{:}0$ and $b{:}0$ are said to be equal if $b\neq 0$. See CORRELATION—correlation ratio, CRITICAL—critical ratio, DEFORMATION—deformation ratio, LIKELIHOOD—likelihood ratio, POINT—point of division, POISSON—Poisson's ratio, and PROPORTIONAL—proportional sets of numbers.

cross ratio (or **anharmonic ratio**). If A, B, C, D are **four distinct collinear points,** the cross ratio (AB, CD) is defined as the quotient of the ratio in which C divides AB by the ratio in which D divides AB; if the abscissas (or ordinates) of four points are x_1, x_2, x_3, x_4, the cross ratio is

$$\frac{(x_3-x_1)(x_4-x_2)}{(x_3-x_2)(x_4-x_1)}.$$

If no ordering of the four points will give a harmonic ratio (see below, harmonic ratio) there are, in general, six distinct values of the cross ratio, depending upon how the

points are ordered. If L_1, L_2, L_3, L_4 are **four distinct concurrent lines** with slopes equal to m_1, m_2, m_3, m_4, respectively, the cross ratio of the four lines is

$$\frac{(m_3-m_1)(m_4-m_2)}{(m_3-m_2)(m_4-m_1)}.$$

harmonic ratio. If the cross ratio of four points (or four lines) is equal to -1, it is called a **harmonic ratio** and the last two points are said to divide the first two **harmonically.**

inverse ratio of two quantities. The ratio of their reciprocals; the reciprocal of their ratio. The *inverse ratio* of 2 to 3 is $(\frac{1}{2})/(\frac{1}{3})=\frac{3}{2}$. *Syn.* Reciprocal ratio.

ratio paper. Same as SEMILOGARITHMIC PAPER. See LOGARITHMIC.

ratio of similitude. The ratio of the lengths of corresponding lines of similar figures; the **ray ratio** (see RADIALLY—radially related figures). Also called *homothetic ratio.*

ratio test. Any of several tests for convergence (or divergence) of an infinite series which make use of the ratio of successive terms of the series. The **ordinary ratio test** (or **Cauchy's ratio test**) states that a series converges or diverges according as the absolute value of the limit, as n becomes infinite, of the ratio of the nth to the $(n-1)$th term is less than or greater than one. If it is equal to one, the test fails. *E.g.*, (1) for the series $1+1/2!+1/3!+\cdots+1/n!+\cdots$, the ratio of the nth to the $(n-1)$th term is

$$(1/n!)/[1/(n-1)!]=1/n,$$

and

$$\lim_{n\to\infty}(1/n)=0.$$

Hence the series converges. (2) For the harmonic series,

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+1/n+\cdots$$

the ratio is

$$(1/n)/[1/(n-1)]=(n-1)/n$$

and

$$\lim_{n\to\infty}(n-1)/n=1;$$

hence the test fails. (However, this series diverges, as can be shown by grouping the terms so that each group equals or exceeds $\frac{1}{2}$, namely,

$$1+\tfrac{1}{2}+(\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8})+\cdots.)$$

The existence of the limit of the ratio of the nth to the $(n-1)$th term is not needed. The **generalized ratio test** (also called **d'Alembert's test**) states that a series converges if after some term the absolute value of the ratio of any term to the preceding is always less than a fixed number less than unity; if this ratio is always greater than unity, the series diverges. **Raabe's ratio test** is a more refined test which states that if the series is $u_1+u_2+u_3+\cdots+u_n+\cdots$, and $u_{n+1}/u_n=1/(1+a_n)$, then the series converges if, after a certain term, the product na_n is always greater than a fixed number which is greater than unity, and it diverges if, after a certain term, the same product is always less than or equal to unity.

reciprocal ratio. Same as INVERSE RATIO.

RA'TION-AL, *adj.* **rational expression or function.** An algebraic expression which involves no variable in an irreducible radical or under a fractional exponent; a function which can be written as a quotient of polynomials. The expressions $2x^2+1$ and $2x+1/x$ are *rational*, but $\sqrt{x+1}$ and $x^{3/2}+1$ are not. See PARTIAL—partial fractions, and FRACTION.

rational integral function. A function containing only rational and integral terms in the variable (or variables). A function may be rational and integral in one or more of the variables while it is not in others; *e.g.*, $w+x^2+2xy^{1/2}+1/z$ is rational and integral in x and in w and x together, but not rational in y and not integral in z. See TERM—rational, integral term. *Syn.* Polynomial.

rational number. A number that can be expressed as an integer or as a quotient of integers; any whole number (integer) or fraction (such as $\frac{1}{2}$, $\frac{4}{3}$, $\frac{6}{2}$). *Tech.* Once integers have been defined (see INTEGER), rational numbers can be defined as the set of all ordered pairs (a, b) for which a and b are integers ($b\neq 0$) and equality, addition, and multiplication are defined as follows:

$$(a, b)=(c, d) \quad \text{if and only if} \quad ad=bc;$$

$$a, (b)+(c, d)=(ad+bc, bd);$$

$$(a, b)\cdot(c, d)=(ac, bd).$$

The usual practice is to write (a, b) as a/b, in which case the above definitions of

equality, addition, and multiplication take the form:

$$\frac{a}{b}=\frac{c}{d} \quad \text{if and only if} \quad ad=bc;$$

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd};$$

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

The rational number $(a, 1)$, or $a/1$, is called an integer and usually written simply as a. See IRRATIONAL—irrational number.

rational root theorem. If a rational number p/q, where p and q have no common factors, is a root of a polynomial equation whose coefficients are integers,

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n=0,$$

then a_0 is divisible by q and a_n is divisible by p.

RA'TION-AL-IZE, *v.* To remove radicals without altering the value of an expression or the roots of an equation.

rationalize an algebraic equation. To remove the radicals which contain the variable (not always possible). A procedure that sometimes suffices is to isolate the radical in one member of the equation (or if there be more than one radical, to arrange them to the best advantage) and raise both sides to a power equal to the index of the radical (or one of the radicals), repeating this process if necessary. Extraneous roots may be introduced by this procedure. *E.g.,* (1) $\sqrt{x-1}=x-2$ rationalizes into $x-1=x^2-4x+4$, or $x^2-5x+5=0$; (2) $\sqrt{x-1}+2=\sqrt{x+1}$ is written $\sqrt{x-1}-\sqrt{x+1}=-2$; squaring gives $x-1-2\sqrt{x^2-1}+x+1=4$ or $\sqrt{x^2-1}=x-2$, whence $x^2-1=x^2-4x+4$ or $4x-5=0$.

rationalize the denominator of a fraction. To multiply numerator and denominator by a quantity that will remove the radical in the denominator. *E.g.,* if the fraction is $\frac{1}{\sqrt{a}+\sqrt{b}}$, a rationalizing factor is $\sqrt{a}-\sqrt{b}$ and we obtain $\frac{\sqrt{a}-\sqrt{b}}{a-b}$; if the fraction is $\frac{1}{\sqrt[3]{c^2}}$, a rationalizing factor is $\sqrt[3]{c}$ and we obtain $\frac{\sqrt[3]{c}}{c}$.

rationalize an integral. To make a substitution (changing variables) so that the radicals in the integrand disappear; the integral

$$\int\frac{x^{1/2}}{1+x^{3/4}}\,dx \text{ is } \textit{rationalized} \text{ into } \int\frac{4z^5}{1+z^3}\,dz$$

by the substitution $x=z^4$ $(dx=4z^3\,dz)$.

RAY, *adj., n.* A straight line extending from a point. The point is called the **origin** of the ray. Same as HALF-LINE.

ray center. Same as CENTER OF PROJECTION. See PROJECTION—central projection, and RADIALLY—radially related figures.

ray ratio. See RADIALLY—radially related figures.

RAYLEIGH-RITZ METHOD. A method for determining approximate solutions of functional equations through the expedient of replacing them by finite systems of equations. Thus, for example, any function of class $C^{(n)}$ on a closed interval (and its first n derivatives) can be approximated arbitrarily closely by polynomials.

RE-AC'TION, *n.* **law of action and reaction.** See ACTION.

REAL, *adj.* **real-number axis** or **real axis.** A straight line upon which the real numbers are plotted; the horizontal axis in an Argand diagram. See COMPLEX—complex numbers.

real number. Any rational or irrational number (see RATIONAL and IRRATIONAL). The *complex numbers* consist of the real and the imaginary numbers (those numbers $a+bi$ for which a and b are real and $b=0$ and $b\neq 0$, respectively). The set of all real numbers is called the **real number system** or the **real continuum** (see CONTINUUM).

real part of a complex number. The term which does not contain the factor i. If the number is $z=x+iy$ (where x and y are real), the real part is x, denoted by $R(z)$, $\text{Re}(z)$, or $\Re(z)$.

real plane. A plane in which all points are assigned ordered pairs of real numbers

for coordinates, as contrasted to the *complex plane.*

real variable. A variable which takes only real numbers for its values.

REAM, *n.* A measure of paper; twenty quires. See DENOMINATE NUMBERS in the appendix.

RE-AR-RANGE'MENT, *n.* **rearrangement of the terms in a series.** See SERIES.

RE-CEIPT', *n.* (*Finance.*) (1) The act of receiving payment in money or goods. (2) A statement acknowledging money or goods having been received.

RE-CEIPTS', *n.* Money or other assets taken in, as contrasted to expenditures.

RE-CEIV'A-BLE, *adj.* See NOTE—note receivable.

RE-CIP'RO-CAL, *adj., n.* The reciprocal of a **number** is the number whose product with the given number is equal to 1; *i.e.,* 1 divided by the number. For a **fraction,** the reciprocal is the fraction formed by interchanging the numerator and denominator in the given fraction. For any set of objects for which multiplication is defined and there is a multiplicative *identity* (whose product with any member x of the set is equal to x), the reciprocal (or **inverse**) of an object x is an object y such that xy and yx are each equal to the identity (provided there is only one y with this property). *E.g.,* the reciprocal of the *polynomial* x^2+1 is $1/(x^2+1)$, since $(x^2+1)\dfrac{1}{x^2+1}=1$. See GROUP.

reciprocal curve of a curve. The curve obtained by replacing each ordinate of a given curve by its reciprocal; the graph of the equation derived from the given equation (in Cartesian coordinates) by replacing y by $1/y$. The graphs of $y=1/x$ and $y=x$ are reciprocals of each other; so are the graphs of $y=\sin x$ and $y=\operatorname{cosec} x$.

reciprocal equation. An equation in one variable whose set of roots remains unchanged if the roots are replaced by their reciprocals; an algebraic equation whose roots are unchanged if the unknown is replaced by its reciprocal. *E.g.,* when x is replaced by $1/x$ and the equations are simplified, $x+1=0$ becomes $1+x=0$, and $x^4-ax^3+bx^2-ax+1=0$ becomes $1-ax+bx^2-ax^3+x^4=0$.

reciprocal of a matrix. Same as the *inverse* of the matrix. See MATRIX.

reciprocal polar figures in the plane. Two figures made up of lines and their points of intersection are reciprocal polar figures if each point in either one of them is the *pole* of a line in the other with respect to some given conic (see POLE—pole and polar of a conic); **polar reciprocal triangles** are two triangles such that the vertices of each of them are the poles of the sides of the other with respect to some conic. **Polar reciprocal curves** are two curves so related that the *polar,* with respect to a given conic, of every point on one of them is tangent to the other (it then follows that the polars of the points on the latter are tangent to the former).

reciprocal ratio. See INVERSE—inverse or reciprocal ratio.

reciprocal spiral. See HYPERBOLIC—hyperbolic spiral.

reciprocal substitution. The substitution of a new variable for the reciprocal of the old; a substitution such as $y=1/x$.

reciprocal system of vectors. Sets of vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ such that $\mathbf{A}_i\cdot\mathbf{B}_i=1$, $i=1, 2, 3$, and $\mathbf{A}_i\cdot\mathbf{B}_j=0$ if $i\neq j$. If the triple scalar product

$$[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]\neq 0,$$

then the set of vectors reciprocal to $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ is $\mathbf{A}_2\times\mathbf{A}_3/[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]$, $\mathbf{A}_3\times\mathbf{A}_1/[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]$, $\mathbf{A}_1\times\mathbf{A}_2/[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]$. See TRIPLE — triple scalar product.

reciprocal theorems. (1) In *plane geometry,* theorems such that the interchanging of two geometric elements, *e.g.,* angles and sides, points and lines, etc., transfers each of the theorems into the other. Two such theorems are not always simultaneously true or false. (2) In *projective geometry,* same as DUAL THEOREMS.

Volterra's reciprocal functions. (*Integral Equations.*) See VOLTERRA.

REC'TAN'GLE, *n.* A parallelogram with one angle a right angle and therefore all of its angles right angles; a quadrilateral whose angles are all right angles. A **diagonal** of a rectangle is a line joining

opposite vertices; if the sides are of length a and b, the length of the diagonal is $\sqrt{a^2+b^2}$. The **altitude** is the perpendicular distance from one side (designated as the **base**) to the opposite side. The **area** of a rectangle is the product of two adjacent sides. If a rectangle has two sides of length 2 and 3, respectively, its area is 6.

REC-TAN′GU-LAR, *adj.* Like a rectangle; mutually perpendicular.

rectangular axes, rectangular (Cartesian) coordinates. See CARTESIAN.

rectangular form of a complex number. The form $x+yi$, as distinguished from the polar or trigonometric form $r(\cos\theta+i\sin\theta)$.

rectangular graph. Same as BAR GRAPH. See GRAPH.

rectangular hyperbola. See HYPERBOLA—rectangular hyperbola.

rectangular parallelepiped. See PARALLELEPIPED.

rectangular solid. A solid all of whose faces are rectangles; a right prism whose bases are rectangles; a rectangular parallelepiped.

REC′TI-FI′A-BLE, *adj.* **rectifiable curve.** (1) A curve whose length can be found. (2) A curve of finite length. See LENGTH—length of a curve.

REC′TI-FY-ING, *adj.* **rectifying plane of a space curve at a point.** The plane of the tangent and binormal to the curve at the point. See DEVELOPABLE—rectifying developable of a space curve.

REC′TI-LIN′E-AR, *adj.* (1) Consisting of lines. (2) Bounded by lines.

rectilinear generators. See RULED—ruled surface, HYPERBOLOID—hyperboloid of one sheet, and PARABOLOID.

rectilinear motion. Motion along a straight line. See VELOCITY.

RE-CUR′RING, *adj.* **recurring continued fraction.** See FRACTION—continued fraction.

recurring decimal. Same as REPEATING DECIMAL.

RE-DEEM′, *v.* To repurchase, to release by making payments. To redeem a note, bond, or mortgage means to pay the sum it calls for. To redeem property means to get ownership by paying off lapsed liability for which it was security.

RE-DEMP′TION, *n.* The act of redeeming.

redemption price. See PRICE.

RE-DUCED′, *p.* **reduced cubic equation.** A cubic equation of the form $y^3+py+q=0$; the form that the general cubic, $x^3+ax^2+bx+c=0$, takes when the x^2 term is eliminated by substituting $x-\frac{1}{3}a$ for x.

reduced differential equation. See DIFFERENTIAL—linear differential equation.

RE-DU′CI-BLE, *adj.* A curve or surface is said to be **reducible** in a given region if it can be shrunk to a point by a continuous deformation without passing outside that region. See DEFORMATION—continuous deformation, and CONNECTED—simply connected region.

reducible set of matrices. A set of matrices which correspond to linear transformations of an n-dimensional vector space V is *reducible* if there is a proper subset V' of V which contains a nonzero element and is such that each point of V' is transformed into a point of V' by any linear transformation corresponding to one of the matrices. See REPRESENTATION—reducible representation of a group.

reducible transformation. A linear transformation T of a **linear space** L into itself is reducible if there are two linear subsets M and N of L such that $T(x)$ belongs to M if x belongs to M, $T(x)$ belongs to N if x belongs to N, and M and N are complementary in the sense that any vector of L can be uniquely represented as the sum of a vector of M and a vector of N. The transformation T can then be completely specified by describing its effect on M and on N. For **Hilbert space**, it is customary to require that M and N be orthogonal complements of each other. Then T is reducible by M and N if and only if T and its *adjoint* T^* map M into M, or if and only if T commutes with the (orthogonal) projection whose range is M.

REDUCTIO AD ABSURDUM PROOF. The method of proof which shows that it

is impossible for that which is to be proved to be false, because if it is false some accepted facts are contradicted; in other words, the method which supposes that the contrary to the fact to be proved is true and then shows that this supposition leads to an absurdity. *E.g.*, accepting the axiom that only one line can be drawn through a given point parallel to a given line, prove that if two lines are parallel to a third line they are parallel to each other. Assume that the two lines are not parallel, *i.e.*, intersect in a point; we then have two lines through a point parallel to a third line, which contradicts the axiom. *Syn.* Indirect proof.

RE-DUC'TION, *adj., n.* (1) A diminution or decreasing, as a reduction of 10% in the price. (2) The act of changing to a different form, by collecting terms, powering equations, simplifying fractions, making substitutions, etc.

reduction ascending. Changing a denominate number into one of higher order, as feet and inches into yards.

reduction descending. Changing a denominate number into one of lower order, as yards and feet into inches.

reduction formulas in integration. See INTEGRATION—reduction formulas in integration.

reduction formulas of trigonometry. See TRIGONOMETRY—identities of trigonometry.

reduction of a common fraction to a decimal. Annexation of a decimal point and zeros to the numerator and dividing (usually approximately) by the denominator. *E.g.*,

$$\frac{1}{4}=\frac{1.00}{4}=.25;\quad \frac{2}{3}=\frac{2.000}{3.}=.667-.$$

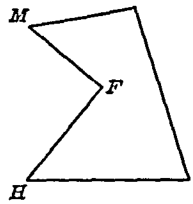
reduction of a fraction to its lowest terms. The process of dividing all common factors out of numerator and denominator.

reduction of the roots of an equation. Same as DIMINUTION of the roots of an equation.

RE-DUN'DANT, *adj.* **redundant equation.** See EQUATION—redundant equation.

redundant number. See NUMBER—perfect number.

RE-EN'TRANT, *adj.* **reentrant angle.** An angle which is an interior angle of a polygon and greater than 180° (angle HFM in figure). The other angles of the figure (interior angles of less than 180°) are called **salient angles.**



REF'ER-ENCE, *n.* **axis of reference.** One of the axes of a Cartesian coordinate system, or the polar axis in a polar coordinate system; in general, any line used to aid in determining the location of points, either in the plane or in space.

frame of reference. See FRAME.

reference angle. Same as RELATED ANGLE. See RELATED.

RE-FLEC'TION, *adj., n.* (*Physics.*) The change of direction which, *e.g.*, a ray of light, radiant heat, or sound, experiences when it strikes upon a surface and is thrown back into the same medium from which it came. Reflection follows two laws: (1) the reflected and incident rays are in a plane normal to the surface; (2) the angle of incidence is equal to the angle of reflection (the **angle of incidence** is the angle the incident ray makes with the normal at the point of incidence; the **angle of reflection** is the angle which the reflected ray makes with this normal).

reflection in a line. Replacing each point in the reflected configuration by a point symmetric to the given point with respect to the line. A reflection in a coordinate axis in the plane is defined by one of the transformations $x'=x, y'=-y$, or $x'=-x, y'=y$. Each given point is replaced by a point symmetric to the given point with respect to the axis in which the reflection is made, the x-axis and y-axis, respectively, in the above transformations.

reflection in the origin. Replacing each point by a point symmetric to the given point with respect to the origin (in the plane, a reflection in the origin is a rotation about the origin through 180°); the result of successive reflections in each axis of a rectangular system of coordinates. See above, reflection in a line.

reflection in a plane. Replacing each point in the reflected configuration by a point symmetric to the given point with respect to the plane; *e.g.*, the reflection of the point (x, y, z) in the (x, y)-plane is the point $(x, y, -z)$.

reflection property of the ellipse, hyperbola, and parabola. See ELLIPSE—focal property of the ellipse, HYPERBOLA—focal property of the hyperbola, PARABOLA—focal property of the parabola.

RE′FLEX, *adj.* **reflex angle.** An angle greater than 180° and less than 360°.

RE-FLEX′IVE, *adj.* **reflexive Banach space.** Let B be a Banach space and B^* and B^{**} be the *first* and *second conjugate spaces* of B (see CONJUGATE—conjugate space). If x_0 is an element of B, then F, defined by $F(f)=f(x_0)$, is a continuous linear functional defined on B^*. B is said to be **reflexive** if every linear functional defined on B^* is of this type, it then following that B and B^{**} are identical if x_0 is identified with the linear functional $F(f)=f(x_0)$. However, there exist non-reflexive Banach spaces B for which there is an isometric correspondence between B and B^{**}. A Banach space is reflexive if and only if the set of all elements x with $\|x\| \leqq 1$ is *weakly compact.* A separable Banach space is reflexive if and only if, for each continuous linear functional f, there is an element $x_0 \neq 0$ such that $f(x_0)=\|f\| \cdot \|x_0\|$. Hilbert space is a reflexive Banach space. *Syn.* Regular Banach space.

reflexive relation. A relation of which it is true that, for any x, x bears the given relation to itself. The relation of equality in arithmetic is *reflexive*, since $x=x$, for all x. A relation such that x does not bear the given relation to itself for any x is said to be **antireflexive.** The relation of being greater than is *antireflexive*, since it is not true for any x that $x>x$. A relation such that there is at least one x which does not bear the given relation to itself is said to be **nonreflexive.** The relation of *being the reciprocal of* is *nonreflexive*, since x may or may not be the reciprocal of x, according as x is equal to unity or is not equal to unity.

RE-FRAC′TION, *n.* (*Physics.*) A change of direction of rays (as of light, heat, or sound) which are obliquely incident upon and pass through a surface bounding two media in which the ray has different velocities (as light going from air to water). It is found that for isotropic media: (1) When passing into a denser medium, the ray is refracted toward a perpendicular to the surface, and, when passing into a less dense medium, it is bent away from the perpendicular; (2) the incident and refracted rays are in the same plane; (3) the sines of the angle of incidence and the angle of refraction bear a constant ratio to each other for any two given media (the **angles of incidence** and **refraction** are the angles which the incident and refracted ray make, respectively, with the perpendicular to the surface). If the first medium is air, this ratio is called the **index of refraction** or the **refractive index** of the second medium. The law stated in (3) is known as **Snell's law.**

RE-GRES′SION, *adj., n.* **edge of regression.** (*Differential Geometry.*) The tangent surface S of a space curve C generally consists of two sheets which are tangent to one another along C, forming a sharp edge there. C is called the *edge of regression* of S.

line of regression. (*Statistics.*) (1) The line determined by the regression function. See below, and LINEAR—equation of linear regression. If the distances to which least squares are applied are parallel to the y-axis, the line is called the *line of regression* of y on x; and, if parallel to the x-axis, the *line of regression* of x on y. (2) Any line which represents the trend of a set of data.

regression coefficient. If two variables y and x are correlated such that $\bar{y}_i=m_y(x_i)$ is the conditional expectation of y given x, then the coefficients in the function $m_y(x_i)$ are the **regression coefficients.** Thus, if $\bar{y}_i=m_y(x_i)=a+bx_i$, a and b are regression coefficients. Sometimes only b is called a regression coefficient. Least-squares estimates of regression coefficients in a linear bivariate regression function $y=a+bx$ may be obtained from

$$a=\frac{\Sigma y}{n}-b\,\frac{\Sigma x}{n}, \quad b=\frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$$

If y, at least, is a random variable, these estimates are minimum variance, unbiased,

and consistent estimates. Also see LINEAR—equation of linear regression.

regression effect. If two jointly correlated variables are compared, it will be noted that the conditional expectation of one of the variables will be closer, in its standard-deviation units, to the mean of its set than are the given values of the 1st variable with which the conditional expectation of the 2nd variable is associated. *E.g.*, for those parents whose height is one standard deviation unit above their average, it will be noticed that their children's average height will be less than one of their standard deviation units above their average. This behavior of the conditional expectation, on the basis of a given value of the first variable, is called the **regression phenomenon** or **effect.** Any two random variables that are linearly correlated with a coefficient of correlation of less than 1 will display this regression. It may be extended to more than two variables.

regression function. (*Statistics.*) Let two random variables x and y be stochastically dependent with a continuous joint frequency function $f(x, y)$. Let $f(y|x)$ be the conditional frequency function of y for given x. Let $y_i = E(y|x_i)$ be the mean expectation of $f(y|x)$ for given x_i. Then $\bar{y}_i = m_y(x_i)$ is the **regression function** of the means of y on x. Conversely, the **regression function** of x on y is $\bar{x}_i = m_x(y_i)$. These two are not in general inverses of each other. If a regression function is linear it is described as a **linear regression function.** See LINEAR—equation of linear regression. To estimate the regression function of y on x from a given set of observations of both variables, the method of least squares applied to $\Sigma[y-g(x)]^2$ yields estimates of the parameters of the regression function for any specified class of equations, and the estimates are best in the sense that they are *minimum-variance unbiased estimates* and also in the sense that the sum of squares of the observed y around the estimated y is a minimum. If $g(x)$ is a linear function of x, the parameter estimates are minimum-variance unbiased estimates.

REGULA FALSI (rule of false position). The method of calculating an unknown (as a root of a number) by making an estimate (or estimates) and working from it and properties of the unknown to secure the value of the latter. If one estimate is used, it is called **simple position;** if two, **double position.** Double position is used in approximating irrational roots of an equation and in approximating logarithms of numbers which contain more significant digits than are listed in the tables being used. The method assumes that small arcs are approximately coincident with the chords which join their extremities. This makes the changes in the abscissas proportional to the changes in the corresponding ordinates; *e.g.*, if $y = f(x)$ has the value -4 when x is 2 and the value 8 when x is 3, then the chord joining the points whose coordinates are $(2, -4)$ and $(3, 8)$ crosses the x-axis at a point whose abscissa x is such that $\frac{1}{4}(x-2) = \frac{1}{12}$, which gives $x = 2\frac{1}{3}$ as an approximate value of a root of $f(x) = 0$. *Newton's method* for approximating roots is an example of *simple position* (see NEWTON).

REG'U-LAR, *adj.* **regular analytic curve.** See ANALYTIC—analytic curve.

regular Banach space. See REFLEXIVE—reflexive Banach space.

regular curve. A curve all of whose points are ordinary points. See POINT—ordinary point on a curve.

regular definition of the sum of a divergent series. A definition which, when applied to convergent series, gives their ordinary sums. **Consistent** is sometimes used to denote the same property. *Regular* is also used to denote not only the above property, but the added property of failing to sum properly divergent series.

regular function of a complex variable. See ANALYTIC—analytic function of a complex variable.

regular permutation group. See PERMUTATION—permutation group.

regular point of a curve. See POINT—ordinary point of a curve.

regular point of a surface. A point of the surface which is not a singular point of the surface. See SINGULAR—singular point of a surface.

regular polygon. See POLYGON.

regular polyhedron. See POLYHEDRON.

regular sequence. (1) A *convergent sequence.* See SEQUENCE—convergent se-

quence. (2) See SEQUENCE — Cauchy sequence.

regular space. A (topological) space such that if U is any neighborhood of a point x of the space, then there is a neighborhood V of x with the closure of V contained in U. A topological space is **normal** if, for any two nonintersecting closed sets P and Q, there are two nonintersecting open sets, one of which contains P and the other Q. A normal space is regular, and a regular space which satisfies the *second axiom of countability* is normal. See METRIC—metric space. A topological space is **completely regular** if, for each x of T and neighborhood U of x, there is a continuous function with values in the interval [0, 1] for which $f(x)=1$ and $f(y)=0$ if y is not in U. A completely regular T_1 space is sometimes called a **Tychonoff space.** A completely regular space is regular.

RE-LAT'ED, *adj.* **related angle.** The acute angle (angle in the first quadrant) for which the trigonometric functions have the same absolute values as for a given angle in another quadrant, with reference to which the acute angle is called the *related angle*; 30° is the related angle of 150° and of 210°.

related expressions or functions. Same as DEPENDENT FUNCTIONS, but less commonly used. See DEPENDENT.

RE-LA'TION, *n.* Equality, inequality, or any property that can be said to hold (or not hold) for two objects in a specified order. *Tech.* A relation is a set R of ordered pairs (x, y), it being said that x is *related* to y (sometimes written xRy) if (x, y) is a member of R. *E.g.*, the relation "less than" for real numbers is the set of all ordered pairs (x, y) for which x and y are real numbers with $x<y$; the relation "sister of" is the set of all ordered pairs (x, y) for which x is a person who is the sister of y.

antireflexive, nonreflexive, and reflexive relation. See REFLEXIVE.

asymmetric, nonsymmetric, and symmetric relations. See SYMMETRIC—symmetric relation.

equivalence relation. See EQUIVALENCE.

intransitive, nontransitive, and transitive relations. See TRANSITIVE.

REL'A-TIVE, *adj.* **relative frequency.** See FREQUENCY.

relative maximum and minimum. See MAXIMUM.

relative velocity. See VELOCITY.

REL'A-TIV'I-TY, *n.* **mathematical theory of relativity.** A special (or restricted) mathematical theory of relativity is based on two postulates: (1) Physical laws and principles can be expressed in the same mathematical form in all reference systems which move relative to one another with constant velocity. (2) The speed of light has the same constant value c (approximately 3×10^{10} cm/sec.) which is independent of the velocity of the source of light. The adoption of these postulates leads to the conclusions that the velocity of an object with nonzero mass must be less than the velocity of light, and that the mass m of the body depends on its velocity and hence on the kinetic energy of the body. It turns out that mass increases with increase in velocity, and this leads to the celebrated mass-energy relation $E=mc^2$. The general theory subsumes that the physical laws and principles are invariant with respect to all possible reference frames. It provides an elegant mathematical formulation of particle dynamics that is essentially geometrical in character. It provides a reasonable explanation of several astronomical phenomena that are not easily explained by Newtonian mechanics, but it fails to provide a satisfactory unified account of electrodynamical phenomena.

RE'LAX-A'TION, *adj., n.* **relaxation method.** In numerical analysis, a method in which the errors, or residuals, resulting from an initial approximation are considered as constraints that are to be relaxed. New approximations are chosen to reduce the worst of the residuals until finally all are within the toleration limit.

RE-LI-A-BIL'I-TY, *n.* (*Statistics.*) (1) Sampling variance. (2) Test reliability is a measure of the precision of measurement. Thus the variance of repeated measurements on the same object is a measure of the reliability of the method of measurement. In the case of a mental test, the correlation coefficient between pairs of test

scores on essentially the same test by the same person is a measure of the test's reliability. The practical problems attendant to obtaining such a hypothetical test-retest measurement enforces modifications in the method of obtaining estimates of the reliability.

RE-MAIN'DER, *adj., n.* The dividend minus the product of the divisor and quotient. If

$$V=D\cdot Q+R, \quad \text{or} \quad \frac{V}{D}=Q+\frac{R}{D},$$

then R is the remainder of the division of V by D with quotient Q. The remainder in division of **integers** is the part of the dividend left when it does not contain the divisor an exact (integral) number of times (17 divided by 5 equals 3, with a *remainder* of 2). The remainder in a division of **polynomials** is usually taken to be the polynomial of lower degree than the divisor which is equal to the dividend minus the product of the quotient and divisor (when the divisor is of the first degree the *remainder* is a constant). See below, remainder theorem. The minuend minus the subtrahend in subtraction is sometimes called the remainder (*difference* is more common).

remainder of an infinite series after the nth term. (1) The difference R_n between the sum, S, of the series and the sum, S_n, of n terms, *i.e.*, $R_n=S-S_n$, when the series is known to converge. (2) The difference between the sum of the first n terms of the series and the quantity (or function) whose expansion is sought; see TAYLOR—Taylor's theorem, and FOURIER SERIES. The series converges and represents the quantity or function for all values of the independent variable for which the remainder converges to zero.

remainder in Taylor's theorem. See TAYLOR—Taylor's theorem.

remainder theorem. When a polynomial in x is divided by $x-h$, the remainder is equal to the number obtained by substituting h for x in the polynomial. More concisely, $f(x)=(x-h)q(x)+f(h)$, where $q(x)$ is the quotient and $f(h)$ the remainder, which is easily verified by substituting h for x. *E.g.*, $(x^2+2x+3)\div(x-1)$ leaves a remainder of $1^2+2\times 1+3$, or 6.

Chinese remainder theorem. See CHINESE.

RE-MOV'A-BLE, *adj.* **removable discontinuity.** See DISCONTINUITY.

RE-MOV'AL, *n.* **removal of a term of an equation.** Transforming the equation into a form having this term missing. See ROTATION—rotation of axes, TRANSLATION—translation of axes, and REDUCED—reduced cubic equation.

RENT, *n.* (1) A sum of money paid at regular intervals in return for the use of property or nonperishable goods. (2) The periodic payments of an annuity. The period between successive payments of rent is the **rent period.**

RENTES, *n.* French perpetuity bonds.

RE-PEAT'ED, *adj.* **repeated root.** See MULTIPLE—multiple root.

RE-PEAT'ING, *adj.* **repeating decimal.** See DECIMAL.

RE-PLACE'MENT, *n.* **replacement cost of equipment.** (1) The cost of new equipment minus the scrap value. (2) The purchase price minus scrap value.

REP'LI-CA'TION, *n.* (*Statistics.*) Repetition of an experiment under conditions which are identical with respect to at least one of the controllable conditions. *E.g.*, the yield of a fruit tree may be observed under a special basis of fertilization. The several trees constitute the **replicates,** since the fertilization is assumed to be invariant from tree to tree.

REP'RE-SEN-TA'TION, *n.* **reducible matrix representation of a group.** Let D_1, D_2, $\cdots$ be the matrices of a representation of a group G by square matrices of order n. This representation is *reducible* if there is a matrix M such that, for each i, $M^{-1}D_iM=E_i$ is a matrix whose elements are all zero except in two or more matrices A_{i1}, $A_{i2}, \cdots, A_{ip}$ having main diagonals along the main diagonal of E_i, where A_{im} is of the same order for all i. When the number of such matrices A_{im} is maximal, the set of

matrices A_{im} for a fixed value of m is said to be an **irreducible representation** of the group; such a set of matrices is isomorphic to a subset of G which contains the (group) product of any two of its members, and G is the direct product of all such subsets. The number of irreducible representations is equal to the number of distinct *conjugate sets* of elements. For an Abelian group, this number of irreducible representations is the order of the group and each matrix of the irreducible representations is of order one; *i.e.*, any finite Abelian group can be represented as the direct sum of cyclic subgroups. This definition of irreducible representation is equivalent to that given for a set of matrices (see REDUCIBLE—reducible set of matrices) when the *set* is a *group.*

representation of a group. (1) A group of a particular type (*e.g.*, a permutation group, or a group of matrices) which is *isomorphic* with the given group. Every finite group can be represented by a permutation group and by a group of matrices. (2) A group H is a **representation** of a group G if there is a homomorphism of G onto H. A set of representations consisting of matrices (or transformations) is a **complete system of representations** of G if for any g of G other than the identity there is a representation for which g does not correspond to the identity matrix (or identity transformation). Any finite group has a complete system of matrix representations and any *locally compact topological group* has a complete system of representations consisting of *unitary transformations* of Hilbert space. The order of the matrices in a matric representation is called the **degree** or **dimension** of the representation. See PERMUTATION—permutation matrix, and above, reducible matrix representation of a group.

spherical representation. See SPHERICAL.

RE-SERVE′, *n.* (*Life Insurance.*) The amount an insurance company needs (at a given time) to add to future net premiums and interest to pay all claims expected according to the particular mortality table being used (this is the difference between the present value of future benefits and the present value of future premiums and is a liability, also called the **reinsurance fund** or **self-insurance fund**). The **reserve per policy** (the **net premium reserve**) is called the **initial reserve** when computed at the beginning of a policy year just after the premium has been received, and the **terminal reserve** when computed at the end of a policy year before the premium has been paid. The average of the initial and terminal reserves is called the **mean reserve.** The **prospective method** for computing reserves makes use of the fact that the reserve (initial or terminal) is the difference between the present value of future benefits and the present value of future premiums. It is also possible to use the past history of the policy to compute the reserve (the **retrospective method**), computing the present value of the past differences of the *net level premiums* and the *natural premiums* (this difference is positive in the early years of the policy and negative in the later years). A **premium deficiency reserve** is an amount equal to the difference between the present value of future net premiums and future gross premiums (required in most states when gross premium is less than net premium).

RE-SID′U-AL, *adj.* **residual set.** See CATEGORY—category of sets.

RES′I-DUE, *adj.*, *n.* If the congruence $x^n \equiv a \pmod m$ has a solution, then a is called a **residue** (in particular, a *power residue* of m of the nth order). If this congruence has no solution, a is called a **non-residue** of m. Thus 4 is a residue of 5 of the second order, since $3^2 \equiv 4 \pmod 5$. The congruence $x^n \equiv a \pmod m$ is solvable if and only if $a^{\phi/d} \equiv 1 \pmod m$, where $\phi = \phi(m)$ is *Euler's ϕ-function* of m and d is the greatest common divisor of n and ϕ. Thus a is a residue of m of order n if and only if $a^{\phi/d} \equiv 1 \pmod m$. This is called *Euler's criterion.*

complete residue system modulo n. Any set of integers which have the property that no two belong to the same number class modulo n is said to form a complete residue system modulo n. This set of integers is also said to form a complete system of incongruent numbers modulo n. *E.g.*, 1, 9, 3, −3, 5, −1, 7 form such a system modulo 7. This set of integers may be also expressed in terms of positive or zero

integers each less than 7 by the numbers 1, 2, 3, 4, 5, 6, 0.

reduced residue system modulo n. A complete residue system modulo n contains some numbers prime to n. This set of numbers is called a reduced residue system modulo n. Thus a reduced residue system modulo 6 is 1, 5; whereas a complete residue system modulo 6 is 1, 2, 3, 4, 5, 0. See RESIDUE—complete residue system modulo n.

residue of an analytic function at an isolated singular point. If $f(z)$ is an analytic function of the complex variable z in the "deleted" neighborhood consisting of all z satisfying $0<|z-z_0|<\epsilon$, then the residue of $f(z)$ at z_0 is $\frac{1}{2\pi i}\int_C f(z)\,dz$, where C is a simple closed rectifiable curve about z_0 in the "deleted" neighborhood. The value of the residue is a_{-1}, where a_{-1} is the coefficient of $(z-z_0)^{-1}$ in the Laurent expansion of $f(z)$ about z_0.

RE-SIST'ANCE, *n.* **electrical resistance.** That property of a conductor which causes the passage of an electric current through it to be accompanied by the transformation of electric energy into heat. See OHM.

RE-SOL'VENT, *adj., n.* **resolvent cubic.** See FERRARI's solution of the general quartic.

resolvent kernel. (*Integral Equations.*) See VOLTERRA—Volterra's reciprocal functions, and KERNEL—iterated kernels.

resolvent of a matrix. The inverse of the matrix $\lambda I-A$, where A is the given matrix, and I is the identity matrix. The resolvent exists for all λ which are not *eigenvalues* of the matrix.

resolvent set of a transformation. See SPECTRUM—spectrum of a transformation.

RES'O-NANCE, *n.* See OSCILLATION.

RE-SULT', *n.* The end sought in a computation or proof.

RE-SULT'ANT, *n.* **resultant of a set of polynomial equations.** A relation between coefficients which is obtained by eliminating the variables and which must be zero if the equations can all be satisfied by the same values of the variables (also called an **eliminant**). In case the equations are linear, this can be accomplished expeditiously by equating to zero the determinant (of order $n+1$) whose columns are the coefficients of the respective variables and the constant terms. *E.g.*, the result of eliminating x and y from

$$\begin{matrix} ax+by+c=0 \\ dx+ey+f=0 \\ gx+hy+k=0 \end{matrix} \quad \text{is} \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}=0$$

The determinant of the coefficients of a system of n homogeneous linear equations in n unknowns is also called a resultant (or eliminant) of the equations (it is zero if and only if the equations have a *nontrivial* simultaneous solution). For two polynomial equations in one variable,

$$f(x)=a_0x^m+a_1x^{m-1}+\cdots+a_m=0,\ a_0\neq 0,$$
$$g(x)=b_0x^n+b_1x^{n-1}+\cdots+b_n=0,\ b_0\neq 0,$$

the resultant is usually taken to be

$$R(f,g)=a_0^n g(r_1)g(r_2)\cdots g(r_m),$$

where $r_1, r_2, \cdots, r_m$ are the roots of $f(x)=0$. This is also equal to the following determinant, which has n rows containing the coefficients of $f(x)$ and m rows containing the coefficients of $g(x)$:

$$\begin{vmatrix} a_0 & a_1 & a_2\cdots a_m & 0\cdots\cdots 0 \\ 0 & a_0 & a_1 & a_2\cdots a_m\,0\ldots 0 \\ 0 & 0 & a_0 & a_1\ a_2\cdots a_m\cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ b_0 & b_1 & b_2\cdots b_n & 0\cdots\cdots 0 \\ 0 & b_0 & b_1 & b_2\cdots b_n\ 0\cdots 0 \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

This determinant is the resultant obtained by Sylvester's dialytic method. See DISCRIMINANT—discriminant of a polynomial equation, and SYLVESTER.

resultant of two functions. Same as the CONVOLUTION of the two functions.

resultant of two or more forces (velocities, accelerations, etc.). That force (velocity, acceleration, etc.) whose effect is equivalent to the several forces (velocities, accelerations, etc.); see PARALLELOGRAM—parallelogram of forces. The resultant is the closing side, with direction reversed, of the open vector polygon whose sides represent the forces and are taken with the initial end of each vector on the terminal end of another.

RET'RO-SPEC'TIVE, *adj.* **retrospective method of computing reserves.** See RESERVE.

RE-VERSE', *adj.* Backward. A series of steps in a computation is taken in *reverse order* when the last is taken first, the next to last second, etc.; a finite sequence of terms is in *reverse order* when the last is made first, etc.

RE-VER'SION, *adj.* **reversion of a series.** The process of expressing x as a series in y, having given y expressed as a series in x.

RE-VER'SION-AR'Y, *adj.* See ANNUITY, and INSURANCE—participating insurance policy.

REV'O-LU'TION, *n.* **axis of revolution.** See SURFACE—surface of revolution, and below, solid of revolution.

cone, cylinder, and ellipsoid of revolution. See CONE, CYLINDER, and ELLIPSOID.

solid of revolution. A solid generated by revolving a plane area about a line (the **axis of revolution**). The volume of a solid of revolution can be computed without multiple integration in two ways. If a plane perpendicular to the axis of revolution intersects the solid in a region bounded by two circles, and the larger and smaller circles have radii r_2 and r_1, then the element of volume $\pi(r_2^2-r_1^2)\,dh$ can be used (the **washer method**), where h is measured along the axis of revolution, r_2 and r_1 are functions of h, and

$$\int_{h_1}^{h_2} \pi(r_2^2-r_1^2)\,dh$$

is the volume of the solid (h_1 and h_2 are the smallest and largest values of h for which the plane intersects the solid). If the axis of revolution is the x-axis and the area is that bounded by the x-axis, the ordinates corresponding to $x=a$ and $x=b$, and the curve $y=f(x)$, the elements of volume are discs ($r_1=0$) and the volume is

$$\int_a^b \pi f^2(x)\,dx.$$

If the plane area is all on the positive side of the axis of revolution and a line parallel to the axis of revolution and at distance x from it intersects the plane area in a line segment (or segments) of total length $L(x)$, then the element of volume $2\pi x L(x)\,dx$ is an approximation to the volume generated by revolving a thin strip of width dx and length $L(x)$ about the axis of revolution. The volume by this method (the **shell method**) is

$$\int_{x_1}^{x_2} 2\pi x L(x)\,dx,$$

where x_1 and x_2 are the minimum and maximum distances of the plane area from the axis of revolution.

surface of revolution. See SURFACE—surface of revolution.

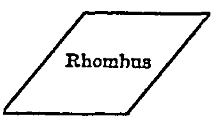
RE-VOLVE', *v.* To rotate about an axis or point. One would speak of revolving a figure in the plane about the origin, through an angle of a given size, or of *revolving* a curve in space about the x-axis with the understanding that the revolution is through an angle of 360° unless otherwise stipulated. See SURFACE—surface of revolution.

RHOMB, *n.* Same as rhombus.

RHOMBOHEDRON, *n.* A six-sided prism whose faces are parallelograms.

RHOM'BOID, *n.* A parallelogram with adjacent sides not equal.

RHOM'BUS, *n.* A parallelogram with adjacent sides equal (all of its sides are then necessarily equal). Some authors require that a rhombus not be a square, but the preference seems to be to call the square a special case of the rhombus.



RHUMB, *adj.* **rhumb line.** The path of a ship sailing so as to cut the meridians at a constant angle; a spiral on the earth's surface, winding around a pole and cutting the meridians at a constant angle. *Syn.* Loxodromic spiral.

RICCATI EQUATION. A differential equation of the form $dy/dx=f_1(x)+2yf_2(x)+y^2f_3(x)$.

RICCI. Ricci tensor. The *contracted curvature tensor* $R_{ij}=R^{\sigma}_{ij\sigma}$, where R^{p}_{ijk} is the *Riemann-Christoffel curvature tensor.* It is often called the **Einstein tensor** in general relativity theory, since it occurs in the Einstein gravitational equations. The Ricci tensor is a symmetric tensor since

$$\frac{\partial \log \sqrt{g}}{\partial x^j}=\begin{Bmatrix} i \\ ij \end{Bmatrix}.$$

RIEMANN. covariant Riemann-Christoffel curvature tensor. The covariant tensor field of rank four:

$$R_{i\alpha\beta\gamma}(x^1,\cdots,x^n)=g_{i\sigma}R^{\sigma}_{\alpha\beta\gamma}(x^1,\cdots,x^n).$$

See below, Riemann-Christoffel curvature tensor.

Riemann-Christoffel curvature tensor. The tensor field

$$R^{i}_{\alpha\beta\gamma}(x^1,x^2,\cdots,x^n)=$$

$$\frac{\partial\begin{Bmatrix} i \\ \alpha\beta \end{Bmatrix}}{\partial x^{\gamma}}-\frac{\partial\begin{Bmatrix} i \\ \alpha\gamma \end{Bmatrix}}{\partial x^{\beta}}+\begin{Bmatrix} \sigma \\ \alpha\beta \end{Bmatrix}\begin{Bmatrix} i \\ \sigma\gamma \end{Bmatrix}-\begin{Bmatrix} \sigma \\ \alpha\gamma \end{Bmatrix}\begin{Bmatrix} i \\ \sigma\beta \end{Bmatrix},$$

where the summation convention applies and $\begin{Bmatrix} i \\ jk \end{Bmatrix}$ are the *Christoffel symbols of the second kind* of the n-dimensional Riemannian space with the fundamental differential form $g_{ij}\,dx^i\,dx^j$. $R^{i}_{\alpha\beta\gamma}$ is a tensor field of rank four, contravariant of rank one and covariant of rank three. Many authors define $R^{i}_{\alpha\beta\gamma}$ as the negative of the above. See above, covariant Riemann-Christoffel curvature tensor.

Riemann hypothesis about the zeros of the zeta function. The zeta function has zeros at $-2,-4,\cdots$. All other complex numbers which are zeros of the zeta function lie in the strip of complex numbers z whose real parts satisfy $0<\text{Re}(z)<1$. Riemann's hypothesis is the (unproved) conjecture that these zeros actually lie on the line $\text{Re}(z)=\frac{1}{2}$. It is known that an infinite number of zeros lie on this line. The proof of Riemann's hypothesis would have extensive consequences in the theory of prime numbers. Riemann's hypothesis is true if and only if $\sum_{1}^{\infty}\mu(n)\,n^{-s}$ converges for the real part of s greater than $\frac{1}{2}$, where μ is the Möbius function.

Riemann integral. See INTEGRAL—definite integral.

Riemann mapping theorem. Any simply connected plane domain whose boundary contains more than one point can be mapped conformally on the interior of the unit circle. A *domain* is a nonempty connected open set.

Riemann sphere (or spherical surface). The surface on a unit sphere corresponding to a (plane) Riemann surface under a stereographic projection.

Riemann-Stieltjes integral. See STIELTJES.

Riemann surface. The relation between complex numbers z and complex numbers w expressed by the monogenic analytic function $w=f(z)$ might be one-to-one, one-to-many, many-to-one, or many-to-many. Respective examples are $w=(z+1)/(z-1)$, $w=z^2$, $w^3=z$, $w^3=z^2$. Riemann surfaces furnish a schematic device whereby the relation is considered as one-to-one (between points of the z- and w-Riemann surfaces) in any case. A suitable number, possibly a denumerably infinite number, of sheets is considered over the z-plane and over the w-plane. These might be joined in a variety of ways at branch points. The sheets are distinguished by imaginary branch cuts joining the branch points or extending to infinity. Thus $w^3=z^2$ gives a one-to-one mapping of a three-sheeted z-surface on a two-sheeted w-surface. Also see under TYPE.

Riemann zeta function. See ZETA.

Riemannian curvature. The scalar determined by a point and two linearly independent directions (contravariant vectors) $\xi_1{}^i$ and $\xi_2{}^i$ at that point:

$$k=\frac{R_{\alpha\beta\gamma\delta}\xi_1{}^{\alpha}\xi_2{}^{\beta}\xi_1{}^{\gamma}\xi_2{}^{\delta}}{(g_{\alpha\delta}g_{\beta\gamma}-g_{\alpha\gamma}g_{\beta\delta})\xi_1{}^{\alpha}\xi_2{}^{\beta}\xi_1{}^{\gamma}\xi_2{}^{\delta}},$$

The $g_{\alpha\beta}$ is the metric tensor of the *Riemannian space*, and $R_{\alpha\beta\gamma\delta}$ is the covariant Riemann-Christoffel curvature tensor (see under RIEMANN). A geometrical construction leading to the Riemannian curvature k is as follows: Consider the two-parameter family of directions $u\xi_1{}^{\alpha}+v\xi_2{}^{\alpha}$ at the given point and form the two-dimensional geodesic surface swept out by the geodesics through the point and having directions in the two-parameter family of directions. The *Gaussian curvature* (total curvature) of the geodesic surface at the given point is the Riemannian curvature of the envelop-

ing n-dimensional Riemannian space at the given point and with respect to the given orientation.

Riemannian space. An n-dimensional coordinate manifold, *i.e.*, a space of points $(x_1, x_2, \cdots, x_n)$, whose element of arc length ds is given by a symmetric quadratic differential form

$$ds^2 = g_{ij}(x^1, \cdots, x^n)\, dx^i\, dx^j,$$

in which the coefficients g_{ij} have a non-vanishing determinant and the *summation convention* applies. It is also often required that the differential form be positive definite, although this restriction is not imposed in applications to general relativity theory. The g_{ij} are the components of a symmetric covariant tensor, called the **fundamental metric tensor.**

Riemannian space of constant Riemannian curvature. A Riemannian space such that the *Riemannian curvature* is the same throughout space and at the same time is independent of the orientation $\xi_1{}^i$, $\xi_2{}^i$. The spaces of constant Riemannian curvature k such that $k>0$, $k<0$, and $k=0$ are locally the **Riemann spherical space, Lobachevski space,** and **Euclidean space,** respectively.

RIESZ-FISCHER THEOREM. Let a measure m be defined on a set Ω and let L_2 be the set of all measurable (real or complex) functions for which $\int |f|^2\, dm$ is finite. The Riesz-Fischer theorem asserts that L_2 is *complete*; *i.e.*, for a sequence $f_1, f_2, \cdots$ of elements of L_2, there is an f in L_2 such that the sequence converges in the mean (of order 2) to f if $\|f_m - f_n\| \to 0$ as m and n become infinite, where $\|f_m - f_n\| = \int |f_m - f_n|^2\, dm$. An immediate consequence of this, also called the Riesz-Fischer theorem, is that if $u_1, u_2, \cdots$ is an orthonormal sequence of functions and $a_1, a_2, \cdots$ is a sequence of complex (or real) numbers for which $\Sigma |a_n|^2$ is convergent, then there exists a function f belonging to L_2 such that $a_n = \int f(x)\overline{u_n(x)}\, dm$ for each n. *E.g.*, a trigonometric series $\frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$ is a Fourier series of some function if (and only if) $\sum_1^\infty (a_n{}^2 + b_n{}^2)$ is convergent.

RIGHT, *adj., n.* **continuous on the right.** See CONTINUOUS.

limit on the right. See LIMIT—limit on the left or right.

right angle. See ANGLE—right angle.

right dihedral angle. See PLANE—plane angle of a dihedral angle.

right-handed coordinate system. See COORDINATE—right-handed coordinate system.

right-handed curve. If the torsion of a directed curve C at a point P is negative, then C is said to be right-handed at P. See LEFT—left-handed curve. *Syn.* Dextrorsum [*Latin*] or dextrorse curve.

right-handed trihedral. See TRIHEDRAL.

right line. A straight line.

right section of a surface. See SECTION—section of a surface.

right triangle. See SPHERICAL—spherical triangle, and TRIANGLE.

RIG'ID, *adj.* **rigid body.** An ideal body which is characterized by the property that the distance between every pair of points of the body remains unchanged. Objects which experimentally are not readily deformable approximate rigid bodies.

rigid motion. Moving a configuration into another position, but making no change in its shape or size; a rotational transformation followed by a translation, or the two taken in reverse order or simultaneously. Superposition of figures in plane geometry is a *rigid motion.*

RI-GID'I-TY, *n.* **modulus of rigidity.** The ratio of the shearing stress to the change in angle produced by the shearing stress. *Syn.* Shearing modulus. See LAMÉ'S CONSTANTS.

RING, *adj., n.* A set with two operations, called addition and multiplication, which have the properties: (1) The set is an *Abelian group* with respect to the operation of addition. (2) Each pair a, b of elements determines a unique product $a \cdot b$, multiplication is *associative*, and multiplication is *distributive with respect to addition*; *i.e.*,

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

and

$$(b + c) \cdot a = b \cdot a + c \cdot a$$

for each a, b, and c of the set. If it is also true that multiplication is *commutative*, the ring is a **commutative ring.** If there is an identity for multiplication (an element 1 for which $1 \cdot x = x \cdot 1 = x$ for all x), the ring is a **ring with unit element** (or a **ring with unity).** A commutative ring with unit element is an integral domain if no product of nonzero elements is zero and a field if each nonzero element has a multiplicative inverse. See DOMAIN—integral domain, FIELD, and IDEAL.

normed vector ring. See ALGEBRA—Banach algebra.

quotient (or factor) ring. See QUOTIENT—quotient space.

ring of sets. A nonempty class of sets which contains the union and the difference of any two of its members. It is a **σ-ring** if it also contains the union of any sequence of its members. A ring of sets is also a *ring* if *symmetric difference* and *intersection* are taken as the *addition* and *multiplication* operations of the ring. For an arbitrary set X, the class of all finite subsets of S is a ring of sets. Another example of a ring of sets is the class of sets of real numbers which are finite unions of intervals which contain their left end-points and do not contain their right end-points. See ALGEBRA—algebra of subsets, MEASURE—measure ring and measure algebra.

ring surface, torus ring. Same as ANCHOR RING. See ANCHOR.

semiring of sets. A class S of sets which contains the empty set and the intersection of any two of its members and which is such that if A and B are members of S with $A \subset B$, then there are a finite number of sets $C_1, C_2, \cdots, C_n$ such that $B - A = \Sigma C_i$, $C_i \cap C_j = 0$ if $i \neq j$, and each C_i is a member of S. Every ring of sets is also a semiring of sets.

RISE, *n.* **rise between two points.** The difference in elevation of the two points. See RUN.

rise of a roof. (1) The vertical distance from the plates to the ridge of the roof. (2) The vertical distance from the lowest to the highest point of the roof.

ROBIN'S FUNCTION. For a region R with boundary surface S, and for a point Q interior to R, the Robin's function $R_{k,h}(P, Q)$ is a function of the form $R_{k,h}(P, Q) = 1/(4\pi r) + V(P)$, where r is the distance PQ, $V(P)$ is harmonic, and $k\,\partial R_{k,h}/\partial n + hR_{k,h} = 0$ on S. The solution $U(Q)$ of the third boundary-value problem of potential theory (the Robin problem) can be represented in the form

$$U(Q) = \int_S f(P) R_{k,h}(P, Q)\, d\sigma_P.$$

See GREEN—Green's function, BOUNDARY—third boundary value problem of potential theory.

RODRIGUES. equations of Rodrigues. The equations $dx + \rho\, dX = 0$, $dy + \rho\, dY = 0$, $dz + \rho\, dZ = 0$ characterizing the lines of curvature of the surface S. The function ρ is the radius of normal curvature in the direction of the line of curvature.

Rodrigues' formula. The equation

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n,$$

where $P_n(x)$ is a *Legendre polynomial.*

ROLLE'S THEOREM. If a continuous curve crosses the x-axis at two points and has a unique tangent at all points between these two x-intercepts, it has a tangent parallel to the x-axis at at least one point between the two intercepts. *Tech.* If $f(x)$ is a single-valued continuous function for $a \leqq x \leqq b$ and vanishes for $x = a$ and $x = b$ and has derivatives at all interior points on (a, b), then $f'(x)$ vanishes at some point between and distinct from a and b. (It may also vanish at a or b or both.) *E.g.*, the sine curve crosses the x-axis at the origin and at $x = \pi$, and has a tangent parallel to the x-axis at $x = \frac{1}{2}\pi$ (radians).

RO'MAN, *adj.* **Roman numerals.** A system of writing integers, used by the Romans, in which I denotes 1; V, 5; X, 10; L, 50; C, 100; D, 500; M, 1000. All integers are then written using the following rules: (1) When a letter is repeated or immediately followed by a letter of lesser value, the values are added. (2) When a letter is immediately followed by a letter of greater value, the smaller is subtracted from the larger. The integers from 1 to 10 are written: I, II, III, IIII or IV, V, VI, VII, VIII, IX, X. The tens are written: X, XX, XXX, XL, L, LX, LXX, LXXX, XC, C. Hundreds are written C, CC, CCC, CD, D, DC, DCC, DCCC, CM, M.

ROOT, *n.* **double, equal, simple, triple and multiple roots.** See MULTIPLE—multiple root of an equation.

infinite root of an equation. An equation of degree $r<n$ which is considered to be an equation of degree n is said to have infinity as a root $n-r$ times. *E.g.*, the equation $ax^2+bx+c=0$ has one infinite root if $a=0$ and $b\neq 0$; two infinite roots if $a=b=0\neq c$. If x is replaced by $1/y$ in this quadratic equation one obtains the equation $a+by+cy^2=0$, which has the same number of zero roots as the original equation had infinite roots. With this convention, it can be said that a line and a hyperbola always intersect in two points; one or both of these may be a point at infinity. See IDEAL—ideal point.

rational root theorem. See RATIONAL—rational root theorem.

root of a congruence. A number which when substituted in the congruence, expressed in a form $f(x)\equiv 0 \pmod{n}$, makes the left member of the congruence divisible by the modulus n. Thus $x=8$ is a root of the congruence $x+2\equiv 0 \pmod 5$, since $8+2$ or 10 is divisible by 5. Another root is $x=3$.

root of an equation. A number which, when substituted for the unknown in the equation, reduces it to an identity (a root of the equation $x^2+3x-10=0$ is 2, since $2^2+3\cdot 2-10=0$). A root of an equation is said to *satisfy the equation* or to be a *solution* of the equation, but *solution* more often refers to the process of finding the root. There are many ways to approximate a root of an equation (see FALSE—method of false position, GRAEFFE, GRAPHICAL—graphical solution of an equation, HORNER'S METHOD, and NEWTON—Newton's method of approximation). Usually, a basic step in approximating a root is to isolate the root, *i.e.*, find two numbers between which there is one and only one root of the equation. The following **location principle** is very useful: If a polynomial or other continuous function of one variable has different signs for two values of the variable, it is zero for some value of the variable between these two values; the equation obtained by equating the given function to zero has a root between two values of the unknown for which the function has different signs. Geometrically, if the graph of a continuous function of a variable x is for one value of x on one side of the x-axis and for another value on the other side (changes sign), it must cross the axis between the two positions. From a given equation, it is possible to derive new equations whose roots are related to those of the given equation in various ways. One can **change the signs of the roots** by replacing the unknown by its negative, giving a new equation whose roots are the negatives of the roots of the original equation. One can **decrease the roots** by the amount a by transforming the equation by the substitution $x=x'+a$, $a>0$, where x is the unknown in the given equation. If the old equation has the root x_1, the new one has a root $x_1'=x_1-a$. The substitution of $x=x'+2$ in $x^2-3x+2=0$, whose roots are 1 and 2, results in the equation $(x')^2+x'=0$ whose roots are -1 and 0. The substitution $x=1/x'$ transforms an equation so that the given equation has roots which are the **reciprocals of the roots** of the transformed equation (see RECIPROCAL—reciprocal equation). The **roots and coefficients** of a polynomial equation are related in the following ways: For a quadratic equation, the sum of the roots is equal to the negative of the coefficient of the first-degree term and the product is equal to the constant term, when the coefficient of the square term is 1. In $ax^2+bx+c=0$, the sum of the roots is $-b/a$ and the product is c/a. If the equation is of the nth degree and the coefficient of the nth degree term is unity, the sum of the roots is the negative of the coefficient of x^{n-1}, the sum of the products of the roots taken two at a time in every possible way is the coefficient of x^{n-2}, the sum of the products of the roots taken three at a time is the negative of the coefficient of x^{n-3}, etc.; finally, the product of all the roots is the constant term with a positive or negative sign according as n is even or odd. If $r_1, r_2, \cdots, r_n$ are the roots of

$$x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=0,$$

then

$$r_1+r_2+\cdots+r_n=-a_1,$$
$$r_1r_2+r_1r_3+\cdots+r_1r_n+r_2r_3+\cdots+r_{n-1}r_n=a_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$r_1r_2r_3\cdots r_n=(-1)^n a_n.$$

See POLYNOMIAL—polynomial equation, QUADRATIC—quadratic formula, CARDAN, FERRARI'S solution of the quartic, and MULTIPLE—multiple root of an equation.

root mean square deviation. See DEVIATION—standard deviation.

root of a number. An nth root of a number is a number which, when taken as a factor n times (raised to the nth power), produces the given number. There are n nth roots of any nonzero number (these may be real or imaginary). If n is odd and the number real, there is one real root; *e.g.*, the cube roots of 27 are 3 and $\frac{3}{2}(-1 \pm \sqrt{-3})$. If n is even and the number positive, there are two real roots, numerically equal but opposite in sign; *e.g.*, the 4th roots of 4 are $\pm\sqrt{2}$ and $\pm\sqrt{-2}$. A **square root** of a number is a number which, when multiplied by itself, produces the given number. A positive (real) number has two real square roots, a negative number two imaginary square roots. A **cube root** of a number is a number whose cube is the given number. Each real number (except zero) has one real cube root and two imaginary cube roots. If a **complex number** (which may be a real number) is written in the form

$$r[\cos\theta + i\sin\theta],$$

or the equivalent form

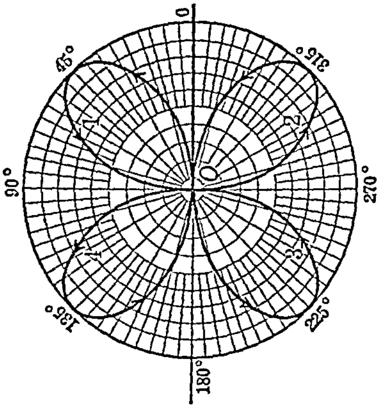
$$r[\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)],$$

its nth roots are the numbers

$$\sqrt[n]{r}\left[\cos\frac{(2k\pi+\theta)}{n} + i\sin\frac{(2k\pi+\theta)}{n}\right],$$

where k takes on the values $0, 1, 2, \cdots, (n-1)$ and $\sqrt[n]{r}$ is an nth root of the nonnegative number r. See DEMOIVRE'S THEOREM, and UNITY—roots of unity.

ROSE, *n.* The graph in polar coordinates of $r = a\sin n\theta$, or $r = a\cos n\theta$, where n is a positive integer. It consists of rose petal-shaped loops with the origin a point common to all of them. When n is odd there are n of the loops; when n is even there are $2n$ of them. The **three-leafed rose** is the graph of the equation $r = a\sin 3\theta$, or $r = a\cos 3\theta$. The curve consists of three loops with their vertices at the pole. The locus of the first equation has the first petal tangent to the positive polar axis, and symmetric about the line $\theta = 30°$, the second petal symmetric about the line $\theta = 150°$, and the third symmetric about the line $\theta = 270°$, each loop thus being tangent to the sides of an angle of 60°. The length of the line of symmetry of each petal, from the pole to the intersection of the curve, is a. The locus of the second equation is the same as that of the first rotated 30° about the origin. The **four-leafed rose** is the graph of the equation $r = a\sin 2\theta$, or $r = a\cos 2\theta$. The graph of the first equation (shown in the figure) has the four petals



symmetric by pairs about each of the lines $\theta = 45°$ and $\theta = 135°$, and tangent to the coordinate axes in the several quadrants. The length of the line of symmetry of each petal, from the pole to the intersection with the curve, is a. The graph of the second equation is the same, except that the petals are symmetric about the coordinate axes and tangent to the lines $\theta = 45°$ and $\theta = 135°$.

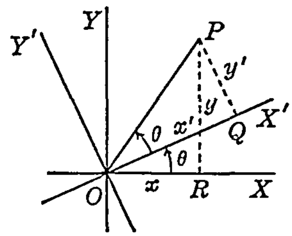
RO-TA'TION, *n.* **rotation about a line.** Rigid motion about the line of such a kind that every point in the figure moves in a circular path about the line in a plane perpendicular to the line.

rotation about a point. Rigid motion in a circular path (in a plane) about the point.

rotation of axes. A rigid motion which leaves the origin fixed. Such a transformation of axes is convenient in studying curves and surfaces, since it does not alter them intrinsically (preserves size and shape). *E.g.*, by a proper rotation of the coordinate axes in the plane, they can be made parallel to the axes of any given ellipse or hyperbola, or one of them parallel to the axes of

any given parabola, thus in each case making the term which contains xy disappear. **In the plane**, the formulas for rotation (rotation formulas) which give the relations between the coordinates (x', y') of a point with reference to a set of axes obtained by rotating a set of rectangular axes through the angle θ, and the coordinates (x, y) relative to the old axes, are

$$x = x' \cos\theta - y' \sin\theta,$$
$$y = x' \sin\theta + y' \cos\theta,$$



where θ is the angle ROQ. **In space**, a rotation moves the coordinate trihedral in such a way as to leave the origin fixed and the axes in the same relative position. The coordinates of a point are transformed from those referred to one system of rectangular Cartesian axes to coordinates referred to another system of axes having the same origin but different directions and making certain given angles with the original axes. If the direction angles, with respect to the old axes, of the new x-axis (the x'-axis) are A_1, B_1, C_1; of the y'-axis are A_2, B_2, C_2; and of the z'-axis, A_3, B_3, C_3, then the formulas for rotation of axes in space are

$$x = x' \cos A_1 + y' \cos A_2 + z' \cos A_3,$$
$$y = x' \cos B_1 + y' \cos B_2 + z' \cos B_3,$$
$$z = x' \cos C_1 + y' \cos C_2 + z' \cos C_3.$$

See ORTHOGONAL—orthogonal transformation.

ROUCHE'S THEOREM. If $F(z)$ and $f(z)$ are analytic functions of the complex variable z in and on a simple rectifiable curve C, and $|F(z)| > |(f(z)|$ at each point on C, then the functions $F(z)$ and $f(z)+F(z)$ have the same number of zeros in the finite domain bounded by C.

ROUND, *adj.* **round angle.** An angle of 360°; a perigon.

ROUND'ING, *n.* **rounding off numbers.** Dropping decimals after a certain significant place. When the first digit dropped is less than 5, the preceding digit is not changed; when the first digit dropped is greater than 5, or 5 and some succeeding digit is not zero, the preceding digit is increased by 1; when the first digit dropped is 5, and all succeeding digits are zero, the commonly accepted rule (computer's rule) is to make the preceding digit even, *i.e.*, add 1 to it if it is odd, and leave it alone if it is already even. *E.g.*, 2.324, 2.316, and 2.315 would take the form 2.32, if rounded off to two places.

ROUND-OFF ERROR. An error in computation resulting from the fact that the computation is not exact but instead is carried out to only a specified number of decimal places.

ROW, *n.* An arrangement of terms in a horizontal line. Used with *determinants* and *matrices* to distinguish horizontal arrays of elements from vertical arrays, which are called **columns**. See DETERMINANT.

RULE, *n.* (1) A prescribed operation or method of procedure; a formula (usually in words, although *rule* is often used synonymously with *formula*). See DESCARTES—Descartes' rule of signs, EMPIRICAL—empirical rule, L'HOSPITAL'S RULE, MECHANIC—mechanic's rule, MERCHANT—merchant's rule, and THREE—rule of three. (2) A graduated straight edge. *Syn.* Ruler.

slide rule. See SLIDE.

RULED, *adj.* **conjugate ruled surface of a given ruled surface.** The ruled surface whose rulings are the lines tangent to the given ruled surface S, at the points of the line of striction L of S, and orthogonal to the rulings of S at the corresponding points of L.

ruled paper. Same as CROSS-SECTION PAPER.

ruled surface. A surface that can be generated by a moving straight line. The generating straight line is called the rectilinear generator. A **doubly ruled surface** is a ruled surface admitting two different sets of generators. Quadric surfaces are the only

doubly ruled surfaces. A **skew ruled surface** is a ruled surface which is not a *developable surface* (see DEVELOPABLE). The various positions of a straight line which generate a ruled surface are the **rulings** of the surface. The cone, cylinder, hyperbolic paraboloid, and hyperboloid of one sheet are ruled surfaces. See RULING.

RUL'ER, *n.* A straight edge graduated in linear units. If English units are used, the *ruler* is usually a foot long, graduated to fractions of an inch. *Syn.* Rule.

RUL'ING, *n.* See RULED—ruled surface.

central plane and point of a ruling. For a fixed ruling L on a ruled surface S, the **central point** is the point in the limiting position of the foot on L of the common perpendicular to L and a variable ruling L' on S, as $L' \rightarrow L$. The plane tangent to a ruled surface S at any point of a ruling L on S necessarily contains L. The plane tangent to S at the central point of L is called the **central plane** of the ruling L on the ruled surface S.

RUN, *n.* A term sometimes used in speaking of the difference between the abscissas of two points. The *run* from the point whose coordinates are (2, 3) to the one whose coordinates are (5, 7) is $5-2$, or 3. The distance between the ordinates is sometimes called the **rise**. Thus the *run* squared plus the *rise* squared is equal to the square of the distance between the two points.

RUNGE-KUTTA METHOD. An approximate method for solving differential equations. To determine an approximate solution of $dy/dx=f(x, y)$ that passes through the point (x_0, y_0), we let $x_1=x_0+h$ and the method determines a corresponding $y_1=y_0+k$ by means of the formulas

$$k_1=h\cdot f(x_0, y_0), \quad k_2=h\cdot f(x_0+\tfrac{1}{2}h, y_0+\tfrac{1}{2}k_1),$$
$$k_3=h\cdot f(x_0+\tfrac{1}{2}h, y_0+\tfrac{1}{2}k_2),$$
$$k_4=h\cdot f(x_0+h, y_0+k_3),$$
$$k=\tfrac{1}{6}(k_1+2k_2+2k_3+k_4).$$

The process is then repeated, starting with (x_1, y_1). This method, which reduces to *Simpson's rule* if f is a function of x alone, can be extended to the approximate solution of systems of linear differential equations and to the solution of higher-order linear differential equations and systems of equations.

RUSSELL. Russell's paradox. Suppose that all sets are divided into two types: A set M is of the *first type* if it does not contain M itself as a member; and it is of the *second type* if it does contain M itself as a member. **Russell's paradox** is that the set N of all sets of the first type must be of the first type, since otherwise the set N of the second type would be a member of N; but N would then be of the second type, since N itself would be a member of N. It thus appears that the concept of all sets which are not members of themselves is not free from contradiction. See BURALI-FORTI PARADOX.

S

SAD'DLE, *adj.* **saddle point.** A point for which the two first partial derivatives of a function $f(x, y)$ are zero, but which is not a local maximum or a local minimum. If the second-order partial derivatives are continuous in a neighborhood of p, $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$ and $\left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2-\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}>0$ at p, then p is a saddle point (see MAXIMUM). At a saddle point, the tangent plane to the surface $z=f(x, y)$ is horizontal, but near the point the surface is partly above and partly below the tangent plane. *Syn.* Minimax.

saddle point of a game. It is easy to see that, for any finite two-person zero-sum game, the elements a_{ij} of the *payoff matrix* satisfy the relation

$$\max_i (\min_j a_{ij}) \leqq \min_j (\max_i a_{ij}).$$

If the sign of equality holds, then $\max_i (\min_j a_{ij})=\min_j (\max_i a_{ij})=v$ and there exist pure strategies i_0 and j_0 for the maximizing and minimizing players, respectively, such that if the maximizing player chooses i_0 then the payoff will be at least v no matter what strategy the minimizing player chooses, and if the minimizing player chooses j_0 then the payoff will be at most v no matter what strategy the maximizing player chooses. Thus

$$v=a_{i_0j_0}=\max_i a_{ij_0}=\min_j a_{i_0j}.$$

In this case, the game is said to have a *saddle point* at (i_0, j_0). There might be more than one saddle point at which the value v is taken on. Similar statements hold for an infinite two-person zero-sum game, for which there might or might not be a saddle point. See BOX—three boxes game, MINIMAX—minimax theorem, and PAYOFF.

saddle point of a matrix. Any finite matrix of real numbers, with element a_{ij} in the ith row and jth column, might be considered to be the *payoff matrix* of a finite two-person zero-sum game. If the game has a saddle point at (i_0, j_0), then the matrix is said to have a saddle point at (i_0, j_0). A necessary and sufficient condition that a matrix have a saddle point is that there exist an element that is both the minimum element of its row and the maximum element of its column. See above, saddle point of a game.

SAIL'ING, *n.* **middle latitude sailing.** Approximating the difference in longitude (DL) of two places from their latitudes (L_1 and L_2) and departure (p) by the formula $p \sec \frac{1}{2}(L_1 + L_2) = DL$ measured in minutes.

parallel sailing. Sailing on a parallel of latitude; using the above formula, putting $L_1 = L_2$.

plane sailing. Sailing on a rhumb line. The constant angle which the rhumb line makes with the meridians is called the **ship's course.** Requires solving a plane right triangle.

triangle of plane sailing. See TRIANGLE —triangle of plane sailing.

SAINT-VENANT. Saint-Venant's compatibility equations. See STRAIN—strain tensor.

Saint-Venant's principle. If some distribution of forces acting on a portion of the surface of a body is replaced by a different distribution of forces acting on the same portion of the body, then the effects of the two different distributions on the parts of the body sufficiently far removed from the region of application of the forces are essentially the same, provided the two distributions of forces have the same resultant force and moment.

SA'LI-ENT, *adj.* **salient angle.** See REENTRANT—reentrant angle.

salient point on a curve. A point at which two branches of a curve meet and stop and have different tangents. The curves

$$y = x/(1 + e^{1/x}) \quad \text{and} \quad y = |x|$$

have *salient points* at the origin.

SAL'TUS, *n.* **saltus of a function.** See OSCILLATION—oscillation of a function.

SAM'PLE, *n.* (*Statistics.*) A finite portion of a population or universe. **Large sample** frequently refers to a sample with more than about 30 observations. **Small sample** usually refers to one with less than about 30 observations.

random sample. See various headings under RANDOM.

stratified random sample. If the population to be sampled is first subclassified into several subpopulations, the sample may be drawn by taking *random samples* from each of the subclasses. The samples need not be proportional to the subpopulation sizes; but, if the combined set of random samples is to be used for purposes of estimating certain population characteristics of the combined population, the assignment of the proportions of the total sample to the subpopulations must be such that

$$\frac{n_1}{N_1\sigma_1^2} = \frac{n_2}{N_2\sigma_2^2} = \cdots = \frac{n_k}{N_k\sigma_k^2},$$

where n_i is the sample size from the ith sub-population with N_i cases and variance σ_i^2. This sample will be the type for which the parameters may be estimated with minimum variance.

stratified sample. Let a population be divided into several subpopulations, called **strata.** If from each of these strata random samples are drawn, the resulting pooled sample is a stratified sample. In effect the original population is divided into several subpopulations and random samples are drawn from each. Thus a stratified sample is basically a group of random samples. Let a population be divided into several strata, within each of which: (1) The standard deviation σ_t of the characteristic under analysis is determinable; (2) the frequency is n_t and is

known. Then for that system of classification, the *stratified sample* which provides the *minimum-variance unbiased estimate* of the mean of the characteristic of the population is the one for which the number of random observations for the ith stratum is proportional to $n_i\sigma_i$. If only the n_i are known, then the sampling procedure which minimizes the variance of the estimates of the mean of the population is one in which the number of observations in the ith stratum is proportional to the n_i. This is sometimes called a **representative** or **proportional sample.**

SAM′PLING, *adj.* **area sampling.** A method of drawing *random samples* in which the sampling units are areas. Usually used because the difficulty of obtaining random sampling units is reduced if each unit (or sets of units) is identified with a geographical location (area).

sampling error. (*Statistics.*) The difference between a random sampling statistic and the parameter of the population from which the *random sample* was drawn is the **sampling error.** Usually refers to the standard deviation of the set of errors that would arise from an infinite set of estimates from random samples. Sampling errors connote the idea of differences between the parameter and an estimate of it, the difference arising solely from the random sampling selection of the items from the population.

SAT′IS-FY, *v.* (1) To fulfill the conditions of, such as to *satisfy* a theorem, a set of assumptions, or a set of hypotheses. (2) A set of values of the variables which will reduce an equation (or equations) to an identity are said to satisfy the equation (or equations); $x=1$ *satisfies* $4x+1=5$; $x=2$, $y=3$ *satisfy* the simultaneous equations

$$x+2y-8=0$$
$$x-2y+4=0.$$

SCA′LAR, *adj., n.* **scalar field.** See TENSOR.

scalar matrix. See MATRIX.

scalar product. See MULTIPLICATION—multiplication of vectors.

scalar quantity. (1) The ratio between two quantities of the same kind, a number. (2) A number, as distinguished from a vector, matrix, quaternion, etc. (3) A tensor of order zero. See TENSOR. *Syn.* Scalar.

SCALE, *n.* A system of marks in a given order and at known intervals. Used on rulers, thermometers, etc., as an aid in measuring various quantities.

binary scale. Numbers written with the base two, instead of ten. Numbers in which the second digit to the left indicates the two's; the third, four's, etc.; 1101 with base 2 means $2^3+2^2+0\times2+1$ or 13 written with base ten. See BASE—base of a system of numbers.

diagonal scale. See DIAGONAL—diagonal scale.

drawing to scale. Making a copy of a drawing with all distances in the same ratio to the corresponding distances in the original; making a copy of a drawing of something with all distances multiplied by a constant factor. *E.g.*, an architect drawing the plan of a house lets feet in the house be denoted by inches, or fractions of an inch, in his drawing, but a bacteriologist might draw at a scale of 4000 to 1.

logarithmic scale. See LOGARITHMIC—logarithmic coordinate paper.

natural scale. The section of the number scale which contains positive integers only.

number scale (complete number scale). The scale formed by marking a point 0 on a line, dividing the line into equal parts, and labeling the points of division to the right of 0 with the integers $1, 2, 3, \cdots$ and those to the left with the negative integers, $-1, -2, -3, \cdots$.

scale of imaginaries. The number scale modified by multiplying each of its numbers by i $(=\sqrt{-1})$. In plotting complex numbers the scale of imaginaries is laid off on a line perpendicular to the line which contains the real number scale. See ARGAND DIAGRAM.

uniform scale. A scale in which equal numerical values correspond to equal distances.

SCA-LENE′, *adj.* **scalene triangle.** A triangle no two of whose sides are equal (the triangle may be either a plane triangle or a spherical triangle).

SCAT′TER-GRAM′, *n.* A diagram showing the frequencies with which joint values of variables are observed. One variable is indicated along the ordinate and the other along the abscissa. The intersections of the rows and columns form cells in which the frequencies are indicated.

SCHLÄFLI. Schäfli's integral for $P_n(z)$. The integral

$$\frac{1}{2\pi i}\int_C \frac{(t^2-1)^n}{2^n(t-z)^{n+1}}\,dt = P_n(z),$$

where $P_n(z)$ is the *Legendre polynomial* of order n and the integration is counterclockwise around a contour C encircling the point z in the complex plane.

SCHLICHT, *adj.* (*German.*) **schlicht function.** Same as SIMPLE FUNCTION. See SIMPLE.

SCHLOEMILCH. Schloemilch's form of the remainder for Taylor's theorem. See TAYLOR—Taylor's theorem.

SCHUR. Schur's lemma. Let S_1 and S_2 be two *irreducible* collections of matrices, corresponding to linear transformations of vector spaces of dimensions n and m, respectively. If there is an $n\times m$ matrix P such that for any A of S_1 there is a B of S_2, and for any B of S_2 an A of S_1, such that $AP=PB$, then either P has all elements zero or P is *square* and *nonsingular*. In the latter case, the two collections S_1 and S_2 are *equivalent* (for any B of S_2 there is an A of S_1 such that $B=P^{-1}AP$).

Schur's theorem. If the *Riemannian curvature* k of an n-dimensional ($n\geqq 2$) Riemannian space is independent of the orientation $\xi_1{}^i$, $\xi_2{}^i$, then k does not vary from point to point. With the aid of Schur's theorem it follows that a necessary and sufficient condition that an n-dimensional ($n\geqq 2$) Riemannian space be of constant Riemannian curvature k is that the metric tensor g_{ij} satisfy the system of second-order partial differential equations

$$R_{\alpha\beta\gamma\delta}=k(g_{\alpha\delta}g_{\beta\gamma}-g_{\alpha\gamma}g_{\beta\delta}).$$

SCHWARZ. Schwarz's inequality. (1) The square of the integral of the product of two real functions over a given interval or region is equal to, or less than, the product of the integrals of their squares over the same intervals or regions, provided these integrals exist. For complex functions, $f(z)$ and $g(z)$,

$$\left|\int_{z_1}^{z_2} fg\,dz\right|^2 \leqq \left[\int_{z_1}^{z_2} f\bar{f}\,|dz|\right]\left[\int_{z_1}^{z_2} g\bar{g}\,|dz|\right],$$

where $\bar{f}$ and $\bar{g}$ are the complex conjugates of f and g. This inequality is easily deduced from *Cauchy's inequality* (see CAUCHY). It is also called the *Cauchy-Schwarz inequality* and *Buniakowski's inequality* (Buniakowski called attention to it earlier than Schwarz). (2) For a *vector space* with an *inner product* (x, y) defined, the inequality $|(x,y)|^2\leqq\|x\|\cdot\|y\|$ is called Schwarz's inequality. For suitable representations of Hilbert space, this inequality is equivalent to the above inequality and to Cauchy's inequality.

Schwarz's lemma. If the function $f(z)$ of the complex variable z is analytic for $|z|<1$, with $|f(z)|<1$ for $|z|<1$, and $f(0)=0$, then either $|f(z)|<|z|$ for $0<|z|<1$ and $|f'(0)|<1$, or $f(z)=e^{i\theta}z$, where θ is a real constant.

SCORE. T score. (*Statistics.*) See T.

SCRAP, *n.* **scrap value of equipment.** Its sale value when it is no longer useful. *Syn.* Salvage value.

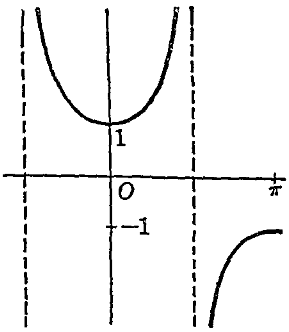
SEAC. An automatic digital computing machine at the National Bureau of Standards. SEAC is an acronym for *Standards Eastern Automatic Computer.*

SEA′SON-AL, *adj.* **elimination of seasonal variation.** (*Statistics.*) Usually accomplished by dividing the actual values of a given time series by the seasonal index of that series. The result is a quantity designed to indicate the annual value that would result if the actual rate prevailing at the time were to continue with only the seasonal fluctuations as causes of further variation. Instead of an annual total, $\frac{1}{12}$ of the total annual rate is indicated for that month's seasonally corrected value. *E.g.*, if the rainfall in February has a *seasonal index* of 2.50 as compared with an average for all months of 1.00, the observed rainfall in February, say 10 inches, is divided by 2.50 to give 4. Twelve months

of rain at the monthly seasonal pattern indicated by the observed February rate would give 48 inches. Alternatively, one may compare the observed values in any two months, after dividing each by its appropriate seasonal index, to see if the change between the two months is greater or less than that amount of fluctuation due to the regular seasonal variation.

SE′CANT, *adj.*, *n.* (1) A line of unlimited length cutting a given curve. (2) One of the trigonometric functions; see TRIGONOMETRIC—trigonometric functions.

secant curve. The graph of $y = \sec x$. Between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, it is concave up. It is asymptotic to the lines $x = -\frac{1}{2}\pi$ and $x = \frac{1}{2}\pi$, and has its y-intercept equal to unity. Similar arcs appear in other intervals of length π radians, being alternately concave upward and concave downward.



SEC′OND, *adj.*, *n.* **second of angle.** One-sixtieth of a minute and one thirty-six hundredth part of a degree. Denoted by a double accent, as 10″, read ten seconds. See SEXAGESIMAL—sexagesimal measure of an angle.

second derivative. The derivative of the first derivative. See DERIVATIVE—derivatives of higher order.

second mean value theorem. See MEAN—mean value theorems for derivatives, mean value theorems for integrals.

second moment. Same as MOMENT OF INERTIA.

second of time. One sixtieth of a minute.

SEC′OND-AR′Y, *adj.* **secondary diagonal of a determinant.** See DETERMINANT.

secondary parts of a triangle. Parts other than the sides and interior angles, such as the altitude, exterior angles, and medians. See PRINCIPAL—principal parts of a triangle.

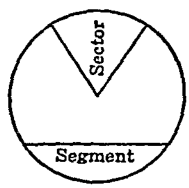
SEC′TION, *n.* **harmonic section of a line.** Four points of the line, harmonically related. See HARMONIC—harmonic division of a line.

method of sections. A method for graphing a surface. Consists of drawing sections of the surface (usually those made by the coordinate planes and planes parallel to them) and inferring the shape of the surface from these sections.

plane section. The plane geometric configuration obtained by cutting any configuration by a plane. A plane section made by a plane containing a normal to the surface is a **normal section.** A **meridian section** of a surface of revolution is a plane section made by a plane containing the axis of revolution. A **right section** of a cylinder is a plane section by a plane perpendicular to the elements of the cylinder, or to the lateral faces of the prism.

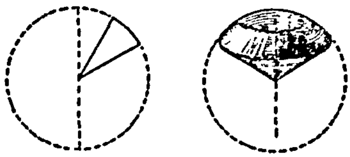
section of a polyhedral angle. See ANGLE—polyhedral angle.

SEC′TOR, *n.* **sector of a circle.** A portion of a circle bounded by two radii of the circle, and one of the arcs which they intercept. The smaller arc is called the **minor arc,** and the larger the **major arc.** The area of a sector is $\frac{1}{2}r^2\phi$, where r is the radius of the circle and ϕ the angle in radian measure subtended at the center of the circle by the arc of the sector.



spherical sector. A solid generated by rotating a sector of a circle about a diameter. Some writers require that this diameter not lie in the sector, while some require that it contain one of the radii bounding the circular sector. Most writers do not restrict the diameter at all, including both of the above cases as spherical sectors. The figure shows a sector of a circle and the spherical sector resulting from rotating it

about a diameter (the dotted line). The volume of a spherical sector is equal to the product of the radius of the sphere and one-third the area of the zone which forms



the base of the sector; or $\frac{2}{3}\pi r^2 h$, where r is the radius of the sphere and h the altitude of the zone (see ZONE).

SEC'U-LAR, *adj.* **secular trend.** See TREND.

SE-CU'RI-TY, *n.* (*Finance.*) Property, or written promises to pay, such as notes and mortgages, used to guarantee payment of a debt. See COLLATERAL.

SEG'MENT, *n.* A part cut off from any figure by a line or plane (or planes). Used most commonly when speaking of a limited piece of a line or of an arc of a curve. See below, segment of a curve, and segment of a line.

addition of line segments. See SUM—sum of directed line segments.

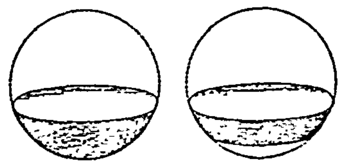
segment of a curve. (1) The part of the curve between two points on it. (2) The area bounded by a chord and the arc of the curve subtended by the chord. A segment of a circle is the area between a chord and an arc subtended by the chord. Any chord bounds two segments, which are different in area except when the chord is a diameter. The larger and smaller segments are called the **major** and **minor segments**, respectively. The area of a segment of a circle is $\frac{1}{2}r^2(\beta - \sin\beta)$, where r is the radius of the circle and β the angle in radians subtended at the center of the circle by the arc. See figure under SECTOR.

segment of a line or line segment. The part of a straight line between two points. The segment may include one or both of the points. A line segment whose end points are identical is a **nil segment**. See DIRECTED—directed line.

spherical segment. The solid bounded by a sphere and two parallel planes intersecting, or tangent to, the sphere, or by a *zone* and the planes (or plane) of its bases (or base). If one plane is tangent to the sphere, the segment is a *spherical segment of one base*; otherwise it is a *spherical segment of two bases.* The bases are the intersections of the parallel planes with the solid bounded by the sphere; the **altitude** is the perpendicular distance between these planes. The volume of a spherical segment is equal to

$$\tfrac{1}{6}\pi h(3r_1{}^2 + 3r_2{}^2 + h^2)$$

where h is the altitude and r_1 and r_2 are the radii of the bases. The formula for the volume of a segment of one base is derived by making one of these r's, say r_2, zero.



spherical segment
of one base *of two bases.*

SEGRE. Segre characteristic of a matrix. See CANONICAL—canonical form of a matrix.

SE-LECT', *adj.* **select mortality table.** See MORTALITY—mortality table.

SELF, *pref.* **self-adjoint transformation.** A linear transformation which is its own *adjoint.* For finite-dimensional spaces, a transformation T, which transforms vectors $x = (x_1, x_2, \cdots, x_n)$ into $Tx = (y_1, y_2, \cdots, y_n)$ with $y_i = \sum_j a_{ij}x_j$ for each i, is self-adjoint if and only if the matrix (a_{ij}) of its coefficients is a *Hermitian matrix.* If (x, y) denotes the inner product of elements x and y of a Hilbert space H, then a bounded linear transformation T of H into H is self-adjoint if and only if $(Tx, y) = (x, Ty)$ for any x and y in H. Any bounded linear transformation T of a (complex) Hilbert space (with domain the entire space) can be uniquely written in the form $T = A + iB$, where A and B are self-adjoint transformations. *Syn.* Hermitian transformation. See SPECTRAL—spectral theorem, and SYMMETRIC—symmetric transformation.

SELL'ING, *adj.* **selling price.** See PRICE.

per cent profit on selling price. See PER CENT.

SEM'I, *pref.* Meaning half; partly; somewhat less than; happening or published twice in an interval or period. A **semicircle** is one-half of a circle (either of the parts of a circle which are cut off by a diameter); a **semicircumference** is one-half of a circumference.

semiannual. Twice a year.

semiconjugate axis of a hyperbola. See HYPERBOLA.

semicontinuous function. See CONTINUOUS—semicontinuous function.

semicubical parabola. See PARABOLA—cubical parabola.

semigroup. A set of elements subject to some rule of combination (which will be called multiplication) such that (1) the product of any two elements (alike or different) is unique and is in the set, (2) the associative law holds, *i.e.*, $a(bc)=(ab)c$ for any elements a, b and c. A semigroup is **Abelian** (or **commutative**) if $ab=ba$ for any elements a and b. Sometimes a cancellation law is assumed (that $x=y$ if there is an element z for which $xz=yz$ or $zx=zy$). A semigroup with a finite number of elements satisfies this cancellation law if and only if it is a *group*.

semimajor and semiminor axes. See ELLIPSE and ELLIPSOID.

semimean axis. See ELLIPSOID.

semiring. See RING—semiring of sets.

semitransverse axis of a hyperbola. See HYPERBOLA.

SENSE, *n.* **sense of an inequality.** See INEQUALITY.

SEN-IOR'I-TY, *n.* **law of uniform seniority.** The following law used in evaluating joint life insurance policies: The difference between the age that can be used in computation (instead of the actual ages) and the lesser of the unequal ages is the same for the same difference of the unequal ages, regardless of the actual ages. (The age used in computation instead of the actual ages is the age which two persons of the same age would have if they were given insurance identical with that given those with the different ages.)

SEN'SI-TIV'I-TY, *adj.* **sensitivity analysis.** An analysis of the variation of the solution of a problem with variations in the values of the parameters involved.

SEP'A-RA-BLE, *adj.* **separable space.** A (topological) space which contains a countable (or finite) set W of points which is *dense* in the space; *i.e.*, every neighborhood of any point of the space contains a point of W. A space which satisfies the *second axiom of countability* is separable. Such a space is sometimes said to be **completely separable, perfectly separable,** or simply **separable.** Hilbert space and Euclidean space of n dimensions are separable.

SEP'A-RA'TION, *n.* **separation of a set.** The division of the set into two classes. A separation of an ordered set (such as the real numbers or the rational numbers) is of the **first kind** if each member of one class is less than every member of the other class and the separating number belongs to one or the other of the classes. The number 3 may be thought of as separating all rational numbers into those less than or equal to 3 and those greater than 3. A separation of an ordered set is of the **second kind** if each member of one class is less than every member of the other and there is no greatest member of the class of lesser objects and no least in the class of larger objects. The separation of the rational numbers into the sets A and B, where x is in A if $x \leqq 0$, and each positive x is in A or B according as $x^2<2$ or $x^2>2$, is of the second kind. See DEDEKIND CUT.

separation of variables. See DIFFERENTIAL—differential equation with variables separable.

SEP'A-RA'TRIX, *n.* Something that separates; a comma that divides a number into periods as in 234,569; a space that divides a number into periods as in 234 569. A decimal point is sometimes called a *separatrix.*

SEP-TIL'LION, *n.* (1) In the U. S. and France, the number represented by one followed by 24 zeros. (2) In England, the number represented by one followed by 42 zeros.

SE'QUENCE, *n.* A set of quantities ordered as are the positive integers. The sets

$$1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}$$

and $x, 2x^2, 3x^3, \cdots, nx^n$, are sequences. If after each term of a sequence there is another term, the sequence is called an *infinite sequence* or just a *sequence* and is written

$$a_1, a_2, a_3, \cdots, a_n, \cdots, \quad \{a_m\}, \quad \text{or} \quad (a_n).$$

accumulation point of a sequence. A point P such that there are an infinite number of terms of the sequence in any neighborhood of P; *e.g.*, the sequence

$$1, \tfrac{1}{2}, 1, \tfrac{1}{3}, 1, \tfrac{1}{4}, 1, \tfrac{1}{5}, \cdots$$

has two accumulation points, the numbers 0 and 1. If, for any number M, there are an infinite number of terms of a sequence of real numbers which are greater (less) than M, then $+\infty(-\infty)$ is said to be an accumulation point of the sequence. An accumulation point of a sequence is also called a **cluster point, or limit point,** of the sequence. For a sequence of real numbers, the largest accumulation point is also called the **limit superior** (or **greatest of the limits, or maximum limit**) and is the number L (or $\pm\infty$) which is the largest number such that there are an infinite number of terms of the sequence greater than $L-\epsilon$ for any positive ϵ ($L=+\infty$ or $L=-\infty$ according as for each number c infinitely many of the terms are larger than c, or for each number c only finitely many of the terms are larger than c); the smallest accumulation point is also called the **limit inferior** (or **least of the limits, or minimum limit**) and is the number l (or $\pm\infty$) which is the least number such that there are an infinite number of terms of the sequence less than $l+\epsilon$ for any positive ϵ ($l=+\infty$ or $l=-\infty$ according as for each number c only finitely many of the terms are less than c, or for each number c infinitely many of the terms are less than c). The *limit superior* (*limit inferior*) is the limit of the *upper* (*lower*) *bounds* of the numbers in the subsequences

$$\begin{array}{l} a_1, a_2, a_3, \cdots, a_n, \cdots \\ a_2, a_3, a_4, \cdots, a_n, \cdots \\ a_3, a_4, a_5, \cdots, a_n, \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots. \end{array}$$

The *limit superior* and *limit inferior* are not always the *least upper* and *greatest lower bounds* of a sequence. The limit superior and limit inferior of the sequence

$$2, -\tfrac{3}{2}, \tfrac{4}{3}, \cdots, (-1)^{n-1}(1+1/n), \cdots$$

are 1 and -1, while the upper and lower bounds are 2 and $-\frac{3}{2}$. The limit superior and limit inferior of any sequence, $\{a_n\}$, are denoted respectively by

$$\overline{\lim_{n\to\infty}}\, a_n \quad \text{and} \quad \underline{\lim_{n\to\infty}}\, a_n,$$

or by $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

Either limit is denoted by

$$\overline{\underline{\lim_{n\to\infty}}}\, a_n.$$

When these two limits are the same the sequence has a *limit* (see below, limit of a sequence).

bound to a sequence. An **upper bound** (**lower bound**) to a sequence of real numbers is a number which is equal to or greater than (equal to or less than) every number in the sequence. If a sequence has both an upper bound and a lower bound, it is said to be a **bounded sequence.** The smallest upper bound is called the **least upper bound** (sometimes simply **the upper bound**) and is the largest term in the sequence if there is a largest, otherwise a number, L, such that there are terms between $L-\epsilon$ and L for every $\epsilon>0$ but no terms greater than L. The largest lower bound is called the **greatest lower bound** (sometimes simply **the lower bound**) and is the least term, or if there is no least, then a number, l, such that there are terms of the sequence between l and $l+\epsilon$ for every $\epsilon>0$, but no terms less than l. Sometimes **limit** is used in place of bound in the above expressions.

Cauchy sequence. A sequence of points $x_1, x_2, \cdots$ such that for any $\epsilon>0$ there is a number N for which $\rho(x_i, x_j)<\epsilon$ if $i>N$ and $j>N$, where $\rho(x_i, x_j)$ is the distance between x_i and x_j. If the points are points of Euclidean space, this is equivalent to the sequence being convergent. If the points are real (or complex) numbers, then $\rho(x_i, x_j)$ is $|x_i-x_j|$ and the sequence is convergent if and only if it is a Cauchy sequence. *Syn.* Convergent sequence, fundamental sequence, regular sequence. See CAUCHY—Cauchy's condition for convergence of a sequence, and COMPLETE—complete space.

cluster point of a sequence. See above, accumulation point of a sequence.

convergent and divergent sequences. See below, limit of a sequence.

limit of a sequence. A sequence of numbers $s_1, s_2, s_3, \cdots, s_n, \cdots$ has the limit s if, for any prescribed accuracy, there is a position in the sequence such that all terms after this position approximate s within this prescribed accuracy; *i.e.*, for any $\epsilon > 0$ there exists an N such that $|s - s_n| < \epsilon$ for all n greater than N. A sequence of points $p_1, p_2, p_3, \cdots$ has the limit p if, for neighborhood U of p, there is a number N such that p_n is in U if $n > N$. A sequence which has a limit is said to be **convergent**; otherwise, it is **divergent.** A sequence of numbers $s_1, s_2, \cdots$ is convergent if and only if the series

$$s_1 + (s_2 - s_1) + (s_3 - s_2) + \cdots + (s_n - s_{n-1}) + \cdots$$

has a *sum.* See SUM—sum of an infinite series.

limit point of a sequence. See above, accumulation point of a sequence.

monotonic (or monotone) sequence. See MONOTONIC.

regular sequence. See above, Cauchy sequence.

SE-QUEN'TIAL, *adj.* **sequential analysis.** (*Statistics.*) Sequential probability ratio analysis is a process by which statistical data are analyzed continuously as the sample accumulates. After each additional item is obtained, and on the basis of a certain calculation, a decision is made whether to accept the hypothesis H_1 under test, or to accept an alternative hypothesis H_2, or to suspend judgment until more data are examined. The decision is based on the probability ratio of the sample under alternative hypotheses where the probabilities of the two types of erroneous conclusions are assigned in advance. Frequently fewer observations are required than under any other known method for the same degree of reliability and discrimination. It is very simple to apply and it requires the analyst to state his problem precisely and determine the alternative answers *in advance* with the attendant probabilities of erroneous conclusion. The calculations involved are the computations of the ratio of the probability of the observations if H_1 is true to the probability of the observations under the hypothesis H_2. If this ratio exceeds $(1-B)/A$, the hypothesis H_1 is accepted; whereas if it is less than $B/(1-A)$, the hypothesis H_2 is accepted; if it is between these two ratios, judgment is suspended. A is the maximum acceptable probability of erroneously rejecting the hypothesis H_2, and B is the maximum acceptable probability of erroneously accepting the hypothesis H_2.

SE'RI-AL, *adj.* **serial bond.** See BOND.

serial plan of building and loan association. See BUILDING—building and loan association.

SE'RI-AL-LY, *adv.* **serially ordered set.** See ORDERED—simply ordered set.

SE'RIES, *n.* The indicated sum of a finite or ordered infinite set of terms. It is said to be a **finite** or an **infinite** series according as the number of terms is finite or infinite. An infinite series can be written in the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots,$$

or Σa_n, where a_n is called the **general term** or the ***n*th term.** *Infinite series* is usual shortened to *series*, as in *convergent series, Taylor's series*, etc. An infinite series need not have a sum; it is said to be **convergent** if it has a sum and **divergent** if it does not (see DIVERGENT, SUM—sum of an infinite series, and various headings under CONVERGENCE). A series is a **positive series** (or a **negative series**) if its terms are all positive (or all negative) real numbers. An **ascending** (or **increasing**) **series** is a series of numbers for which the numerical value of each term is greater than that of the preceding (compare with *monotonic increasing*). Such a series is always divergent. A **decreasing** (or **descending**) **series** is a series of numbers for which the numerical value of each term is less than that of the preceding (compare with *monotonic decreasing*). See various headings below.

Abel's theorem on power series. See ABEL.

addition of infinite series. The addition of corresponding terms of the two series. If two convergent series of constant terms,

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

and

$$b_1+b_2+b_3+\cdots+b_n+\cdots,$$

have the sums S and S', then the series

$$(a_1+b_1)+(a_2+b_2)+(a_3+b_3)+\cdots+(a_n+b_n)+\cdots$$

converges and has the sum $S+S'$. If the series

$$u_1+u_2+u_3+\cdots+u_n+\cdots$$

and the series

$$v_1+v_2+v_3+\cdots+v_n+\cdots,$$

whose terms are functions of x, converge in certain intervals, the term by term sum of these series, namely

$$(u_1+v_1)+(u_2+v_2)+(u_3+v_3)\cdots+(u_n+v_n)+\cdots,$$

converges in any interval common to the two intervals.

alternating series. See ALTERNATING.

arithmetic series. The indicated sum of the terms of an arithmetic progression. The sum to n terms is denoted by S_n, and $S_n=\frac{1}{2}n(a+l)$ or $\frac{1}{2}n[2a+(n-1)d]$. See ARITHMETIC—arithmetic progression.

asymptotic series. See ASYMPTOTIC—asymptotic expansion.

autoregressive series. See AUTOREGRESSIVE.

binomial series. The binomial expansion with infinitely many terms; the expansion of a binomial raised to a power that is not a positive integer or zero. The expansion of $(a+x)^n$ by the binomial theorem results in a convergent series for all values of n, provided the absolute value of x is less than the absolute value of a.

differentiation of an infinite series. The term-by-term differentiation of the series. This is permissible, *i.e.*, the resulting series represents the derivative of the function represented by the given series in the same interval, if the resulting series is uniformly convergent in this interval. This condition is always satisfied by a power series in any interval within its interval of convergence; *e.g.*, the series

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\pm\frac{x^n}{n}\mp\cdots$$

converges for $-1<x\leqq 1$ and represents $\log(1+x)$ in this interval; the derived series

$$1-x+x^2-\cdots\pm x^{n-1}\mp\cdots$$

converges uniformly for $-a<x<a$ if $a<1$, and represents

$$\frac{1}{1+x}$$

in any such interval.

discount series. See DISCOUNT—discount series.

division of two power series. The division of the two series as if they were polynomials arranged in ascending powers of the variable. Their quotient converges and represents the quotient of the sums of the series for all values of the variable within a region of convergence common to both their regions and numerically less than the numerically least value for which the series in the denominator is zero.

entire series. See ENTIRE.

Euler's transformation of series. See EULER.

exponential series. See EXPONENTIAL—exponential series.

factorial series. The series

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!}+\cdots.$$

The sum of this series is the number *e*. See e.

Fourier series. See FOURIER.

geometric series. A series whose terms form a geometric progression. The general form of a geometric series is

$$a+ar+ar^2+ar^3+\cdots+ar^{n-1}+\cdots.$$

Its sum to n terms is

$$S_n=\frac{a(1-r^n)}{1-r}.$$

When r is numerically less than one, the series converges, since

$$\lim_{n\to\infty} r^n=0,$$

and its sum is $a/(1-r)$. *E.g.*, the sum of

$$1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$$

is $[1-(\frac{1}{2})^n]/(1-\frac{1}{2})=2[1-(\frac{1}{2})^n]$. The limit of this sum [the sum of the infinite series $1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}+\cdots$] is 2.

harmonic series. A series whose terms are in *harmonic progression*; a series the reciprocals of whose terms form an *arithmetic series*. See HARMONIC—harmonic progression.

hypergeometric series. See HYPERGEOMETRIC.

integration of an infinite series. The term-by-term integration (definite integration) of an infinite series. Any series of continuous functions which converges uniformly on an interval may be integrated term by term and the result will converge and equal the integral of the function represented by the original series, provided the limits of integration are finite and lie within the interval of uniform convergence. Any **power series** satisfies this condition in any interval within its interval of convergence and may be integrated term by term provided the limits of integration lie within the interval of convergence. The series

$$1-x+x^2-\cdots(-1)^{n+1}x^{n-1}\cdots$$

converges when $|x|<1$. Hence term by term integration is permissible between the limits 0 and $\frac{1}{2}$, for instance, or between x_1 and x_2, provided $|x_1|<1$ and $|x_2|<1$. Actually, one of x_1 or x_2 can be 1. This is a special case of the following more general theorem: Let $S_n(x)$ be the sum of the first n terms of an infinite series for which there is a set of *measure zero* such that, on the complement of this set in the interval $[a, b]$, $|S_n(x)|$ is uniformly bounded and the series is convergent to a sum $S(x)$. If $\int_a^b S(x)\,dx$ and $\int_a^b S_n(x)\,dx$ exist for each n, then $\lim \int_a^b S_n(x)\,dx = \int_a^b S(x)\,dx$. If Lebesgue (instead of Riemann) integration is used, then it is not necessary to assume the existence of $\int_a^b S(x)\,dx$ and the assumption of the existence of $\int_a^b S_n(x)\,dx$ can be replaced by the assumption that each $S_n(x)$ is *measurable*.

Laurent series. See LAURENT.

logarithmic series. The expansion in Taylor's series of $\log(1+x)$, namely,

$$x-x^2/2+x^3/3-x^4/4+\cdots +(-1)^{n+1}x^n/n\cdots.$$

From this series is derived the relation

$$\log(n+1)=\log n+2[(2n+1)^{-1}+\tfrac{1}{3}(2n+1)^{-3}+\tfrac{1}{5}(2n+1)^{-5}+\cdots],$$

which is convenient for approximating logarithms of numbers because it converges rapidly.

Maclaurin's series. See TAYLOR—Taylor's theorem.

multiplication of infinite series. The multiplication of the series as if they were polynomials, multiplying each term of one series by all the terms of the other. If each series converges absolutely, the terms of the product series have a sum equal to the product of the sums of the given series, whatever the order of the terms in the product series. This need not be the case if one series is conditionally convergent. The **Cauchy product** (usually called the *product*) of two series $a_0+a_1+a_2+\cdots$ and $b_0+b_1+b_2+\cdots$ is the series $c_0+c_1+c_2+\cdots$ for which

$$c_n=a_0b_n+a_1b_{n-1}+\cdots+a_nb_0$$

is the sum of all products a_ib_j for which $i+j=n$. For power series, the nth term of a product is the sum of all terms of the nth degree which are products of a term of one series by a term of the other. If two series are convergent and one (or both) is absolutely convergent, then their Cauchy product is convergent and has a sum which is the product of the sums of the given series. Also, if two series and their Cauchy product are convergent, then the sum of the Cauchy product is the product of the sums of the given series. A power series converges absolutely within its interval of convergence; hence two power series can always be multiplied, and the result will be valid within their common interval of convergence.

oscillating series. See DIVERGENT—divergent series.

power series. A series whose terms contain ascending positive integral powers of a variable, a series of the form

$$a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots,$$

where the a's are constants and x is a variable; or a series of the form

$$a_0+a_1(x-h)+a_2(x-h)^2+\cdots +a_n(x-h)^n+\cdots.$$

See TAYLOR—Taylor's theorem.

rearrangement of the terms of a series. Defining another series which contains all the terms of the original series, but not necessarily in the same order. *I.e.*, for any n the first n terms of the new series are all

terms of the old series, and the first n terms of the old series are all terms of the new series. If a series is absolutely convergent, all the rearrangements have the same sum. If it is conditionally convergent, rearrangements can be made such as to give any arbitrary sum, or to diverge.

reciprocal series. A series whose terms are each reciprocals of the corresponding terms of another series, of which it is said to be the *reciprocal series.*

remainder of an infinite series. See REMAINDER—remainder of an infinite series.

reversion of a series. See REVERSION.

sum of an infinite series. See SUM.

Taylor's series. See TAYLOR—Taylor's theorem.

telescopic series. The series

$$\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}+\cdots+\frac{1}{(k+n-1)(k+n)}+\cdots,$$

where k is not a negative integer. It is called *telescopic* because it can be written in the form

$$\left[\frac{1}{k}-\frac{1}{k+1}\right]+\left[\frac{1}{k+1}-\frac{1}{k+2}\right]+\cdots+\left[\frac{1}{k+n-1}-\frac{1}{k+n}\right]+\cdots$$

which sums to $\frac{1}{k}$.

the p series. The series

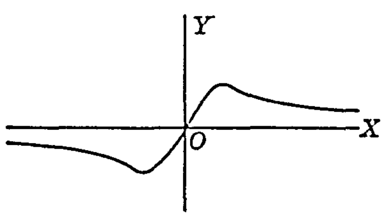
$$1+(\tfrac{1}{2})^p+(\tfrac{1}{3})^p+\cdots+(1/n)^p+\cdots.$$

It is of importance in applying the *comparison test*, since it converges for all values of p greater than one and diverges for p equal to or less than one. When p equals 1, it is the *harmonic series.*

time series. See TIME—time series.

trigonometric series. See TRIGONOMETRIC—trigonometric series.

SER'PEN-TINE, *adj.* **serpentine curve.** The curve defined by the equation $x^2y+b^2y-a^2x=0$. It is symmetric about the origin, passes through the origin, and has the x-axis as an asymptote.



SER'VICE, *adj.* **service table.** A table showing (for various convenient ages) the number of lives in the service of a company, the total *decrement*, and the *decrements* due to specific causes.

SER'VO-MECH'A-NISM, *n.* An amplifying device that effects a certain relation between an input signal and an output signal. Examples are steering devices, automatic stabilizers, and components of computing machines.

hunting of a servomechanism. The output of a servomechanism is designed to follow the instructions of the input. Errors (or deviations) in the output, which ideally should be self-correcting, are called *hunting motions.*

SET, *n.* A number of particular things, as the *set* of numbers between 3 and 5, the *set* of points on a segment of a line, or within a circle, etc. See SUBSET.

bounded set of numbers. A set such that the absolute value of each of its members is less than some constant. All proper fractions constitute a bounded set, for they are all less than 1 in absolute value.

F_σ, G_δ, and Borel sets. See BOREL—Borel set.

finite and infinite sets. See FINITE and INFINITE.

intersection and sum of sets. See INTERSECTION and SUM.

ordered set. See ORDERED.

SEX'A-GES'I-MAL, *adj.* Pertaining to the number sixty.

sexagesimal measure of an angle. The system in which one complete revolution is divided into 360 parts, written 360° and called degrees; one degree into 60 parts, written 60′ and called minutes; and one minute into 60 parts, written 60″ and called seconds. See RADIAN.

sexagesimal system of numbers. A number system using sixty for a base instead of ten. See BASE—base of a system of numbers.

SEX'TIC, *adj.* Of the sixth degree; of the sixth order (when speaking of curves or

surfaces). A **sextic curve** is an algebraic curve of the sixth order; a **sextic equation** is a polynomial equation of the sixth degree.

SEX-TIL'LION, *n.* (1) In the U. S. and France, the number represented by one followed by 21 zeros. (2) In England, the number represented by one followed by 36 zeros.

SHEAF, *n.* **sheaf of planes.** All the planes that pass through a given point. The point is called the **center** of the sheaf. The equations of all the planes in the sheaf can be found by multiplying the equations of three planes not having a line in common and passing through the point by different parameters (arbitrary constants), adding these equations, and letting the parameters take all possible values. See PENCIL—pencil of planes. *Syn.* Bundle of planes.

SHEAR, *n.* **modulus of shear.** See RIGIDITY—modulus of rigidity.

simple shear transformation. See TRANSFORMATION.

SHEAR'ING, *adj.* **shearing force.** One of two equal forces acting in opposite directions and not in the same line, causing, when acting upon a solid, a distortion known as a *shearing strain.*

shearing motion. The motion that takes place when a body gives way due to a shearing stress.

shearing strain. See STRAIN.

shearing stress. See STRESS.

SHEET, *n.* **sheet of a surface.** A part of the surface such that one can travel from any point on it to any other point on it without leaving the surface. See HYPERBOLOID—hyperboloid of one sheet, hyperboloid of two sheets.

sheet of a Riemann surface. Any portion of a Riemann surface which cannot be extended without giving a multiple covering of some part of the plane over which the surface lies. Thus for the function $w=z^{1/2}$ a sheet of the Riemann surface of definition consists of the z-plane cut by any simple curve extending from the origin to the point at infinity.

SHEPPARD. Sheppard's correction. (*Statistics.*) The calculation of moments from a grouped distribution of a variable contains an error because the frequencies are assumed to be concentrated at the midpoint (or some unique point) in the interval. A correction may be made so that *on the average* a correct estimate is obtained. Denote the ith moment of the continuous distribution by u_i and the grouped distribution moment by u_i'. Then, under fairly widely acceptable conditions, $u_1=u_1'$ and $u_2=u_2'-\frac{h^2}{12}$, where h is the uniform width of the group intervals. A general formula for the correction on the ith moment is known.

SHOCK, *adj.* **shock wave.** In fluid dynamics, a discontinuous solution of a nonlinear hyperbolic equation or system of equations, arising from continuous initial and boundary conditions.

SHORT, *adj.* **short arc of a circle.** The shorter of the two arcs subtended by a chord of the circle.

short division. See DIVISION—short division.

SHRINK'ING, *n.* **shrinking of the plane.** See SIMILITUDE—transformation of similitude, and STRAIN—one-dimensional strain.

SIDE, *n.* **side of an angle.** See ANGLE.

side opposite an angle in a triangle or polygon. The side separated from the vertex of the angle by the same number of sides in whichever direction they are counted around the triangle or polygon.

side of a polygon. Any one of the line segments forming the polygon.

SI-DE'RE-AL, *adj.* Pertaining to the stars.

sidereal clock. A clock that keeps sidereal time.

sidereal time. Time as measured by the apparent diurnal motion of the stars. It is equal to the hour angle of the vernal equinox (see HOUR). The sidereal day, the fundamental unit of sidereal time, is assumed to begin and to end with two successive passages over the meridian of the vernal equinox. There is one more sidereal day than mean solar days in a sidereal year.

sidereal year. The time during which the earth makes one complete revolution around the sun with respect to the stars. Its length is 365 days, 6 hours, 9 minutes, 9.5 seconds. See YEAR.

SIERPINSKI. Sierpinski set. (1) Let G be the class of all uncountable G_δ sets on a line (see BOREL—Borel set). A Sierpinski set is a set S on the line which has the property that both S and its complement contain at least one point from each set belonging to G. Such a set can be shown to exist by using the *well-ordering principle* (or *axiom of choice*) to obtain a well-ordering of G with the property that the set of all predecessors of an element of G has *cardinal number* less than the cardinal number c of the real numbers (G itself has cardinal number c). Zorn's lemma can then be used to choose two points from each set of G, such that for any set neither of the two points chosen from that set were chosen from any previous set. A Sierpinski set can then be formed by choosing one of the members of each of these two-point sets. A Sierpinski set S has the properties that, for every set E, either E is of *measure zero* or one of the intersections of E with S and with the complement of S is *nonmeasurable*, and either E is of *first category* or one of the intersections of E with S and with the complement of S does not have the *property of Baire*. If S is a Sierpinski set and the exterior measure of a set A is $m_e(A)$, the set function $M(A)=m_e(A \cap S)$ defines a measure on a σ-algebra which includes S and all measurable sets. Also, $M(A)=m(A)$ if A is measurable. (2) A set S of points in the plane is a Sierpinski set if S contains at least one point of each closed set of nonzero measure and no three points of S are collinear. Such a set S is not measurable, although no line contains more than two points of S (see FUBINI—Fubini's theorem). The class C of closed sets of nonzero measure has cardinal number c and can be well-ordered so that each member of C has fewer than c predecessors. The set S can then be constructed by use of Zorn's Lemma, choosing a point x_α from each C_α of C in such a way that x_α is not collinear with any two points chosen from previous members of C.

SIEVE, *n.* **number sieve.** See NUMBER—number sieve.

sieve of Eratosthenes. See ERATOSTHENES.

SIG'MA, *n.* The name of the Greek letter σ, Σ, equivalent to the English s, S. See SUMMATION—summation sign.

σ-algebra and σ-ring. See ALGEBRA—algebra of subsets, RING—ring of sets.

σ-finite. See MEASURE—measure of a set.

SIGN, *n.* **algebraic sign.** A positive or negative sign.

continuation of sign in a polynomial. See CONTINUATION.

Descartes' rule of signs. See DESCARTES.

law of signs. In *addition and subtraction*, two successive like signs give a positive result, and two unlike signs give a negative result. We have

$$2-(-1)=3,$$

while

$$2-(+1)=2-1=1$$

and

$$2+(-1)=1.$$

In *multiplication and division*, the product or quotient of two factors with like signs is positive; with unlike signs, negative. We have

$$(-4)(-2)=2,$$

and

$$(-4)/2=4/(-2)=-2.$$

See SUM—sum of real numbers, PRODUCT—product of real numbers.

summation sign. See SUMMATION.

SIG'NA-TURE, *n.* **signature of a quadratic form, Hermitian form, or matrix.** See INDEX—index of a quadratic form.

SIGNED, *adj.* **signed numbers.** Positive and negative numbers. *Syn.* Directed numbers.

SIG-NIF'I-CANCE, *n.* **statistical significance.** Deviations between hypothesis and observations which are so improbable under the hypothesis as to cause one to believe that the difference is not merely due to sampling errors or fluctuations are said to be **statistically significant.** The failure of a difference to fall in the acceptable realm of sampling deviations brands it as statistically significant. Clearly, the decision of significance is arbitrary and discretionary in that

the size of the probability which is deemed too small to permit the acceptance of the hypothesis is arbitrary and discretionary. See HYPOTHESIS—test of hypothesis.

test of significance. See HYPOTHESIS—test of hypothesis.

SIG-NIF′I-CANT, *adj.* **significant digit** or **figure.** See DIGIT.

SIG′NUM, *n.* **signum function.** *Signum* x is defined as that function whose value is 1 for $x>0$, -1 for $x<0$, and 0 for $x=0$. It is denoted by sgn x or sg x.

SIM′I-LAR, *adj.* **similar ellipses (or hyperbolas).** Ellipses (or hyperbolas) which have the same eccentricity; ellipses (or hyperbolas) whose semiaxes are in the same ratio. If the axes of one are a, b and of the other a', b', then $a/b=a'/b'$.

similar ellipsoids. Ellipsoids whose principal sections are similar ellipses. Thus the ellipsoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\mu,$$

where μ is a parameter greater than zero, are all similar.

similar figures in plane geometry. Figures having all corresponding angles equal and all corresponding line segments (sides) proportional.

similar fractions. See FRACTION.

similar hyperboloids and paraboloids. Hyperboloids and paraboloids whose principal sections are similar. The hyperboloids whose equations are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\mu$$

with μ taking different positive values (different negative values) are similar. The paraboloids whose equations are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\mu z,$$

with μ taking different values, are similar elliptic paraboloids. The paraboloids whose equations are

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\mu z,$$

with μ taking different values, are similar hyperbolic paraboloids.

similar matrices. Matrices which are transforms of each other by a nonsingular matrix. See TRANSFORMATION—collineatory transformation.

similar polygons. Two polygons having the angles of one equal to the corresponding angles of the other and the corresponding sides proportional; two polygons whose vertices are respectively the points of two *similar sets of points.* See below, similar sets of points.

similar sets of points. Points so situated on a pencil of lines (two points on each line) that all the ratios of the distances from the vertex of the pencil to the two points (one in each of the two sets) on a given line are equal. The set of points (one on each line) whose distances from the vertex are the antecedents of the ratios, and the set of points whose distances are the consequents, are called *similar sets* of points, or *similar systems* of points. Two such sets of points are also said to be **homothetic** and any figures formed by joining corresponding pairs of points in each set are said to be **homothetic.** See SIMILITUDE—transformation of similitude.

similar solids. See SOLID.

similar surfaces. Surfaces which can be made to correspond point to point in such a way that the distance between any two points on one surface is always the same multiple of the distance between the two corresponding points on the other. The areas of similar surfaces are to each other as the squares of corresponding distances.

similar terms in one (or more) unknowns. Terms which contain the same power (or powers) of the unknowns. The terms $3x$ and $5x$, ax and bx, axy and bxy, are similar terms. *Syn.* Like terms.

similar triangles. Triangles with corresponding angles equal. Corresponding sides are then proportional.

SIM′I-LAR′I-TY, *n.* The property of being similar.

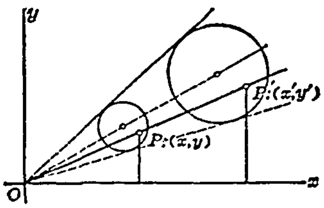
general similarity transformation. A transformation (composed possibly of a translation, a rotation, and a homothetic transformation), which transforms figures into similar figures.

SI-MIL′I-TUDE, *n.* **center of similitude.** See RADIALLY—radially related figures.

ratio of similitude. See RATIO.

transformation of similitude. The transformation $x'=kx$, $y'=ky$, in rectangular

coordinates. It multiplies the distance between every two points by the same constant k, called the ratio of similitude. If k is less than one the transformation is said to shrink the plane. In the figure, the circumference of the larger circle is k times



the circumference of the smaller, and the point P' is k times as far from the origin as the point P. This transformation is also called the **homothetic transformation.** See RADIALLY—radially related figures.

SIM'PLE, *adj.* **simple arc.** A set of points which can be put into one-to-one correspondence with the points of the closed interval [0, 1] in such a way that the correspondence is continuous in both directions. See TOPOLOGICAL—topological transformation. A continuum (of at least two points) such that there are not more than two points whose omission does not destroy the connectedness of the set is a simple arc. See below, simple closed curve.

simple closed curve. A set of points which can be put into one-to-one correspondence with the points of a circle in such a way that the correspondence is continuous in both directions. See TOPOLOGICAL—topological transformation. A continuum (of at least two points) which is no longer connected if any two arbitrary points are removed is a simple closed curve. *Syn.* Jordan curve.

simple cusp. See CUSP—cusp of the first kind.

simple elongations and compressions. Same as ONE-DIMENSIONAL STRAINS. See STRAIN.

simple event. See EVENT.

simple fraction. See FRACTION.

simple function of a complex variable. A function $f(z)$ is *simple* in a region D if it is analytic in D and does not take on any value more than once in D. *Syn.* Schlicht function.

simple harmonic motion. See HARMONIC.

simple hexagon. See HEXAGON—simple hexagon.

simple integral. A single integral, as distinguished from multiple and iterated integrals.

simple interest. See INTEREST.

simple pendulum. See PENDULUM.

simple point on a curve. Same as ORDINARY POINT. See POINT.

simple polyhedron. See POLYHEDRON.

simple root. A root of an equation that is not a repeated root. If the equation is algebraic, $f(x)=0$, a simple root is a root r such that $f(x)$ is divisible by the first power of $x-r$ and by no higher power. See MULTIPLE—multiple root of an equation.

SIM'PLEX, *n.* An n-dimensional simplex (or simply an **n-simplex**) is a set which consists of $n+1$ linearly independent points $p_0, p_1, \cdots, p_n$ of an Euclidean space of dimension greater than n together with all the points of type

$$x=\lambda_0 p_0+\lambda_1 p_1+\cdots+\lambda_n p_n,$$

where $\lambda_0+\lambda_1+\cdots+\lambda_n=1$ and $0\leqq\lambda_i$ for each i (see BARYCENTRIC—barycentric coordinates). Such a set is sometimes called a **closed simplex,** while the set of all such points x for which each λ_i is positive is an **open simplex.** A set of points of one of these two types is sometimes called a **degenerate simplex** if the points $p_0, \cdots, p_n$ are not linearly independent (or if two or more of these points coincide). Each of the points $p_0, \cdots, p_n$ is said to be a **vertex** of the simplex, and any simplex whose vertices are $r+1$ of these points is an **r-dimensional face,** or an **r-face,** of the simplex. An n-simplex is its own n-face, while faces of dimension less than n are called **proper faces.** A simplex of dimension 0 is a single point; a simplex of dimension 1 has 2 vertices and consists of the straight line segment joining these vertices (its vertices are its only proper faces); a simplex of dimension 2 has 3 vertices and is a triangle with its interior [illegible] 1-dimensional faces are its sides [illegible] 0-dimensional faces are its [illegible] simplex of dimension 3 has 4 [illegible] is a tetrahedron together [illegible] (its 2-dimensional faces are [illegible] set of all the vertices of a [illegible]

its **skeleton.** Any $n+1$ objects can be called an **abstract** ***n*-simplex** (see COMPLEX—simplicial complex). A **topological simplex** is any topological space (such as a solid sphere) which is homeomorphic to a simplex. A simplex is **oriented** if an order has been assigned to its vertices. If $(p_0p_1 \cdots p_n)$ is an orientation of a simplex with vertices $p_0, \cdots, p_n$, this is regarded as being the same as any orientation obtained from it by an even permutation of the vertices and as the negative of any orientation obtained by an odd permutation of the vertices. *E.g.*, a 2-simplex with vertices p_0 and p_1 has the two orientations (p_0p_1) and (p_1p_0). A 3-simplex, or triangle, has the two orientations corresponding to the two directions for enumerating vertices around the triangle. If $(p_0p_1 \cdots p_n)$ is an orientation of an n-simplex, then this simplex and the $(n-1)$-simplex $p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_n$ obtained by discarding the vertex p_i are **coherently** (or **concordantly**) **oriented** if the orientation of the $(n-1)$-simplex is $(-1)^i (p_0 \cdots p_{i-1}p_{i+1} \cdots p_n)$. *E.g.*, if (ABC) is an orientation of a triangle with vertices A, B, C, then this triangle is coherently oriented with each of its sides if the sides have the orientations (AB), (BC), $-(AC)$, (CA).

simplex method. A standard finite iterative algorithm for solving a linear programming problem by successively determining basic *feasible solutions*, if any exist, and testing them for optimality. See PROGRAMMING—linear programming.

SIM-PLIC′I-AL, *adj.* **simplicial complex.** See COMPLEX.

simplicial mapping. A mapping of a simplicial complex K_1 into a simplicial complex K_2 for which the images of simplexes of K_1 are simplexes of K_2. If the mapping is one-to-one and the image of K_1 is all of K_2, then K_1 and K_2 are said to be *isomorphic*, or *combinatorially equivalent*.

SIM′PLI-FI-CA′TION, *n.* The process of reducing an expression or a statement to a briefer form, or one easier to work with. See SIMPLIFIED.

SIM′PLI-FIED, *adj.* The simplified form of an expression, quantity, or equation can mean either (1) the briefest, least complex form, or (2) the form best adapted to the next step to be taken in the process of seeking a certain result. Probably the most indefinite term used seriously in mathematics. Its meaning depends upon the operation as well as the expression at hand and its setting. *E.g.*, if one desired to factor $x^4+2x^2+1-x^2$, to collect the x^2 terms would be foolish, since it would conceal the factors. Usually a **radical** is said to be in simplified form when there is no fraction under the radical and no factor under the radical which possesses the root indicated by the index; $\sqrt{2}$ and $2\sqrt{3}$ are in simplest form, but $\sqrt{\frac{2}{3}}$ and $\sqrt{12}$ are not. A **fraction** whose numerator and denominator are rational numbers is usually said to be in simplified form when written so that numerator and denominator are integers with no common factors other than ± 1.

SIM′PLY, *adv.* **simply connected region.** See CONNECTED.

SIMPSON'S RULE. A rule for approximating the area bounded by a curve (whose equation is given in rectangular coordinates), the x-axis, and the ordinate corresponding to two abscissas, say a and b. It assumes that small arcs of the curve are very nearly coincident with the arc of the parabola through the midpoint and terminal points of the arc. In algebraic terms, it makes use of Taylor's series, dropping all terms after the quadratic term. The formula is

$$A=\frac{(b-a)}{6n}[y_a+4y_1+2y_2+4y_3 +2y_4+\cdots+4y_{2n-1}+y_b]$$

where $2n$ equal subintervals have been laid off on the x-axis by

$$a, x_1, x_2, \cdots, x_{2n-1}, b,$$

and

$$y_a, y_1, y_2, \cdots, y_{2n-1}, y_b$$

are the respective ordinates of these points. The numerical difference between the number given by this formula and the actual area is known to be less than

$$\frac{M(b-a)^5}{180(2n)^4},$$

where M is the greatest numerical value of the 4th derivative of the function whose graph is the given curve. This rule can be used to approximate the value of any definite integral. If the curve is not of higher order than 3, this formula in the form

$$\frac{b-a}{6}[y_a+4y_1+y_b],$$

where $n=1$, gives the exact area and is called the **prismoidal formula** (for areas). See NEWTON—Newton's three-eighths rule.

SI'MUL-TA'NE-OUS, *adj.* **simultaneous equations.** Two or more equations which may or may not have common solutions, but are conditions imposed simultaneously on all the variables. *E.g.*, $x+y=2$ and $3x+2y=5$, treated as simultaneous equations, are satisfied by $x=1$, $y=1$, these values being the coordinates of the point of intersection of the straight lines which are the graphs of the two equations. The number of solutions of two **simultaneous polynomial equations** in two variables is equal to the product of their degrees (provided they have no common factor), infinite values (see homogeneous coordinates) being allowed, and equal solutions being counted to the degree of their multiplicity. *E.g.*, (1) the equations $y=2x^2$ and $y=x$ have the two common solutions $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$; (2) the equations $y-2x^2=0$ and $y^2-x=0$ have two real and two imaginary common solutions. **Simultaneous linear equations** are simultaneous equations which are linear (of the first degree) in the variables (see CONSISTENCY—consistency of linear equations).

simultaneous inequalities. Two or more inequalities, which may or may not have common solutions, but are conditions imposed simultaneously on all the variables. The simultaneous inequalities $x^2+y^2<1$, $y>0$ are satisfied simultaneously by the points above the x-axis and inside the unit circle about the origin.

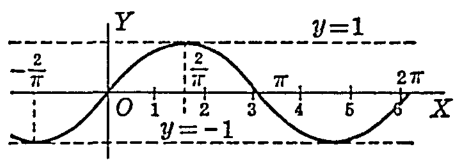
SINE, *adj.*, *n.* **exponential values of sin x and cos x.** See EXPONENTIAL—exponential values of $\sin x$ and $\cos x$.

law of sines. For a plane triangle, the sides of a triangle are proportional to the sines of the opposite angles. If the angles are A, B, C, and the sides opposite these angles are a, b, c, this law is

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

For a **spherical triangle**, the law of sines states that the sines of the sides are proportional to the sines of the opposite angles.

sine curve. The graph of $y=\sin x$. The curve passes through the origin and all points on the x-axis whose abscissas are multiples of π (radians), is concave toward the x-axis, and the greatest distance from the x-axis to the curve is unity.



sine of a number or angle. See TRIGONOMETRIC—trigonometric functions.

sine series. See FOURIER—Fourier's half-range series.

SIN'GLE, *adj.* **single-address system.** A method of coding problems for machine solution, whereby each separate instruction is restricted to telling what to do with a single item at a specified address or memory position. See MULTIADDRESS.

single premium. See PREMIUM.

single-valued function of one or more variables. A function which has one and only one value corresponding to each value (or set of values) of its variable (or variables). *E.g.*, the function x^2+1 is a *single-valued function.* See MULTIPLE—multiple valued function.

SIN'GU-LAR, *adj.* **singular curve on a surface.** A curve C on a surface S such that every point of C is a singular point of S. See below, singular point of a surface.

singular point of an analytic function. A point at which the function (of a complex variable) is not analytic, but in every neighborhood of which there are points of analyticity. An **isolated singular point** is a point z_0 on the Riemann surface of existence of the function at which it is not analytic, but such that there exists on the surface a neighborhood

$$|z-z_0|<\epsilon$$

of z_0 at each point $z(\neq z_0)$ of which $f(z)$ is analytic. An isolated singular point can be of any one of three types. (1) **Removable singular point.** An isolated singular point z_0 such that $f(z)$ can be defined, or re-defined, at z_0 in such a way as to be analytic at z_0. *E.g.*, if $f(z)=z$ for $0<|z|<1$ and $f(0)=1$, then $f(z)$ has a removable singular point at $z_0=0$. (2) **Pole.** An isolated singular point z_0 such that $f(z)$ can be represented by an expression of the form $f(z)=\frac{\phi(z)}{(z-z_0)^k}$, where k is a positive integer, $\phi(z)$ is analytic at z_0, and $\phi(z_0)\neq 0$. The integer k is called the **order** of the pole. *E.g.*, $f(z)=(z-1)/(z-2)^3$ has a pole of order 3 at $z_0=2$ with $\phi(z)=z-1$. (3) **Essential isolated singular point.** An isolated singular point which is neither a removable singularity nor a pole. In any neighborhood of an essential singular point, and for every finite complex number α with the exception of at most one number α, the equation $f(z)-\alpha=0$ has an infinitude of roots. This is the **second theorem of Picard.** *E.g.*, $f(z)=\sin\frac{1}{z}$ has an essential isolated singular point at $z_0=0$. An **essential singular point** is any singular point which is not a pole and which is not a removable singularity. *E.g.*, for $f(z)=\tan\frac{1}{z}$, the origin is an essential singular point, but is not an *isolated singular point*, since it is a limit point of poles of $f(z)$.

singular point of a curve. See POINT—ordinary point of a curve.

singular point of a surface. A point of the surface S: $x=x(u,v)$, $y=y(u,v)$, $z=z(u,v)$, at which $H^2\equiv EG-F^2=0$. See SURFACE—fundamental coefficients of a surface. Since

$$H^2=\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2+\left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2+\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2,$$

it follows that, for real surfaces and parameters, the quantity H^2 is nonnegative; and H^2 is positive at a point unless all three Jacobians vanish there. H appears in the denominator of several important formulas in differential geometry. See REGULAR—regular point of a surface.

singular solution of a differential equation. See DIFFERENTIAL—solution of a differential equation.

singular transformation. See LINEAR—linear transformation.

SINISTRORSUM [*Latin*] **or SIN'IS-TRORSE,** *adj.* Same as LEFT-HANDED. See LEFT.

SINK, *n.* A negative source. See SOURCE.

SINKING FUND. See FUND—sinking fund.

SI'NUS-OID, *n.* The *sine curve.* See SINE.

SKEL'E-TON, *n.* See COMPLEX—simplicial complex, and SIMPLEX.

SKEW, *adj.* **distance between two skew lines.** See DISTANCE.

skew lines. Nonintersecting, nonparallel lines in space.

skew quadrilateral. The figure formed by joining four noncoplanar points by line segments, each point being joined to two, and only two, other points.

skew-symmetric determinant. A determinant having its conjugate elements numerically equal but opposite in sign. If the element in the first row and second column is 5, the element in the first column and second row would be -5. A skew-symmetric determinant of odd order is always equal to zero.

skew-symmetric matrix. A matrix which is equal to the negative of its transpose; a square matrix such that $a_{ij}=-a_{ji}$, where a_{ij} is the element in the ith row and jth column. *Syn.* Skew matrix.

skew-symmetric tensor. See TENSOR.

SLANT, *adj.* **slant height.** The slant height of a **right circular cone (cone of revolution)** is the common length of the elements of the cone; the slant height of a **frustum of a right circular cone** is the length of the segment of an element of the cone intercepted by the bases of the frustum. The slant height of a **regular pyramid** is the common altitude of its lateral faces; the slant height of a **frustum of a regular pyramid** is the common altitude of its faces (the perpendicular distance between the parallel edges of the faces).

SLIDE, *adj.* **slide rule.** A mechanical device to aid in calculating by the use of logarithms. It consists essentially of two rules, one sliding in a groove in the other, containing logarithmic scales by means of which products and quotients are calculated by adding and subtracting logarithms. Detailed explanations of the construction and use of any particular slide rule can be secured from its manufacturer.

SLOPE, *n.* **slope angle.** Same as angle of inclination. See ANGLE—angle of inclination.

point-slope and slope-intercept forms of the equation of a line. See LINE—equation of a straight line.

slope of a curve at a point. The slope of the tangent line at that point; the derivative, dy/dx, evaluated at the point. See DERIVATIVE.

slope of a straight line. The tangent of the angle that the line makes with the positive x-axis; the tangent of the smallest positive angle through which the positive axis of abscissas can be revolved in order to be parallel to the given line; the rate of change of the ordinate with respect to the abscissa, *i.e.* (in rectangular Cartesian coordinates),

$$\frac{y_2-y_1}{x_2-x_1},$$

where (x_1, y_1) and (x_2, y_2) are points on the line. In *calculus* the slope at (x_1, y_1) is the derivative of the ordinate with respect to the abscissa,

$$\lim_{x_2\to x_1} \frac{y_2-y_1}{x_2-x_1} \quad \text{or} \quad \left(\frac{dy}{dx}\right)_{x=x_1}$$

(which is the same for all points on the line). The slope of $y=x$ is 1; of $y=2x$, 2; of $y=3x+1$, 3. See DERIVATIVE—derivative of a function of one variable. Slope is not defined for lines perpendicular to the x-axis.

SMALL, *adj., n.* **group without small subgroups.** See under GROUP.

in the small. In the neighborhood of a point. *E.g.*, when studying properties such as curvature of a curve at a point, one is concerned only with the behavior of the curve in the neighborhood of the point. Classical differential geometry is a study **in the small or im kleinen.** The study of a geometric object in its entirety, or the study of definite sections of it, or the study of a function in a given fixed interval, is a study **in the large or im grossen.** Algebraic geometry is a study in the large.

small arcs, angles, or line segments. Arcs, lines, or line segments, which are small enough to satisfy certain conditions, such as making the difference between two ordinates of a curve less than a stipulated amount, or the quotient of the sine of an angle by the angle (in radians) differ from 1 by less than a given amount.

small circle. See CIRCLE—small circle.

SN, *n.* See ELLIPTIC—Jacobian elliptic functions.

SNELL'S LAW. See REFRACTION.

SO'LAR, *adj.* **solar time.** See TIME.

SO'LE-NOI'DAL, *n.* **solenoidal vector in a region.** A vector F such that its integral over every reducible surface S in the region is zero; *i.e.*, $\int_S \mathbf{F}\cdot\mathbf{n}\, da=0$, where n is the unit vector in the direction of the outer normal to the element of area da. The divergence of a vector is zero at every point in a region if, and only if, the vector is solenoidal in the region, or if, and only if, the vector is the curl of some vector function (vector potential). See EQUATION—equation of continuity.

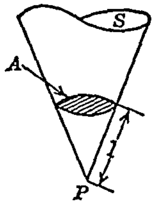
SOL'ID, *adj., n.* See GEOMETRIC—geometric solid.

frustum of a solid. See FRUSTUM.

similar solids. Solids bounded by similar surfaces; solids whose points can be made to correspond in such a way that the distances between all pairs of points of the one are a constant multiple of the distances between corresponding points of the other. Volumes of similar solids are proportional to the cubes of the distances between corresponding points. All spheres are similar solids; so are all cubes.

solid angle. The solid angle at any point P (see figure) subtended by a surface S is equal to the area A of the portion of the surface of a sphere of unit radius, with center at P, which is cut by a conical surface with vertex at P and the perimeter

of S as a generatrix. The unit solid angle is called the **steradian.** The total solid angle about a point is equal to 4π steradians. See SPHERICAL—spherical degree.



solid geometry. See GEOMETRY—solid geometry.

solid of revolution. See REVOLUTION.

SO-LU'TION, *n.* (1) The process of finding a required result by the use of certain given data, previously known facts or methods, and newly observed relations. (2) The *result* is also spoken of as the *solution. E.g.*, a root of an equation is called a *solution* of the equation it satisfies, although a *solution* of the equation may refer either to the process of finding a root or to the root itself.

algebraic, analytic and geometric solutions. See headings under ALGEBRAIC, ANALYTIC, and GEOMETRIC.

solution of a differential equation. See DIFFERENTIAL—solution of a differential equation.

solution of equations. For a single equation, solution may mean either (1) the process of finding (or approximating) a root of the equation, or (2) a root of the equation. The solution of a set of **simultaneous equations** is the process of finding a set of values of the unknowns which satisfy all the equations (this set of values of the unknowns is also called a solution) (see SIMULTANEOUS—simultaneous equations). The **geometric** (or **graphical**) solution of an equation $f(x)=0$ is the process of finding the roots by graphing $y=f(x)$ and estimating where its graph crosses the x-axis (see ROOT—root of an equation). A **solution by inspection** consists of guessing a root and testing it by substitution in the equation. See ROOT—root of an equation, and POLYNOMIAL—polynomial equation.

solution of linear programming problem. See PROGRAMMING—linear programming.

solution of a two-person zero-sum game. See GAME.

solution of a triangle. Finding the remaining angles and sides when sufficient of these have been given. For a **plane right triangle,** it is sufficient to know any two sides, or to know one of the acute angles and one side. The unknown parts are found by use of trigonometric tables and the definitions of the trigonometric functions (see TRIGONOMETRIC): if a, b, c represent the legs and hypotenuse, respectively, and A, B are the angles opposite sides a and b, then $a=b\tan A=c\sin A$, $b=c\cos A$, $A=\tan^{-1} a/b$, $B=90°-A$. For an **oblique plane triangle,** it is sufficient to know all three sides, two angles and one side, or two sides and one angle (except that when two sides and the angle opposite one of them is given there may be two solutions; see AMBIGUOUS). See SINE—law of sines, COSINE—law of cosines, TANGENT—tangent law, TRIGONOMETRY—half-angle formulas of plane trigonometry, and HERO'S FORMULA. For a right spherical triangle, *Napier's rules* supply all the formulas needed. For formulas providing solutions of an **oblique spherical triangle** in cases when solutions exist, see COSINE—law of cosines, GAUSS—Gauss' formulas, NAPIER—Napier's analogies, SINE—law of sines, TRIGONOMETRY—half-angle formulas and half-side formulas of spherical trigonometry. Also see QUADRANT—laws of quadrants, and SPECIES—law of species.

SOURCE, *n.* In hydrodynamics, potential theory, etc., a point at which additional fluid is considered as being introduced into the region occupied by the fluid. If fluid is being removed at the point, the negative source is called a **sink.**

SOUTH, *adj.* **south declination.** See DECLINATION—declination of a celestial point.

SPACE, *adj., n.* (1) A three-dimensional region. See DIMENSION. (2) Any set or accumulation of things, the members being called elements or points and usually assumed to satisfy a set of postulates of some kind. See EUCLIDEAN—Euclidean space, and METRIC—metric space.

coordinate in space. See CARTESIAN—

Cartesian coordinates, CYLINDRICAL—cylindrical coordinates, and SPHERICAL—spherical coordinates.

enveloping space. The space in which a configuration lies. The configuration is then said to be embedded in the enveloping space. Thus the circle $x=\cos\theta$, $y=\sin\theta$ is embedded in the two-dimensional Euclidean (x, y)-space.

space curves. Curves that may or may not be plane curves; the intersection of two distinct surfaces is usually a space curve. Space curves do not lie in a plane (*i.e.*, they are twisted) except when their *torsion* is zero.

SPAN, *n.* **span of a roof.** The length of the plates of the building; the width of the building.

SPE'CIES, *n.* **law of species.** (*Spherical Trigonometry.*) One-half the sum of any two sides of a spherical triangle and one-half the sum of the opposite angles are the same species. Two angles, two sides, or an angle and a side are said to be of the *same species* if they are both acute or both obtuse, and of *different species* if one is acute and one obtuse.

species of a set of points. Let G' be the derived set of a set G, G'' the derived set of G', and in general $G^{(n)}$ the derived set of $G^{(n-1)}$. If one of the sets $G', G'', \cdots$ is the null set (contains no points), then G is said to be of the first species. Otherwise G is of the second species. The set G of all numbers of the form $m+1/n$ with m and n integers is of the first species, since $G''=0$. The set of all rational numbers is of the second species, since all derived sets consist of all real numbers.

SPE-CIF'IC, *adj.* **specific gravity.** The ratio of the weight of a given volume of any substance to the weight of the same volume of a standard substance. The substance taken as the standard for solids and liquids is water at 4°C, the temperature at which water has the greatest density.

specific heat. (1) The number of calories required to raise the temperature of one gram of a substance 1°C, or the number of B.T.U.'s required to raise one pound of the substance 1°F. Sometimes called *thermal capacity.* (2) The ratio of the quantity of heat necessary to change the temperature of a given mass 1° to the amount necessary to change an equal mass of water 1°.

SPEC'TRAL, *adj.* **spectral measure and integral.** Let H be a Hilbert space and S a set with a specified σ-algebra A of subsets. A spectral measure on S is a function which assigns a projection $P(X)$ to each member X of A in such a way that $P(S)$ is the identity transformation on H and $P\left(\bigcup_1^\infty X_k\right)=\sum_1^\infty P(X_k)$ for any sequence of pairwise disjoint sets $X_1, X_2, \cdots$ belonging to A. It follows that if $X_1 \subset X_2$, then $P(X_2-X_1)=P(X_2)-P(X_1)$; also, $P(X_1)\leqq P(X_2)$ in the sense that the range of $P(X_1)$ is contained in the range of $P(X_2)$, or that $P(X_1)\cdot P(X_2)=P(X_1)$. For any two members X_1 and X_2 of A, $P(X_1\cup X_2)+P(X_1\cap X_2)=P(X_1)+P(X_2)$ and $P(X_1\cap X_2)=P(X_1)\cdot P(X_2)$. If X_1 and X_2 are disjoint, then the ranges of $P(X_1)$ and $P(X_2)$ are orthogonal. If S is the complex plane (or a subset of the complex plane) and A is the σ-algebra of Borel sets, then the spectral measure has the additional property that, for X a member of A, the range of $P(X)$ is the union of the ranges of projections $P(X_a)$ for X_a a compact subset of X. The spectrum of a spectral measure is the complement of the union of all the open sets U for which $P(U)=0$. If the spectrum is bounded and $f(\lambda)$ is a bounded (Borel) measurable function (real or complex valued), then $T=\int f(\lambda)\,dP$ defines a bounded transformation T in the sense that the approximating sums for the integral define operators which converge in norm to T. Also, for any two elements x and y of the Hilbert space, $m(X)=(P(X)x, y)$ defines a complex valued *measure* on A, and $(Tx, y)=\int f(\lambda)\,dm$. It follows that $\int f\cdot g\,dP=\int f\,dP\cdot\int g\,dP$ and that, if f is continuous, $\left\|\int f(\lambda)\,dP\right\|$ is the least upper bound of $|\lambda|$ for λ belonging to the spectrum; the spectrum of the transformation $T=\int \lambda\,dP$ is

coincident with the spectrum of the spectral measure. If the spectrum is not bounded, but $f(\lambda)$ is bounded on bounded sets, then $\int f(\lambda)\,dP$ is the unique transformation which coincides with $\int f_X(\lambda)\,dP$ on the range of the projection $P(X)$ for each bounded X of A, where $f_X(\lambda)$ coincides with $f(\lambda)$ on X and is zero on the complement of X.

spectral theorem. For any *Hermitian, normal,* or *unitary* transformation T defined on a Hilbert space, there is a unique spectral measure defined on the Borel sets of the complex plane for which $T=\int \lambda\,dP$. If T is Hermitian, $P(X)=0$ if X does not intersect the real line, and $\int \lambda\,dP$ can be regarded as an integral along the real line; if T is unitary, $P(X)=0$ if X does not intersect the circle $|z|=1$ and $\int \lambda\,dP$ can be regarded as an integral around this circle.

SPEC'TRUM, *n.* **spectrum of a matrix.** See EIGENVALUE—eigenvalue of a matrix.

spectrum of a transformation. Let T be a linear transformation of a vector space L into itself and I be the identity transformation, $I(x)\equiv x$. The *spectrum* of T then consists of three pair-wise disjoint sets: the **point spectrum,** which is the set of numbers λ for which $T-\lambda I$ does not have an inverse (is not one-to-one); the **continuous spectrum,** which is the set of numbers λ for which $T-\lambda I$ has an inverse which is not bounded (*i.e.* not continuous) and whose domain is dense in L; the **residual spectrum,** which is the set of numbers λ for which $T-\lambda I$ has an inverse whose domain is not dense in L. The set of numbers which do not belong to the spectrum is called the **resolvent set** and consists of those numbers λ for which $T-\lambda I$ has a bounded inverse with dense domain. If L is a finite-dimensional vector space and T is the transformation which transforms vectors $x=(x_1, x_2, \cdots, x_n)$ into vectors $T(x)=(y_1, y_2, \cdots, y_n)$ with $y_i=\sum_j a_{ij}x_j$, then the point spectrum is the entire spectrum of T and is the set of *eigenvalues* of the matrix (a_{ij}). If λ_0 is in the point spectrum of T, then there is a vector $x\neq 0$ such that $T(x)=\lambda_0 x$; λ_0 is called a **characteristic value** of T and x a **characteristic element** (or **vector**) of T. The linear set of characteristic elements corresponding to λ_0 is the **characteristic manifold** corresponding to λ_0. If L is a Banach space, the spectrum is a nonempty set. If T is a bounded linear transformation and $|\lambda|>\|T\|$, then λ belongs to the resolvent set and the inverse of $T-\lambda I$ is $-\sum_1^\infty \lambda^{-n}T^{n-1}$. If L is a (complex) Hilbert space and λ belongs to the residual spectrum of T, then $\bar{\lambda}$ belongs to the point spectrum of T^*; if λ belongs to the point spectrum of T, then $\bar{\lambda}$ belongs to either the point spectrum or the residual spectrum of T^*. If T is *Hermitian, normal,* or *unitary,* then the residual spectrum of T is empty. If T is Hermitian, all numbers in the spectrum are real; if T is unitary, all numbers in the spectrum are on the circle $|z|=1$. *E.g.,* let $u_1, u_2, \cdots$ be a complete orthonormal sequence in Hilbert space, $\lambda_1, \lambda_2, \cdots$ a sequence of numbers with limit 1 ($\lambda_n\neq 1$), and T the linear transformation defined by $T\left(\sum_1^\infty a_i u_i\right)=\sum_1^\infty a_i\lambda_i u_i$. Then the numbers $\lambda_1, \lambda_2, \cdots$ constitute the point spectrum. The transformation $T-I$ has an inverse which is not bounded, but has dense domain. Thus 1 belongs to the continuous spectrum. All numbers other than 1 and $\lambda_i (i=1, 2, \cdots)$ belong to the resolvent set. See SPECTRAL—spectral theorem.

SPEED, *n.* Distance passed over per unit of time. Speed is concerned only with the length of the path passed over per unit of time and not with its direction (see VELOCITY). The **average speed** of an object during a given interval of time is the quotient of the distance traveled during this time interval and the length of the time interval. The speed (or **instantaneous speed**) is the limit of the average speed as the time interval approaches zero. If the distance the object has traveled at time t is $h(t)$, then the average speed between times t_0 and t is the absolute value of the ratio

$$\frac{h(t)-h(t_0)}{t-t_0}.$$

The speed at time t_0 is the absolute value of the limit of this ratio as t approaches t_0. For instance, if the distance passed over is equal to the cube of the time, the speed at a time t_0 is the limit of $(t_1^3 - t_0^3)/(t_1 - t_0)$ as t_1 approaches t_0, which is $3t_0^2$. If the distance traveled is represented as a function of the time, speed is the absolute value of the *derivative* of this function with respect to the time.

angular speed (in a plane). Relative to a point O, the **average angular speed** of a point, during a time interval of length t, is A/t, where A is the measure of the angle through which the line joining O to the point passes during this interval of time. The angular speed (the **instantaneous angular speed**) is the limit of the average speed over an interval of time, as that interval approaches zero. *Tech.* If the angle between some fixed line through O and the line joining O to the point is a function of time, the angular speed is the absolute value of the derivative of this function with respect to time.

constant speed. See CONSTANT—constant speed and velocity.

SPHERE, *n.* (1) The locus of points in space at a given distance from a fixed point. (2) The locus of points in space at a distance not greater than a given distance from a fixed point. The fixed point is the **center,** the given distance the **radius.** The **diameter** of a sphere is equal to twice the radius (the diameter may be either the segment intercepted by the sphere on a line passing through the center or the length of this segment). The **volume** of a sphere is equal to $\frac{4}{3}\pi r^3$, where r is the radius. The **area** of the surface of a sphere is equal to four times the area of a great circle of a sphere; *i.e.*, $4\pi r^2$. The set of points $(x_1, x_2, \cdots, x_{n+1})$ of $(n+1)$-dimensional space, for which $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$, is said to be an ***n*-sphere.**

celestial sphere. The spherical surface in which the stars appear to move.

chord of a sphere. The segment cut out of a secant by the surface of the sphere; a line segment joining two points on the sphere.

circumscribed and inscribed spheres. See CIRCUMSCRIBED.

equation of a sphere. In rectangular coordinates, the equation of a sphere of radius r is

$$x^2 + y^2 + z^2 = r^2$$

when the center is at the origin and

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

when the center is at the point whose coordinates are (a, b, c). In spherical coordinates, $\rho = r$ when the center is at the pole. See DISTANCE—distance between two points.

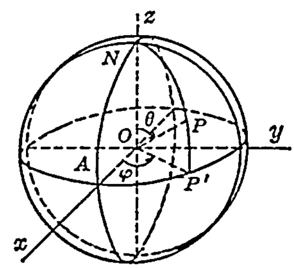
family of spheres. See FAMILY—one-parameter family of surfaces.

secant of a sphere. Any line cutting the sphere.

SPHER'I-CAL, *adj.* **spherical angle.** See ANGLE—spherical angle.

spherical cone. See CONE—spherical cone.

spherical coordinates. A system of coordinates in space. The position of any point P (see the figure) is assigned by its



radius vector $OP = r$ (*i.e.*, the distance of P from a fixed origin or pole O), and two angles: the **colatitude** θ, which is the angle NOP made by OP with a fixed axis ON, the **polar axis**; and the **longitude** ϕ, which is the angle AOP' made by the plane of θ with a fixed plane NOA through the polar axis, called the **initial meridian plane.** A given radius vector r confines the point P to the sphere of radius r about the pole O. The angles θ and ϕ serve to determine the position of P on this sphere. The angle θ is always taken between 0 and π radians, while ϕ can have any value (r being taken as negative if ϕ is measured to $P'O$ extended). The relations between the spherical and Cartesian coordinates are:

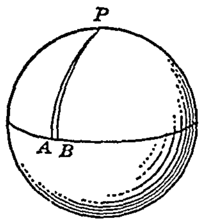
$$x = r \sin\theta \cos\phi,$$

$$y = r \sin\theta \sin\phi,$$

$$z = r \cos\theta.$$

Sometimes ρ is used in place of r, and θ and ϕ are often interchanged. *Syn.* Geographical coordinates, polar coordinates in space.

spherical degree. The area of the *birectangular spherical triangle* whose third angle is one degree. The area of the triangle APB in the figure is one spherical degree. See SOLID—solid angle.



spherical excess. Of a **spherical triangle:** The difference between the sum of the angles of a spherical triangle and 180° (the sum of the angles of a spherical triangle is greater than 180° and less than 540°). Of a **spherical polygon** of n sides: The difference between the sum of the angles of the spherical polygon and $(n-2)180°$ (the sum of the angles of a plane polygon of n sides).

spherical harmonic. See HARMONIC—spherical harmonic.

spherical image (or representation) of curves and surfaces. For a **curve**, see various headings under INDICATRIX. The spherical image of a point on a surface is the extremity of the radius of the unit sphere parallel to the positive direction of the normal to the surface at the point. The spherical representation (or image) of a **surface** is the locus of the spherical images of points on the surface. *Syn.* Gaussian representation of a surface.

spherical polygon. A portion of a spherical surface bounded by three or more arcs of great circles. Its area is

$$\frac{\pi r^2 E}{180},$$

where r is the radius of the sphere and E is the spherical excess of the polygon.

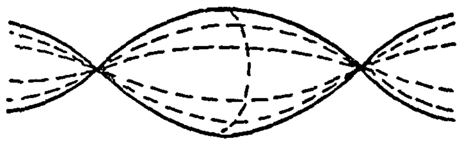
spherical pyramid. See PYRAMID.

spherical sector and segment. See SECTOR and SEGMENT.

spherical surface. A surface whose total curvature K has the same positive value at all its points. See PSEUDOSPHERICAL—pseudospherical surface, and SURFACE—surface of constant curvature. Not all spherical surfaces are spheres, but all are applicable to spheres. Hence all spherical surfaces have the same intrinsic properties. A spherical surface is of **elliptic type** if its linear element is reducible to the form

$$ds^2 = du^2 + c^2 \sin^2 (u/a)\, dv^2,\ c < a;$$

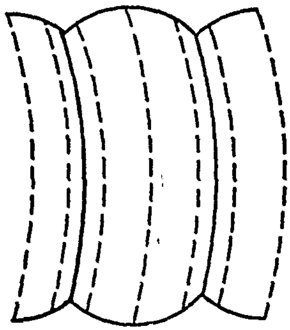
the coordinate system is a geodesic one.



A spherical surface of revolution of elliptic type consists of a succession of congruent spindle-shaped zones. A spherical surface is of **hyperbolic type** if its linear element is reducible to the form

$$ds^2 = du^2 + c^2 \sin^2 (u/a)\, dv^2,\ c > a;$$

the coordinate system is a geodesic one.



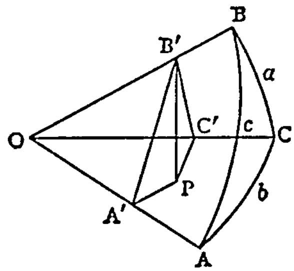
A spherical surface of revolution of hyperbolic type consists of a succession of congruent cheese-shaped zones each of which is bounded by parallels of minimum radius. A spherical surface is of **parabolic type** if its linear element is reducible to the form

$$ds^2 = du^2 + a^2 \sin^2 (u/a)\, dv^2;$$

the coordinate system is a geodesic polar one. The only spherical surfaces of revolution of parabolic type are spheres.

spherical triangle. A spherical polygon with three sides; a portion of a sphere bounded by three arcs of great circles. In the spherical triangle ABC (in figure),

the sides of the triangle are a=angle BOC, b=angle AOC, and c=angle AOB. The angles of the triangle are A=angle $B'A'P$, B, and C=angle $B'C'P$. A spherical triangle is a **right spherical triangle** if it has at least one right angle (it may have two or three and is **birectangular** if it has two right



angles and **trirectangular** if it has three right angles), a **quadrantal spherical triangle** if it has one side equal to 90° (a quadrant), an **oblique spherical triangle** if none of its angles are right angles, an **isosceles spherical triangle** if it has two equal sides, a **scalene spherical triangle** if no two sides are equal. The area of a spherical triangle is $\pi r^2E/180$, where r is the radius of the sphere and E is the triangle's spherical excess. See SOLUTION—solution of a triangle.

spherical trigonometry. The study of spherical triangles—finding unknown sides, angles, and areas by the use of trigonometric functions of the plane angles which measure angles and sides of the triangles. See TRIGONOMETRY.

spherical wedge. A solid the shape of a slice (from stem to blossom) of a spherical watermelon; the solid bounded by a lune of a sphere and the two planes of its great circles. Its volume is

$$\frac{\pi r^3 A}{270},$$

where r is the radius of the sphere and A is the dihedral angle (in degrees) between the plane faces of the wedge.

SPHE'ROID, *n.* Same as ELLIPSOID OF REVOLUTION. See ELLIPSOID.

SPI'NODE, *n*, Same as CUSP.

SPI'RAL, *adj., n.* See HYPERBOLIC—hyperbolic spiral, LOGARITHMIC—logarithmic spiral and PARABOLIC—parabolic spiral.

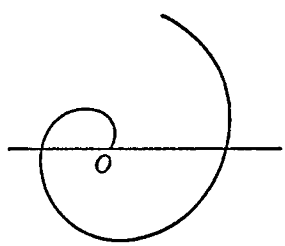
cornu spiral. The plane curve which has parametric equations

$$x=\int_0^s \cos \tfrac{1}{2}\pi\theta^2 d\theta,\ y=\int_0^s \sin \tfrac{1}{2}\pi\theta^2 d\theta.$$

The curvature of this curve at a point P is πs, where s is the length of the curve from the origin to P. See FRESNEL—Fresnel integrals.

equiangular spiral. Same as LOGARITHMIC SPIRAL.

spiral of Archimedes. The plane curve which is the locus of a point that moves with uniform speed, starting at the pole, along the radius vector while the radius vector moves with uniform angular speed. Its polar equation is $r=a\theta$. The figure shows the portion of the curve for which r is positive. See POLAR—polar coordinates in the plane.



spiral surface. A surface generated by rotating a curve C about an axis A and simultaneously transforming C homothetically relative to a point of A in such a way that for each point P of C the angle between A and the points on the locus described by P remains constant.

SPREAD'ING, *adj.* **spreading method for the potential of a complex.** See POTENTIAL.

SPUR, *n.* (*German.*) **spur of a matrix.** The sum of the elements in the principal diagonal. *Syn.* Trace.

SQUARE, *adj., n.* In *arithmetic* or *algebra*, the result of multiplying a quantity by itself. In *geometry*, a quadrilateral with equal sides and equal angles; a rectangle with two adjacent sides equal. The area of a square is equal to the square of the length of a side.

difference of two squares. See DIFFERENCE.

magic squares. See MAGIC.

method of least squares. See METHOD.

perfect trinomial square. The square of a binomial; an expression of the form $a^2+2ab+b^2$, which is equal to $(a+b)^2$.

square matrix. See MATRIX.

square numbers. Numbers which are the squares of integers, as 1, 4, 9, 16, 25, 36, 49, etc.

square root. See ROOT—root of a number.

pooled sum of squares. (*Statistics.*) See POOLED.

SQUAR'ING, *n.* **squaring the circle.** See QUADRATURE—quadrature of a circle.

STA'BLE, *adj.* **stable oscillations.** See OSCILLATION.

stable system. A physical system described by a system of differential equations

$$\frac{dx_i}{dt}=f_i(x_1,\cdots,x_n);\ x_i(t_0)=c_i;\ i=1,\cdots,n,$$

is said to be **stable** if it returns to a stationary state under perturbations of sufficiently small magnitude. It is said to be **totally stable** if it returns to a stationary state from arbitrary perturbations. See STATIONARY—stationary state.

STAND'ARD, *adj., n.* **standard deviate.** See DEVIATE.

standard deviation. See DEVIATION.

standard form of an equation. A form that has been universally accepted by mathematicians as such, in the interest of simplicity and uniformity. *E.g.,* the *standard form* of a rational integral (polynomial) equation of the nth degree in x is

$$a_0x^n+a_1x^{n-1}+\cdots+a_n=0;$$

the *standard form* in rectangular Cartesian coordinates of the equation of an ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

standard (or primary) infinitesimal and infinite quantities. The infinitesimal or infinite quantity relative to which *orders* are defined. If x is the *standard* or *primary infinitesimal,* then x^2 is an infinitesimal of higher (second) order with respect to x. Similarly if x is infinite, then x^2 is an infinite quantity of higher (second) order with respect to the *standard* or *primary infinite quantity* x. See INFINITESIMAL—order of an infinitesimal, and INFINITY—order of infinities.

standard time. See TIME.

STAR, *n.* For a member P of a family of sets, the *star* of P consists of all sets which contain P as a subset. The star of a simplex S of a *simplicial complex* K is the set of all simplexes of K for which S is a face (the star of a vertex P is the set of all simplexes which have P as a vertex). *E.g.,* the star of a vertex of a tetrahedron is the set of all edges and faces which contain P.

star-shaped set. A set B in Euclidean space of any number of dimensions, or in any linear space, is *star-shaped with respect to a point P of B* provided that, for every point Q of B, all points of the linear segment PQ are points of B.

STAT'IC, *adj.* **static moment.** Same as MOMENT OF MASS.

STAT'ICS, *n.* The branch of mechanics of solids and fluids dealing with those situations wherein the forces acting on a body are so arranged that the body remains at rest relative to a given frame of reference. See FRAME—frame of reference.

STA'TION-AR'Y, *adj.* **stationary point.** A point on a curve at which the slope (or rate of change of the ordinate) is equal to zero.

stationary state. For a physical system described at time t by a set of state variables $x_1(t),\cdots,x_n(t)$ that vary with time in accordance with a system of differential equations

$$\frac{dx_i}{dt}=f_i(x_1,\cdots,x_n);\ x_i(t_0)=c_i;\ i=1,\cdots,n;$$

a stationary state is a set of values $a_1,\cdots,a_n$ of $x_1,\cdots,x_n$ such that $f_i(a_1,\cdots,a_n)=0$ for $i=1,\cdots,n$. See STABLE—stable system.

stationary value of an integral. (*Calculus of Variations.*) See VARIATION.

STA-TIS'TIC, *n.* Sometimes refers to a quantitative datum. In statistical literature, it almost universally means an estimate of a parameter obtained from a sample. See ESTIMATE.

sufficient statistic. Let $f(x_1, x_2,\cdots,x_n, T)$ be a frequency function in the n-dimen-

sional sample space. If there is a statistic t such that

$$f(x_1, x_2, \cdots, x_n, T) = f(t, T)g(x_1, x_2, \cdots, x_n),$$

where $f(t, T)$ is not a function of the $x_1, x_2, \cdots, x_n$ except in so far as they determine t, and where $g(x_1, x_2, \cdots, x_n)$ is independent of T, then t is a **sufficient statistic.** The arithmetic mean of a random sample of a normal distribution is a **sufficient estimate** of the mean. Roughly, a statistic which contains all the information in a sample about a population parameter is considered *sufficient* in that no other information can be obtained from the sample to improve the estimate of the parameter.

STA-TIS'TI-CAL, *adj.* **statistical control.** See CONTROL.

statistical graphs. See GRAPHING—statistical graphing.

statistical independence. See INDEPENDENCE.

statistical inference. See INFERENCE.

statistical record. A record of many events of the same type, such as the number of deaths per year at specified age, the price of steel at given intervals over a long period, or the weights of a large number of people of a given height.

statistical significance. See SIGNIFICANCE.

STATISTICS, *n.* 1. Methods of obtaining and analyzing quantitative data. The following aspects are applicable only in reference to some phase of the experimental logic of quantitatively measured, variable, multiple phenomena: (a) Inference from samples to population by means of probability (commonly called **statistical inference**); (b) characterizing and summarizing a given set of data without direct reference to inference (called **descriptive statistics**); (c) methods of obtaining samples for statistical inference (called **sampling statistics**). 2. Sometimes used in all three of above senses. 3. A set of quantitatively measured data (obsolete, technically). 4. Plural of *statistic.*

g statistics. The quantities $g_1 = k_3/k_2^{3/2}$ and $g_2 = k_4/k_2^2$, where the k_i are *k statistics.* Used in tests of departure from normality; g_1 is used for measure of *skewness*, and g_2 for *kurtosis*. Symmetry implies $g_1 = 0$ and normal curve kurtosis implies $g_2 = 0$.

k statistics. A system of statistics related to moments and used for characterizing the distribution function of a variable:

$$k_1 = S_1/n,\ k_2 = S_2/(n-1),$$

$$k_3 = \frac{nS_3}{(n-1)(n-2)},$$

$$k_4 = \frac{n(n+1)S_4 - 3(n-1)S_2^2}{(n-1)(n-2)(n-3)},$$

where S_i is the sum of the ith powers of the differences between the n values of the variable and the mean of these values of the variable. Used for analysis of a distribution *via* moments, *e.g.*, to approximate the sampling distribution of certain statistics drawn from the parent distribution.

unbiased statistics. See BIASED.

vital statistics. See VITAL.

STEP, *adj., n.* **step function.** A function which is defined throughout some interval I and is constant on each one of a finite set of nonintersecting intervals whose sum is I. See INTERVAL.

successive steps. See SUCCESSIVE.

STERADIAN, *n.* See SOLID—solid angle.

STERE, *n.* One cubic meter, or 35.3156 cubic feet. Used mostly in measuring wood. See METRIC—metric system.

STER'E-O-GRAPH'IC, *adj.* **stereographic projection.** See PROJECTION.

STIELTJES. Lebesgue-Stieltjes integral. Let a *summable* function $f(x)$ and a monotonically increasing function $\phi(x)$ be defined on an interval $[a, b]$. Define $F(\xi)$ for $\phi(a) \leqq \xi \leqq \phi(b)$ by the relations: (1) $F(\xi) = f(x)$ if there is a point x such that $\xi = \phi(x)$; (2) if $\xi_0 \neq \phi(x)$ for any x, then it follows that there is a unique point of discontinuity x_0 of $\phi(x)$ such that

$$\phi(x_0 - 0) \leqq \xi_0 \leqq \phi(x_0 + 0),$$

and $F(\xi_0)$ is defined as $f(x_0)$. If the *Lebesgue integral* $\int_{\phi(a)}^{\phi(b)} F(\xi)\, d\xi$ exists, its value is defined to be the Lebesgue-Stieltjes integral of $f(x)$ with respect to $\phi(x)$, written

$$\int_a^b f(x)\, d\phi(x).$$

If $\phi(x)$ is of *bounded variation*, it is the difference $\phi_1 - \phi_2$ of monotonic increasing

functions ϕ_1 and ϕ_2, and the Lebesgue-Stieltjes integral $\int_a^b f(x)\,d\phi(x)$ is defined as $\int_a^b f(x)\,d\phi_1(x) - \int_a^b f(x)\,d\phi_2(x)$. If $F(\xi)$ as defined above is summable on $[\phi(a), \phi(b)]$, $f(x)$ is summable on $[a, b]$, and

$$\phi(x) = \int_a^x \theta(x)\,dx$$

for some summable function $\theta(x)$, then

$$\int_a^b f(x)\theta(x)\,dx = \int_a^b f(x)\,d\phi(x),$$

the former being a Lebesgue integral.

Riemann-Stieltjes integral. Let $a = x_0, x_1, x_2, \cdots, x_n = b$ be a subdivision of the interval $[a, b]$ and let

$$s_n = \max |x_i - x_{i-1}| \ (i = 1, 2, \cdots, n).$$

Let $f(x)$ and $\phi(x)$ be bounded real-valued functions defined on $[a, b]$ and

$$S_n = \sum_{i=1}^n f(\xi_i)[\phi(x_i) - \phi(x_{i-1})],$$

where ξ_i are arbitrary numbers satisfying $x_{i-1} < \xi_i < x_i$. If $\lim S_n$ exists as n becomes infinite in such a way that s_n approaches zero, and if the limit is independent of the choice of ξ_i and of the manner of the successive subdivisions, then this limit is the **Riemann-Stieltjes integral** of $f(x)$ with respect to $\phi(x)$, written

$$\int_a^b f(x)\,d\phi(x).$$

If $\int_a^b f(x)\,d\phi(x)$ exists, then $\int_a^b \phi(x)\,df(x)$ exists and

$$\int_a^b f(x)\,d\phi(x) + \int_a^b \phi(x)\,df(x) = f(b)\phi(b) - f(a)\phi(a).$$

If $f(x)$ is bounded on $[a, b]$ and $\phi(x)$ is of bounded variation on $[a, b]$, then

$$\int_a^b f(x)\,d\phi(x)$$

exists if, and only if, the total variation of $\phi(x)$ over the set of points of discontinuity of $f(x)$ is zero (*i.e.*, the set of all points $\phi(x)$ on the interval $[\phi(a), \phi(b)]$ for which f is discontinuous at x is of measure zero, $\phi(x)$ being taken as the interval between $\phi(x-0)$ and $\phi(x+0)$ if ϕ is discontinuous at x).

STIRLING. Stirling's formula. (1) The formula $(n/e)^n\sqrt{2\pi n}$ which is asymptotic to $n!$; *i.e.*, $\lim_{n\to\infty} n!/[(n/e)^n\sqrt{2\pi n}] = 1$. More precisely,

$$n! = (n/e)^n\sqrt{2\pi n}\, e^{\theta_n/(12n)},$$

where θ_n is a number for which $0 < \theta_n < 1$. (2) Maclaurin's series, discovered by Stirling, but published first by Maclaurin.

Stirling's series. Either of the two asymptotic expansions:

$$\log \Gamma(x) = (x - \tfrac{1}{2}) \log x - x + \tfrac{1}{2} \log (2\pi) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}B_k}{2k(2k-1)x^{2k-1}};$$

$$\Gamma(z) = e^{-x}x^{x-1/2}(2\pi)^{1/2}\left\{1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} + 0\left(\frac{1}{x^4}\right)\right\};$$

where $\Gamma(x)$ is the *gamma function*; $B_1, B_2, \cdots$ are the *Bernoulli numbers* $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \cdots$; and $0(1/x^4)$ is a function such that $x^4 \cdot 0(1/x^4)$ is bounded as $x \to \infty$ (the second series can be extended through successive powers of x^{-1}).

STO-CHAS'TIC, *adj.* **stochastic variable.** Same as CHANCE VARIABLE. See CHANCE.

STOCK, *n.* The capital (or certain assets) of a corporation or company which is in the form of transferable shares. **Preferred stock** is stock upon which a fixed rate of interest (dividend) is paid, after the dividends on bonds have been paid and before any dividends are paid on common stock. If the company fails, the order of redemption is the same as the above order of payments. **Common stock** is stock which receives as dividends its proportionate share of the proceeds after all other demands have been met. A **stock certificate** is a written statement that the owner of the certificate has a certain amount of capital in the corporation which issued the certificate. A **stock company** is a company composed of the purchasers of certain stock. A **stock insurance company** is a stock company whose business is insurance. The policies they sell are nonparticipating. The profits go to the stockholders. However, some stock companies, because of the competition of mutual companies, write participat-

ing policies. *Syn.* Nonparticipating insurance company.

STOKES' THEOREM. Let S be a surface which can be oriented by a unit normal vector $\mathbf{v}$ at each point of S, where $\mathbf{v}$ varies continuously over S. Let R be a region of S, and C be the boundary of R. Then the line integral of $P\,dx+Q\,dy+R\,dz$ around C, taken in the direction such that the interior of R is always on the observer's left as he moves around the curve on the positive side of S, is equal to the integral over R of

$$\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dy\,dz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dz\,dx+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy,$$

provided P, Q, R, and their first partial derivatives are continuous throughout R and C. In vector notation, with $\mathbf{F}=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$, this is

$$\int_C \mathbf{F}\cdot\boldsymbol{\tau}\,ds=\iint_S \nabla\times\mathbf{F}\cdot\mathbf{v}\,dS,$$

where $\boldsymbol{\tau}$ is the unit vector tangent to ds and in the direction of integration on C and $\nabla\times\mathbf{F}$ is the *curl* of $\mathbf{F}$. It is necessary to restrict S and R. Sufficient restrictions are that S can be divided into a finite number of parts each of which can be represented in each of the forms $x=f(y, z)$, $y=g(x, z)$, $z=h(x, y)$, where f, g, and h are continuous and single-valued, and that the projections on the coordinate planes of the portions of R in each of these parts of S satisfy the conditions stated for *Green's theorem*.

STOR'AGE, *adj.* **storage component.** In a computing machine, any component that is used in storing information for later use. The storage might be permanent or temporary, of quick or slow access, etc. Magnetic drums and tapes, television tubes, mercury delay lines, etc., are used for this purpose. *Syn.* Memory component.

STRAIGHT, *adj.* Continuing in the same direction; not swerving or turning; in a *straight line*. See LINE.

straight angle. See ANGLE—straight angle.

straight line. See LINE—straight line.

STRAIN, *adj.*, *n.* The change in the relative positions of points in a medium, the change being produced by a deformation of the medium as a result of *stress*.

coefficient of strain. See below, one-dimensional strains.

homogeneous strains. The concept in dynamics represented approximately by the *homogenous affine transformation*; the forces acting internally in an elastic body when it is deformed.

longitudinal strain. See below, strain tensor.

one-dimensional strains. The transformations $x'=x$, $y'=Ky$, or $x'=Kx$, $y'=y$. These transformations elongate or compress a configuration in the directions parallel to the axes, according as $K>1$ or $K<1$. The constant K is called the *coefficient of the strain*. *Syn.* Simple elongations and compressions, one-dimensional elongations and compressions.

principal directions of strain. At each point of an undeformed medium there exists a set of three mutually orthogonal directions which remain mutually orthogonal after the deformation has taken place. These directions are called *principal directions of strain*.

principal strains. The elongations in the directions of the principal directions of strain (*q.v.*).

shearing strain. Strain due to the distortion of the angles between the initially orthogonal directions in a medium which has been deformed. See below, strain tensor.

simple strains. A general name given to *simple elongations* and *compressions*, and *simple shears*.

strain tensor. In the linear theory of elasticity, a set of six functions e_{xx}, e_{yy}, e_{zz}, e_{xy}, e_{zy}, e_{xz}, related to the displacements u, v, w along the Cartesian axes x, y, z, respectively, by the formulas $e_{xx}=\frac{\partial u}{\partial x}$,

$$e_{yy}=\frac{\partial v}{\partial y},\ e_{zz}=\frac{\partial w}{\partial z},\ e_{xy}=\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right),$$

$$e_{zy}=\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right),\quad e_{xz}=\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right).$$

These six quantities (or alternatively, the set of three principal strains) characterize the state of strain of a body. The quantities e_{xx}, e_{yy}, e_{zz} are called the longitudinal

strains, and the remaining ones are the **shearing strains** (*q.v.*). The integrability conditions for the six components of the strain tensor (called **Saint-Venant's compatibility equations**) are

$$(e_{ij})_{kl}+(e_{kl})_{ij}-(e_{ik})_{jl}-(e_{jl})_{ik}=0,$$

where i, j, k, l take on any of the values x, y, z and subscripts on the outside of parentheses indicate partial differentiation.

STRAT'E-GY, *n.* **strategy for a game.** A **pure strategy** is any specifically determined plan, covering all possible contingencies but not involving the use of random devices, that a player might make in advance for a complete play of a game. If a player of a game has m possible pure strategies, then any probability vector $X=(x_1, x_2, \cdots, x_m)$, with each $x_i \geq 0$ and with $\Sigma xi=1$, is a **mixed strategy** for the player. If the player chooses this mixed strategy, then with probability x_i as determined by a random device he will employ his ith pure strategy for a given play of the game. Similarly, for continuous games a mixed strategy is a probability distribution over the continuum $[0, 1]$ of pure strategies. Since, for example, the mixed strategy $(1, 0, \cdots, 0)$ is equivalent to the player's first pure strategy, it follows that any pure strategy can be considered as a special mixed strategy. The word **strategy** is often used (when the meaning is clear from the context) to denote a *pure strategy*; it is also sometimes used to denote a *mixed strategy.* A pure strategy is a **dominant strategy** for one player of a game, relative to a second pure strategy for the same player, if the first strategy has, for each pure strategy of the opponent, at least as great *payoff* as the second (it is a **strictly dominant strategy** if its payoff is always greater than that of the second). For a two-person zero-sum game having *value* v, a strategy—either a pure strategy or a mixed strategy given by a probability vector or a probability distribution function—for the *maximizing player* that will make the expected value of the payoff at least v (or for the *minimizing player* that will make the expected value of the payoff at most v), regardless of the strategy chosen by the opponent, is said to be an **optimal strategy.**

STRAT'I-FIED, *adj.* **stratified sample.** See SAMPLE.

STREAM, *adj.* **stream function and stream lines.** See FUNCTION—stream function.

STRESS, *n.* A material body is said to be *stressed* when the action of external forces is transmitted to its interior. The average stress $\bar{T}$ is the average force F per unit area a of the planar element passing through a given point in the medium. The actual stress at the point is the limit of the ratio $\bar{T}=F/a$ as the area a containing the point in question is made to approach zero. The magnitude and the direction of the stress vector **T** depend not only on the choice of the point in the body but also on the orientation of the planar element at the chosen point. The component $\mathbf{T}_n$ of the stress vector **T** in the direction of the normal to the planar element is called the **normal stress**, while the component T in the plane of the element is the **shearing stress.**

internal stress. The resistance of a physical body to external forces; the unit internal resistance set up by external forces.

STRETCH'ING, *adj.* **stretching and shrinking transformations.** See SIMILITUDE—transformation of similitude.

STRICTLY, *adv.* **strictly convex space.** See CONVEX.

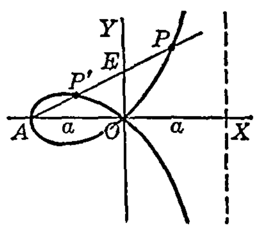
strictly increasing and strictly decreasing. See INCREASING—increasing function, DECREASING—decreasing function.

STRIC'TION, *n.* **line of striction of a ruled surface.** The locus of the central points of the rulings on the surface. See RULING—central point of a ruling on a ruled surface.

STRONG, *adj.* **strong topology.** See TOPOLOGY—topology of a space.

STROPH'OID, *n.* The plane locus of a point on a variable line passing through a fixed point when the distance from the describing point to the intersection of the line with the y-axis is equal to the y-intercept. If the coordinates of the fixed point are taken as $(-a, 0)$, the equation of the curve is $y^2=x^2(x+a)/(a-x)$. In the

figure, $P'E=EP=OE$, A is the point through which the line always passes, and the dotted line is the asymptote of the curve.



STUDENT'S t. See T.

STURM. Sturm-Liouville differential equation. A differential equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+[\lambda\rho(x)-q(x)]y=0,$$

where $p(x)>0$, $\rho(x)$ and $q(x)$ are continuous functions of x, and λ is an arbitrary parameter.

Sturm's functions. A sequence of functions derived from a given polynomial, $f(x)$; explicitly, the sequence of functions $f_0(x)$, $f_1(x), \cdots, f_n(x)$, where $f_0(x)\equiv f(x)$, $f_1(x)\equiv f'(x)$, and $f_2(x)$, $f_3(x)$, etc., are the negatives of the remainders occurring in the process of finding the highest common factor of $f(x)$ and $f'(x)$ by Euclid's algorithm. This sequence is called a **sequence of Sturm's functions.**

Sturm's theorem. A theorem determining the number of real roots of an algebraic equation which lie between any two arbitrarily chosen values of the unknown. The theorem states that the number of real roots of $f(x)=0$ between two values a and b of the unknown, x, is equal to the difference between the number of variations of sign in the sequence of *Sturm's functions* [derived from $f(x)$] when $x=a$ and when $x=b$, vanishing terms not being counted and multiple roots being counted to the degree of their multiplicity. See VARIATION—variation of sign in an ordered set of numbers.

SUB-AD'DI-TIVE, *adj.* See ADDITIVE—additive function, additive set function.

SUB'BASE, *n.* See BASE—base for a topological space.

SUB'CLASS', *n.* Same as SUBSET. See SET.

subclass numbers. (*Statistics.*) The numbers of observations in subclasses. **Disproportionate subclass numbers** exist in the analysis of variance when the number of observations in each subclass is not proportional to the row and column marginal totals, where a subclass is a cell identified by the row and column in which it occurs. Also known as **nonorthogonality.** The *addition theorem* of the analysis of variance (Cochrane's theorem) is annulled by disproportionate subclass numbers.

SUB-FAC-TO'RI-AL, *n.* **subfactorial of an integer.** If n is the integer, *subfactorial* n is $n!\times\left[\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots\frac{(-1)^n}{n!}\right]$. This is equal to $n!E$, where E is the sum of the first $n+1$ terms in the Maclaurin expansion of e^x with $x=-1$. *E.g.*, *subfactorial* 4 is equal to

$$4!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)=24(\tfrac{1}{2}-\tfrac{1}{6}+\tfrac{1}{24})=9.$$

SUB'GROUP', *n.* See GROUP.

invariant subgroup. See INVARIANT—invariant subgroup.

SUB'HAR-MON'IC, *adj.* **subharmonic function.** A real function $u(x,y)$, defined in a domain D, is said to be subharmonic in D provided $u(x,y)$ satisfies the following conditions in D: (1) $-\infty\leqq u(x,y)<+\infty$. (The condition $u(x,y)\not\equiv-\infty$ is sometimes added.) (2) $u(x,y)$ is upper semicontinuous in D. (3) For any subdomain D' included, together with its boundary B', in D, and for any function $h(x,y)$ harmonic in D', continuous in $D'+B'$, and satisfying $h(x,y)\geqq u(x,y)$ on B', we have $h(x,y)\geqq u(x,y)$ in D'. A subharmonic function $u(x,y)$ which satisfies $u(x,y)\not\equiv-\infty$ necessarily is summable. If a function $u(x,y)$ satisfies $u(x,y)\not\equiv-\infty$ and is upper semicontinuous in its domain D of definition, then $u(x,y)$ is subharmonic if and only if it satisfies either of the following mean-value inequalities for each circular disc in D:

$$u(x_0,y_0)\leqq \frac{1}{2\pi}\int_0^{2\pi}u(x_0+\rho\cos\theta,\,y_0+\rho\sin\theta)\,d\theta,$$

$$u(x_0, y_0) \leqq \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} u(x_0+\rho\cos\theta, y_0+\rho\sin\theta)\,\rho\,d\rho\,d\theta.$$

If a function $u(x, y)$ has continuous second partial derivatives in its domain D of definition, then it is subharmonic in D if, and only if, the following differential inequality is satisfied at each point of D:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \geqq 0.$$

The notion of subharmonic function extends immediately to functions of n variables. See CONVEX—convex function.

SUB-NOR'MAL, *n.* The projection on the axis of abscissas (x-axis) of the segment of the normal between the point of the curve and the point of intersection of the normal with the x-axis. The length of the subnormal is $y(dy/dx)$, where y and dy/dx (the derivative of y with respect to x) are evaluated at the given point on the curve. See TANGENT—length of a tangent.

polar subnormal. See POLAR—polar tangent.

SUB'RE'GION, *n.* A region within a region.

SUB'SCRIPT, *n.* A small number or letter written below and to the right or left of a letter as a mark of distinction or as part of an operative symbol; used on a variable to denote a constant value of that variable or to distinguish between variables. The symbols a_1, a_2, etc., denote different constants; $D_x f$ denotes the derivative of f with respect to x; (x_0, y_0), (x_1, y_1), etc., denote coordinates of fixed points; $f(x_1, x_2, \cdots, x_n)$ denotes a function of n variables, $x_1, x_2, \cdots, x_n$; ${}_nC_r$ denotes the number of possible combinations of n things r at a time. **Double subscripts** are used, for example, in writing determinants with general terms (the general term might be denoted by a_{ij}, where the first subscript denotes the row number and the second the column number).

SUB-SE'QUENCE, *n.* A sequence within a sequence; $\frac{1}{2}, \frac{1}{4}, \cdots, 1/(2n), \cdots$ is a subsequence of $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, 1/n, \cdots$.

SUB'SET, *n.* A set contained within a set; a set whose members are members of another set. A subset is said to be a **proper subset** of (or **contained properly** in) another set if it is a subset of the set and does not contain all the members of it.

SUB'SINE, *adj.* **subsine function of order** ρ. A function f that is dominated by functions of the form $F(x) \equiv A\cos\rho x + B\sin\rho x$ in the way that convex functions are dominated by linear functions. Thus for $f(x)$ to be a subsine function of order ρ, if the values x_1, x_2 lie in the interval of definition and satisfy $0 < x_2 - x_1 < \pi/\rho$, and the above function $F(x)$ satisfies $F(x_1) = f(x_1)$, $F(x_2) = f(x_2)$, we must have $f(x) \leq F(x)$ for $x_1 < x < x_2$. See PHRAGMÉN-LINDELÖF FUNCTION.

SUB'STI-TU'TION, *n.* **elimination by substitution.** See ELIMINATION.

integration by substitution. See INTEGRATION—change of variables in integration.

inverse substitution. The substitution which exactly undoes the effect of a given substitution. For examples, see TRANSFORMATION—inverse transformation.

substitution group. Same as PERMUTATION GROUP.

substitution of one quantity for another. Replacing the one quantity by the other. Substitutions are made in order to simplify equations, simplify integrands (in the calculus), and to change (transform) geometric configurations into other forms or to different positions. See TRANSFORMATION.

trigonometric substitution. See TRIGONOMETRIC.

SUB-TAN'GENT, *n.* The projection on the axis of abscissas (x-axis) of the segment of the tangent joining the point of tangency on the curve and the point of intersection of the tangent with the x-axis; the segment of the axis of abscissas between the foot of the ordinate at the point on the curve and the x-intercept of the tangent. The length of the subtangent is $y(dx/dy)$, where y and dx/dy are evaluated at the point on the curve. See DERIVATIVE, and TANGENT—length of a tangent.

SUB-TEND', *v.* To be opposite to, or measure off, as a side of a triangle sub-

tends the opposite angle, and an arc of a circle subtends the central angle of the arc. The angle is also said to be *subtended by* the side of the triangle or arc of the circle.

SUB-TRAC′TION, *adj., n.* The process of finding a quantity which when added to one of two given quantities will give the other. These quantities are called, respectively, the **subtrahend** and **minuend,** and the quantity found is called the **difference** or **remainder.** *E.g.*, 2 subtracted from 5 is written $5-2=3$; 5 is the *minuend,* 2 the *subtrahend,* and 3 the *difference* or *remainder.* Subtraction of signed numbers is called **algebraic subtraction** (this is equivalent to changing the sign of the subtrahend and adding it to the minuend). *E.g.*, 5 minus 7 is equal to 5 plus negative 7, written $5-7=5+(-7)$, which is -2. See SUM—sum of real numbers.

higher decade subtraction. The process of subtracting a one-place number from a two-place number to give a two-place number; also the process of subtracting a two-place number from a two-place number to give a one-place number. These operations are assumed to be done mentally. Examples of higher-decade subtraction are $28-4=24$, $23-7=16$, $48-42=6$, $54-49=5$.

subtraction formulas. See TRIGONOMETRY —identities of plane trigonometry.

SUB′TRA-HEND′, *n.* A quantity to be subtracted from another.

SUC-CES′SIVE, *adj.* **successive terms or steps.** Terms or steps following one after the other.

successive trials. (*Probability.*) Successive trials (occurrences of a given event) in which one is interested in the number of times the favorable occurrence of the event is likely to take place in a certain number of trials. The probability of an event happening exactly r times in n trials is

$$_nC_r p^r q^{n-r},$$

where p and q are the probabilities of the event happening and failing, respectively, in a single trial, and $_nC_r$ is the number of combinations of n things taken r at a time (see COMBINATION). By this formula, the probability of a six coming up exactly once in three throws of a die is $3(\frac{1}{6})(\frac{5}{6})^2=\frac{25}{72}$, the probability of its coming up twice is $3(\frac{1}{6})^2(\frac{5}{6})=\frac{5}{72}$, and the probability of its coming up three times is $(\frac{1}{6})^3=\frac{1}{216}$.

SUF-FI′CIENT, *adj.* **sufficient condition.** See CONDITION.

sufficient statistic. A statistic, derived from a set of observations, which contains all the information in that set of observations relevant to the estimate being made. The arithmetic mean is an example of a sufficient statistic of observations from a population with a normal distribution.

SUM, *n.* The **sum** of two or more objects is the object which is determined from these objects by a given operation called **addition.** Usually this addition operation is related (sometimes remotely) to some process of accumulation. *E.g.*, $2+3=5$ is equivalent to stating that putting together two piles, one containing 2 things and the other 3 things, will yield a pile containing 5; the sum of vectors which represent forces is the vector which represents a force equivalent to all the individual forces operating together; the sum of several sets of "points" is the set which is obtained by accumulating all the "points" which are in any of the individual sets. See the various headings below.

algebraic sum. The combination of terms either by addition or subtraction in the sense that adding a negative number is equivalent to subtracting a positive one. The expression $x-y+z$ is an *algebraic sum* in the sense that it is the same as $x+(-y)+z$.

arithmetic sum of two quantities. The quantity obtained by adding two positive quantities. Five is the sum of two and three, written $2+3=5$. Sometimes *arithmetic sum* is used for the sum of the numerical values of signed numbers; the *arithmetic sum* of 5 and -3 is 8.

limit of a sum. See LIMIT—fundamental theorems on limits.

partial sum of an infinite series. The sum of a finite number of consecutive terms of the series, beginning with the first term. If the series is $a_1+a_2+a_3+\cdots$, then each of the quantities S_n is a partial sum, where

$$S_n=a_1+a_2+\cdots+a_n$$

is the sum of the first n terms of the series.

sum of angles. *Geometrically*, the angle determined by a rotation from the initial side through one angle, followed by a rotation, beginning with the terminal side of this angle, through the other angle; *algebraically*, the ordinary algebraic addition of the same kind of measures of the angles (*e.g.*, degrees plus degrees or radians plus radians).

sum of complex numbers. See COMPLEX—complex numbers.

sum of directed line segments. The line segment which extends from the initial point of the first line segment to the terminal point of the last line segment when the line segments are placed so that the terminal point of each is coincident with the initial point of the next. *E.g.* 5 miles east plus 2 miles west is 3 miles east. This is a special case of the sum of vectors; see below.

sum of an infinite series. The limit of the sum of the first n terms of the series, as n increases. This is not a sum in the ordinary sense of arithmetic, because the terms of an infinite series can never all be added term by term. The sum of the series

$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\cdots+(\tfrac{1}{2})^n+\cdots$$

is 1, because that is the limit approached by the sum of the first n of these terms, namely $1-1/2^n$, as n becomes infinite. The sum of the series is precisely 1, even though the actual arithmetic sum of a finite number of terms of the series is always less than 1. *Tech.* The sum of the infinite series $a_1+a_2+a_3+\cdots$ is

$$S=\lim_{n\to\infty}(a_1+a_2+a_3+\cdots+a_n).$$

That is, S is the sum of the series if and only if it is true that for any positive number ϵ there exists a number N such that, whenever $n>N$, the *partial sum* $a_1+a_2+a_3+\cdots+a_n$ differs from S by less than ϵ. See SERIES—geometric series.

sum of like powers of two quantities. A sum such as a^2+b^2 and a^3+b^3. Such sums are of interest in factoring, because when the power is odd the sum is divisible by the sum of the quantities. See DIFFERENCE—difference of like powers of two quantities.

sum of matrices. The sum $A+B$ of matrices A and B is the matrix whose elements are formed by the rule that the element in row r and column s is the sum of the elements a_{rs} and b_{rs} in row r and column s of A and B, respectively. This sum is defined only if A and B have the same number of rows and the same number of columns.

sum of order t. For positive numbers a_i, the expression $(\Sigma a_i^t)^{1/t}$. For the definition of the analogous mean of order t, see AVERAGE.

sum of real numbers. Positive integers (and zero) can be thought of as symbols used to describe the "manyness" of sets of objects (also see PEANO—Peano's postulates). Then the **sum of two integers** A and B is the integer which describes the "manyness" of the set of objects obtained by combining a set of A objects with a set of B objects. This means that addition of integers is the process of finding the number class which is composed of the number classes denoted by the addends (see CARDINAL—cardinal number). The **sum of fractions** is obtained by the same process, after a common denominator (common unit) has been established. *E.g.*, $\frac{1}{2}$ and $\frac{2}{3}$ are the same as $\frac{3}{6}$ and $\frac{4}{6}$, and 3 sixths plus 4 sixths is 7 sixths. In general,

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}.$$

The **sum of mixed numbers** can be determined by adding the integral parts and fractions separately, or by reducing each mixed number to a fraction. *E.g.*, $2\frac{1}{2}+3\frac{1}{4}=2+3+\frac{1}{2}+\frac{1}{4}=5\frac{3}{4}$, or $2\frac{1}{2}+3\frac{1}{4}=\frac{10}{4}+\frac{13}{4}=\frac{23}{4}$. When the numbers being added have been given signs, the addition is done as follows (this is sometimes called **algebraic addition**). Two positive numbers are added as above; two negative numbers are added by adding their numerical values and making the result negative; a positive and a negative number are added by subtracting the lesser numerical value from the greater and giving the difference the sign of the number which has the greater numerical value. *E.g.*, $(-2)+(-3)=-5$; $(-2)+3=1$. The significance of this definition becomes apparent when we let positive numbers denote distances eastward and negative numbers distances westward and think of their sum as the distance from the starting point to the place reached by travelling in succession the paths measured

by the addends. *E.g.*, an interpretation of $(-3)+2=-1$ is that one would finish 1 mile west of the starting point if one traveled 3 miles west and then 2 miles east. The sum of irrational numbers may be left in indicated form, after similar terms have been combined, until some specific application indicates the degree of accuracy desired. Such a sum as $(\sqrt{2}+\sqrt{3})-(2\sqrt{2}-5\sqrt{3})$ would thus be left in the form $6\sqrt{3}-\sqrt{2}$. A sum such as $\pi+\sqrt{2}$ can be approximated as

$$3.1416+1.4142=4.5558.$$

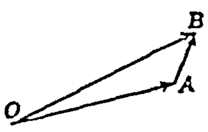
It is necessary to have a specific definition of irrational numbers before one can specifically define the sum of numbers, one or more of which is irrational. See DEDEKIND CUT.

sum of sets. The set containing all the points belonging to at least one of the sets forming the sum. The sum of two sets U and V is usually denoted by $U+V$ or $U \cup V$. *Syn.* Join, Union.

sum of vectors. *Algebraically*, the vector obtained by the addition of corresponding components. *E.g.*,

$$(2i+3j)+(i-2j)=3i+j,$$
$$(2i+3j+5k)+(i-2j+3k)=3i+j+8k.$$

Geometrically, the sum of two vectors is the third side of the triangle of which the addends form the other two sides, the initial point of one addend being on the terminal point of the other, the initial point of the latter coinciding with the initial point of the sum. In the figure, $OA+AB=OB$. See PARALLELOGRAM—parallelogram of forces, and VECTOR—vector components.



SUM'MA-BLE, *adj.* **absolutely summable series.** Refers to summability by Borel's integral method. A series Σa_n is said to be absolutely summable if the integrals

$$\int_0^\infty e^{-x}|a(x)|\,dx \quad \text{and} \quad \int_0^\infty e^{-x}|a^{[m]}(x)|\,dx$$

all exist, where $m=1, 2, 3, \cdots$ denotes derivatives of these orders, and $a(x)=a_0+a_1x+a_2x^2/2!+\cdots$.

summable divergent series. Series to which a sum is assigned by some *regular* definition of the sum of a divergent series. Better usage is to speak of a series as, *e.g.*, Cesàro summable. *I.e.*, indicate the method by which the series is summable. See SUMMATION—summation of divergent series.

summable function. A function which is Lebesgue integrable, *i.e.*, whose Lebesgue integral exists. The function is said to be summable over the region of integration. See LEBESGUE—Lebesgue integral.

uniformly summable series. A series of variable terms is *uniformly summable* on (a, b) by a given definition of the sum of a divergent series if the sequence which defines the sum converges uniformly on (a, b). The series $\Sigma(-x)^n$ diverges for $x=1$, but is uniformly summable for $0 \leqq x \leqq 1$ by any of the standard definitions, such as Hölder's, Cesàro's, and Borel's. By Hölder's definition, we have for the sum

$$\lim_{n\to\infty} [1+(1-x)+(1-x+x^2)$$
$$+\cdots+\sum_0^n(-x)^n]/n$$
$$=\lim_{n\to\infty}\left[\frac{1}{n}-x\frac{(n-1)}{n}+x^2\frac{(n-2)}{n}\right.$$
$$\left.-\cdots+\frac{(-1)^nx^n}{n}\right],$$

which evidently converges uniformly, with regard to x, on the closed interval $(0, 1)$.

SUM-MA'TION, *adj., n.* **integration as a summation process.** See INTEGRAL—definite integral.

summation convention. The convention of letting the repetition of an index (subscript or superscript) denote a summation with respect to that index over its range. *E.g.*, if 6 is the range of the index i, then a_ix^i stands for $\sum_{i=1}^{6} a_ix^i = a_1x^1+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6$. The superscript i in x^i is not the ith power of the number x, but merely an index which denotes that x^i is the ith object of the six objects $x^1, x^2, \cdots, x^6$. A repeated index, such as i in a_ix^i, is called a dummy index or an umbral index, since the value of the expression does not depend on the symbol used for this index. An index which is not repeated, such as i in $a_{ij}x^j$, is called a free index.

summation of divergent series. Attributing sums to divergent series by transforming them into convergent series, or by other devices. *E.g.*, the sum of $1-1+1-1+\cdots$ is defined as the sum of $1+x+x^2+x^3+\cdots$, with x put equal to -1 in the sum (or limit) function, or as

$$\lim_{n\to\infty}\frac{S_1+S_2+\cdots+S_n}{n} = \lim_{n\to\infty}\frac{1+0+1+\cdots+\frac{1}{2}[1-(-1)^n]}{n},$$

where S_n denotes the sum of the first n terms. In both cases the sum is $\frac{1}{2}$. The former method is an illustration of the use of *convergence factors*, in this instance, $1, x, x^2, \cdots$. The latter method is an illustration of the method of *arithmetic means*. See ABEL—Abel's method of summation, BOREL, CESÀRO—Cesàro's summation formula, HÖLDER—Hölder's definition of the sum of a divergent series.

summation of an infinite series. The process of finding the sum of the series. See SUM—sum of an infinite series.

summation sign. Sigma, the Greek letter corresponding to the English S, written Σ. When the process of summing includes the first to the nth terms of a set of numbers $a_1, a_2, a_3, \cdots, a_n, \cdots$, the sum is written

$$\sum_{i=1}^{n} a_i, \quad \text{or} \quad \sum_{1}^{n} a_i.$$

When the summation includes infinitely many terms, it is written

$$\sum_{i=1}^{\infty} a_i, \quad \sum_{1}^{\infty} a_i, \quad \text{or simply} \quad \sum a_i.$$

SU'PER-AD'DI-TIVE, *adj.* **superadditive function.** See ADDITIVE—additive function.

SU'PER-HAR-MON'IC, *adj.* **superharmonic function.** A function that is related to subharmonic functions in the way that concave functions are related to convex functions; *i.e.*, a real function f, of any number of variables, such that $-f$ is subharmonic. See SUBHARMONIC.

SU-PE'RI-OR, *adj.*, *n.* **limit superior.** (1) See SEQUENCE—accumulation point of a sequence. (2) The **limit superior** of a function $f(x)$ at a point x_0 is the largest number L such that for any $\epsilon>0$ and neighborhood U of x_0 there is a point $x\neq x_0$ of U for which $f(x)> L-\epsilon$; this definition applies to the case $L=+\infty$ if $f(x)> L-\epsilon$ is replaced by $f(x)>\epsilon$, while $L=-\infty$ if for any $\epsilon>0$ there is a neighborhood U of x_0 in which $f(x)<-\epsilon$ for each $x\neq x_0$. This limit is denoted by $\limsup_{x\to x_0} f(x)$ or $\overline{\lim_{x\to x_0}} f(x)$. The limit superior of $f(x)$ at x_0 is equal to the limit as $\epsilon\to 0$ of the l.u.b. of $f(x)$ for $|x-x_0|<\epsilon$ and $x\neq x_0$, and may be positively or negatively infinite. (3) The **limit superior** of a sequence of point sets $U_1, U_2, \cdots$ is the set consisting of all points belonging to infinitely many of the sets U_n. It is equal to the intersection of all sums of the form $U_p+U_{p+1}+\cdots$; *i.e.*, $\prod_{p=1}^{\infty}\sum_{n=p}^{\infty} U_n$. For sequences of sets, the limit superior is also called the **complete limit.** See INFERIOR—limit inferior. *Syn.* Upper limit.

SU'PER-OS'CU-LAT-ING, *adj.* **superosculating curves on a surface.** Normal sections of a surface which are *superosculated* by their circles of curvature. See below, SUPEROSCULATION.

SU'PER-OS'CU-LA-TION, *n.* The property of some pairs of curves or surfaces of having contact of higher order than other pairs, which are said to osculate.

SU'PER-POSE', *v.* To place one configuration upon another in such a way that corresponding parts coincide. To superpose two triangles which have their corresponding sides equal is to place one upon the other so that corresponding sides coincide.

SU'PER-POS'A-BLE, *adj.* **superposable configurations.** Two configurations which can be superposed. *Syn.* Congruent.

SU'PER-PO-SI'TION, *adj.*, *n.* **axiom of superposition.** See AXIOM—axiom of superposition.

superposition principle for electrostatic intensity. See ELECTROSTATIC.

SU'PER-SCRIPT, *n.* A number written above and to the right or left of a letter, usually denoting a power or a derivative, but sometimes used in the same sense as subscript. See ACCENT, PRIME—prime as a symbol, and EXPONENT.

SUP′PLE-MEN′TAL, *adj.* **supplemental chords of a circle.** The chords joining a point on the circumference to the two extremities of a diameter.

SUP′PLE-MEN′TA-RY, *adj.* **supplementary angles.** Two angles whose sum is 180°; two angles whose sum is a straight angle. The angles are said to be *supplements* of each other.

SUP-PORT′, *adj., n.* **line of support.** Relative to a convex region B in the plane, a line of support is a line containing at least one point of B but such that one of the two open half planes determined by the line contains no point of B. The equation of such a line can be written in the form

$$x \cos \theta + y \sin \theta = S(Q),$$

where Q is the point with coordinates $(\cos \theta, \sin \theta)$ and $S(Q)$ is the *normalized support function.* The function $S(Q)$ is a *subsine function* of the angle θ. For a convex or concave function, a line of support can be defined similarly in terms of the graph of the function.

plane (and hyperplane) of support. Relative to a *convex body* B in three-dimensional space, a plane of support is a plane containing at least one point of B, but such that one of the two open half-spaces determined by the plane contains no point of B. For a *normed vector space* T and a convex body B contained in T, a hyperplane of support is a hyperplane H whose distance from B is zero and which is the separating hyperplane between two open halves of T, one of which contains no points of B. This means that H is a hyperplane of support of B if and only if there is a *continuous linear functional* f and a constant c for which $f(P) \leqq c$ if P belongs to B and H is the set of all P with $f(P) = c$. A *separable Banach space* is *reflexive* if and only if the distance between H and B being zero implies H contains a point of B, for any convex body B and hyperplane of support H. For a space with an inner product, a hyperplane of support must contain a point of the convex body, and there is a point P for which the hyperplane of support consists of all those points Q for which $(P, Q) = S(P)$, where $S(P)$ is the support function. See below, support function.

support function. Relative to a convex body B in any space with a real *inner product* (*e.g.*, a *Euclidean space* of any dimension, or a real *Hilbert space*), the support function $S(P)$ is defined (for all points P of the space other than $P=0$) as

$$S(P) = \max (P, Q)$$

for Q in B, where (P, Q) is the inner product of P and Q. Thus for each point Q of B we have $(P, Q) \leqq S(P)$, with the sign of equality holding for some point Q_0 of B. All of B lies in one of the two closed half spaces bounded by the hyperplane consisting of all points R for which $(P, R) = S(P)$. The function $S(P)$ is a convex function of P. The support function satisfies the relation $S(kP) = kS(P)$ for $k \geqq 0$. Accordingly, $S(P)$ is completely determined by its values $S(Q)$ on the unit sphere consisting of all points Q with $(Q, Q) = 1$. With its independent variable thus restricted, the function $S(Q)$ is called a **normalized support function** of B. See MINKOWSKI—Minkowski distance function, and above, plane of support.

SURD, *n.* A sum of one or more irrational indicated roots of numbers. Sometimes used for *irrational number.* A surd of one term is called **quadratic, cubic, quartic, quintic,** etc., according as the index of the radical is two, three, four, five, etc. It is an **entire surd** if it does not contain a rational factor or term (*e.g.*, $\sqrt{3}$ or $\sqrt{2} + \sqrt{3}$); a **mixed surd** if it contains a rational factor or term (*e.g.*, $2\sqrt{3}$ or $5 + \sqrt{2}$); a **pure surd** if each term is a surd (*e.g.*, $3\sqrt{2} + \sqrt{5}$). A **binomial surd** is a binomial, at least one of whose terms is a surd, such as $2 + \sqrt{3}$ or $\sqrt[3]{2} - \sqrt{3}$. **Conjugate binomial** surds are two binomial surds of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$, where a, b, c and d are rational and $\sqrt{b}$ and $\sqrt{d}$ are not both rational. The product of two conjugate binomial surds is rational, *e.g.*, $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$. A **trinomial surd** is a trinomial at least two of whose terms are surds which cannot be combined without evaluating them; $2 + \sqrt{2} + \sqrt{3}$ and $3 + \sqrt{5} + \sqrt[3]{2}$ are trinomial surds.

SUR'FACE, *adj., n.* A surface is the geometric figure consisting of those points whose coordinates satisfy an equation such as $z=f(x, y)$, or $F(x, y, z)=0$, or parametric equations $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, with conditions such as continuity or nonvanishing of a Jacobian imposed to insure nondegeneracy. *E.g.*, the surface of the sphere of radius 2 with center at (0, 0, 0) has the equation $x^2+y^2+z^2=4$; it also has the parametric equations $x=2\sin\phi\cos\theta$, $y=2\sin\phi\sin\theta$, $z=2\cos\phi$, *Tech.* In a rather restrictive sense, a *closed surface* might be defined as a connected, compact metric space which is homogeneous in the sense that each point has a neighborhood which is homeomorphic with the interior of a circle in the plane; a *surface with boundary curves* would then be defined by changing the neighborhood condition so that each point on a boundary curve has a neighborhood which is homeomorphic with half of a 2-cell with the diameter included and lying along the boundary curve. An equivalent definition is that a surface is a geometric figure which can be subdivided into a finite number of "triangles" (each of which has vertices and edges and is homeomorphic with a plane triangle) such that (i) if two triangles intersect, their intersection is a side of each of them; (ii) no side belongs to more than two triangles; (iii) for any two triangles R and S, there is a sequence of triangles $T_1, T_2, \cdots, T_n$ such that $T_1=R$, $T_n=S$, and any two adjacent triangles, T_i and T_{i+1}, have a side in common. Such a surface is **closed** if each side of a triangle also belongs to another triangle; otherwise the surface has a finite number of closed boundary curves. A surface is **orientable** if the above triangles can be oriented so that a direction around the perimeter is prescribed in such a way that two intersecting triangles assign opposite orientations (directions) to their common side (this is equivalent to requiring that the surface not contain a Möbius strip, or that a small oriented "circle" can not be moved in the surface so as to return to its initial position with its orientation reversed). See GENUS—genus of a surface.

algebraic surface. A surface which admits a parametric representation such that the coordinate functions are algebraic functions of the parameters u, v.

applicable surfaces. Surfaces such that there exists a length-preserving map of one on the other. See ISOMETRIC—isometric map.

canal surface. The envelope of a one-parameter family of spheres of equal radii having their centers on a given space curve. For any point on the curve, the *characteristic* is the great circle in the plane normal to the curve at the point.

curvature of a surface. See various headings under CURVATURE.

curved surface. A surface no part of which is a plane surface.

cylindrical surface. See CYLINDRICAL—cylindrical surface.

equation of a surface. See SURFACE, EQUATION—equation of a surface, and PARAMETRIC—parametric equations.

fundamental coefficients of a surface. The **fundamental coefficients of the first order** of a surface are the coefficients E, F, G of the first fundamental quadratic form of the surface. *Syn.* Fundamental quantities of the first order of a surface. The **fundamental coefficients of the second order** of a surface are the coefficients D, D', D'' of the second fundamental quadratic form of the surface. See below, fundamental quadratic forms of a surface.

fundamental quadratic forms of a surface. The expression $E\,du^2+2F\,du\,dv+G\,dv^2$ is the **first fundamental quadratic form of a surface.** Also written $g_{\alpha\beta}\,du^\alpha\,du^\beta$ in tensor notation. See LINEAR—linear element. The **second fundamental quadratic form of a surface** is the expression $\Phi=D\,du^2+2D'\,du\,dv+D''\,dv^2$. Also written

$$\Phi=L\,du^2+2M\,du\,dv+N\,dv^2,$$

and $\Phi=e\,du^2+2f\,du\,dv+g\,dv^2$, and, in tensor notation, $d_{\alpha\beta}\,du^\alpha\,du^\beta$. See DISTANCE—distance from a surface to a tangent plane. The **third fundamental quadratic form of a surface** is the fundamental quadratic form of the spherical representation of the surface.

fundamental quantities of the first order of a surface. See above, fundamental coefficients of a surface.

Gaussian representation of a surface. Same as spherical representation of a surface. See SPHERICAL.

imaginary surface. See IMAGINARY—imaginary curve (surface).

material surface. See MATERIAL.

minimal surface. See MINIMAL.

molding surface. A surface generated by a plane curve whose plane rolls without slipping over a cylinder. If the cylinder is a line, the molding surface is a surface of revolution. See below, surface of Monge.

normal to a surface. See NORMAL—normal to a curve or surface.

one-sided surface. A surface which has only one side, in the sense that any two bugs on the surface can get to each other without going around an edge, regardless of where put. See MÖBIUS—Möbius strip, KLEIN BOTTLE, and MINIMAL—double minimal surface. *Tech.* A surface is one-sided if it is *nonorientable*; it is nonorientable if and only if it contains a Möbius strip (also see SURFACE).

parallel surfaces. See PARALLEL.

plane surface. A plane.

principal direction on a surface. See DIRECTION.

pseudospherical surface. See PSEUDOSPHERICAL.

quadric surface. See QUADRIC.

ruled surface. See RULED.

similar surfaces. See SIMILAR—similar surfaces.

spherical representation of a surface. See SPHERICAL.

spherical surface. See SPHERICAL.

surface area. (1) The limit of the sum of the areas of the polygons formed by the intersections of tangent planes at neighboring points distributed over the entire surface, as the area of the largest of these polygons approaches zero. Each of these plane tangential areas is obtainable by projecting some area lying in one of the coordinate planes onto the tangent plane. (2) Let a plane P be such that no line perpendicular to P cuts a given surface S in more than one point, and let A be the projection of S into P. Let A be subdivided and let each subdivision be projected into a tangent plane along lines perpendicular to P, where the tangent plane is tangent to S at a point on a line perpendicular to P at a point of the subdivision. The area of S can then be defined as the limit of the sum of the areas of these projections as $\Delta \to 0$, where Δ is the least upper bound (for all subdivisions) of distances between points belonging to the same subdivision. If P is the (x, y)-plane, and if β is the angle between the tangent plane and the xy-plane, the area of S is equal to the integral of $(\sec\beta)\,dx\,dy$ (the **element of area** or **differential of area**) over the area in the xy-plane which is bounded by the projection on that plane of the boundary of the given surface. If the equation of the surface is in the form $z=f(x, y)$, then

$$\sec\beta=\sqrt{(1+D_x{}^2z+D_y{}^2z)},$$

where D_xz and D_yz denote the partial derivatives of z with respect to x and y. It is assumed here that $\sec\beta$ is finite, *i.e.*, none of the tangent planes are perpendicular to the plane on which the surface is being projected. In more advanced theory, many definitions of area have been given, a frequently used one being that given by Lebesgue: The area of a surface is the smallest value which is the limit of the sum of the areas of polyhedrons converging (in the sense of Fréchet) to the surface.

surface of constant curvature. A surface whose total curvature is the same at all its points. These are *developable surfaces*, for $K=0$; *spherical surfaces* (not just the sphere), for $K>0$; and *pseudospherical surfaces*, for $K<0$. See PSEUDOSPHERICAL and SPHERICAL.

surface of Enneper. The real minimal surface for which $\phi(u)=\text{const.}$ See WEIERSTRASS—equations of Weierstrass. If we take $\phi(u)=3$, and let $u=s+it$, the parametric curves are the lines of curvature and the coordinate functions are

$$x=3s+3st^2-s^3,$$
$$y=3t+3s^2t-t^3,$$
$$z=3s^2-3t^2;$$

the *map* is a conformal one, and the coordinate functions are harmonic.

surface harmonic. See HARMONIC—surface harmonic.

surface of Henneberg. The real minimal surface for which $\phi(u)=1-1/u^4$. See WEIERSTRASS—equations of Weierstrass. The surface of Henneberg is a *double minimal surface*. See MINIMAL.

surface integral. The integral of some

function, $f(x, y, z)$, over a surface. It is usually written

$$\int_S f(x, y, z)\, dS.$$

If the surface is divided into a number of smaller nonoverlapping areas and the sum of the products of each of these areas by a value of $f(x, y, z)$ at a point in that area is formed, then the integral over the surface is the limit of this sum as the number of subdivisions becomes infinite in such a way that the upper bound of all distances between two points in the same subdivision approaches zero. If the equation of the surface is $z=g(x, y)$, then the surface integral can be written

$$\int_a^b \int_{x_1}^{x_2} \sec\beta f[x, y, g(x, y)]\, dx\, dy,$$

the integration being taken over the projection of the surface on the xy-plane. Similar forms can be derived for the projections on the xz and yz planes. See above, surface area, for the value of $\sec\beta$ and an example of a surface integral. If the surface can be oriented by a unit normal vector **n** at each point, then, for a vector function $\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$,

$$\int_S \mathbf{F}\cdot\mathbf{n}\, dS=\int_S (F_1\lambda+F_2\mu+F_3\nu)dS$$

is the limit of the sum of scalar products $\mathbf{F}\cdot\mathbf{n}\Delta S$ (when the number of subdivisions becomes infinite as described above), where **n** is the unit normal vector at a point of the subdivision of area ΔS. The numbers λ, μ, ν are the cosines of the angles between **n** and the positive directions of the x, y and z axes. See STOKES' THEOREM.

surface of Joachimsthal. A surface such that all the members of one of its two families of lines of curvature are plane curves, and such that all these planes are coaxial.

surface of Liouville. A surface which admits a parametric representation such that the first fundamental quadratic form reduces to

$$ds^2=[f(u)+g(v)][du^2+dv^2].$$

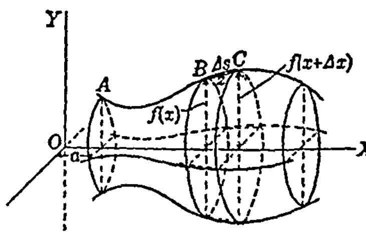
surface of Monge. A surface generated by a plane curve whose plane rolls without slipping over a developable surface. See above, molding surface.

surface patch. A surface, or part of a surface, bounded by a closed curve in contradistinction to a surface of infinite extent or a closed surface such as a sphere.

surface of revolution. A surface which can be generated by revolving a plane curve about an axis in its plane. Sections of a surface of revolution perpendicular to this axis are circles, called **parallel circles** or simply **parallels**; sections containing the axis are called **meridian sections**, or simply **meridians.** The earth is a surface of revolution which can be generated by revolving a meridian about the line through the north and south poles. A surface of revolution can also be generated by a circle moving always perpendicular to a fixed line with its center on the fixed line and expanding or contracting so as to continually pass through a curve which lies in a plane with the straight line. The **element of area** of a surface of revolution can be taken as $2\pi r ds$, where r is the distance from the axis of revolution of any point in the element of arc, ds, of the curve which is rotated to form the surface. If the curve $y=f(x)$ is revolved about the x-axis (as in the figure), $2\pi r ds=2\pi f(x)\, ds$ and the area of the surface of revolution, between the values a and b of x, is

$$\int_a^b 2\pi f(x)\sqrt{1+(dy/dx)^2}\, dx.$$

From the figure, it can be seen that $2\pi r\Delta s$ is the area derived by rotating the arc $BC=\Delta s$ about the x-axis, and hence $2\pi r ds$ is an approximation to this area (see ELEMENT—element of arc of a curve).



surface of Scherk. The real minimal surface for which $\phi(u)=\dfrac{2}{1-u^4}$. See WEIERSTRASS—equations of Weierstrass. The surface of Scherk is *doubly periodic.*

surface of translation. A surface admitting a representation of the form $x=$

$x_1(u)+x_2(v)$, $y=y_1(u)+y_2(v)$, $z=z_1(u)+z_2(v)$. It might be considered as being generated by translating the curve C_1: $x=x_1(u)$, $y=y_1(u)$, $z=z_1(u)$ parallel to itself in such a way that each point of C_1 describes a curve congruent to C_2: $x=x_2(v)$, $y=y_2(v)$, $z=z_2(v)$; or equally well by a translation of C_2 parallel to itself in such a way that each point of C_2 describes a curve congruent to C_1. The loci described by the points of C_1 (or of C_2) are called the **generators** of the surface. *Syn.* Translation surface.

surface of Voss. A surface with a conjugate system of geodesics.

surfaces of center relative to a given surface. The loci of the centers of principal curvature of the given surface. See CENTER—centers of principal curvature of a surface at a point. The surfaces of center of S are also surfaces of center of any surface parallel to S. See above, parallel surfaces, and COMPLEMENTARY—surface complementary to a given surface.

traces of a surface. See TRACE—traces of a surface.

Weingarten surface. A surface such that each of the principal radii is a function of the other. *E.g.*, surfaces of constant total curvature and surfaces of constant mean curvature are Weingarten surfaces. *Syn.* *W*-surface.

SUR-REN'DER, *adj.* **surrender charge.** (*Insurance.*) The deduction that is made from the terminal reserve to determine the cash surrender value (not over $2\frac{1}{2}\%$ of the terminal reserve is allowed by law in most states).

surrender value of an insurance policy. The amount the insurance company is willing to pay the insured for the return (cancellation) of the policy; the difference between the terminal reserve and the surrender charge.

SUR'TAX', *n.* Tax, additional to the normal tax, levied on incomes above a certain level.

SUR-VEY'OR, *n.* **surveyor's measure.** See DENOMINATE NUMBERS in the appendix.

SWAC. An automatic digital computing machine at Numerical Analysis Research, University of California, Los Angeles. SWAC is an acronym for *Standards Western Automatic Computer.*

SYL'LO-GISM, *n.* A logical statement that involves three propositions, usually called the *major premise*, *minor premise*, and *conclusion*, the conclusion necessarily being true if the premises are true. *E.g.*, the three propositions might be: "Any man likes fishing or likes singing"; "John does not like fishing"; "John likes singing." A **hypothetical syllogism** is a particular type of syllogism which relates three implications (p, q, r) and states: "If p implies q, and q implies r, then p implies r." This is frequently written as: $[(p \to q) \wedge (q \to r)] \to (p \to r)$. A **categorical syllogism** relates implications with universal quantifiers, an example of which is: If the propositions "*For any quadrilateral T, if T is a square, then T is a rectangle*" and "*For any quadrilateral T, if T is a rectangle, then T is a parallelogram*" are true, then the proposition "*For any quadrilateral T, if T is a square, then T is a parallelogram*" is true. See IMPLICATION.

SYLVESTER. Sylvester's dialytic method. A method of eliminating a variable from two algebraic equations. It consists essentially of multiplying each of the equations by the variable, thus getting two more equations and only one higher power of the variable, doing the same with the two new equations, etc., until the number of equations is one greater than the number of powers of the variable, then eliminating the various powers of the variable between these equations as if the powers were different unknowns (see ELIMINATION—elimination of n variables from $n+1$ equations). *Sylvester's method* is equivalent to the procedure illustrated by the following example, which does not require determinants: It is desired to eliminate x from

(1) $$x^2+ax+b=0$$

and

(2) $$x^3+cx^2+dx+e=0.$$

Multiply equation (1) by x and subtract the result from equation (2). This results in an equation of the second degree. Eliminate x^2 between this equation and equation (1), and so on. Finally one reaches

two linear equations, subtraction of which eliminates the variable entirely. See RESULTANT—resultant of a set of polynomial equations.

SYM'BOL, *n.* A letter or mark of any sort representing quantities, relations, or operations. See the appendix for a list of mathematical symbols.

algebraic symbols. Symbols representing numbers, and algebraic combinations and operations with these numbers.

SYM-MET'RIC, SYM-MET'RI-CAL, *adj.* Possessing symmetry. See various headings below and under SYMMETRY.

cyclosymmetric function. A function which remains unchanged under a *cyclic* change of the variables. See ABSOLUTE—absolute symmetry.

elementary symmetric functions. For n variables $x_1, x_2, \cdots, x_n$, the elementary symmetric functions are

$$\begin{aligned} \sigma_1 &= x_1+x_2+\cdots+x_n, \\ \sigma_2 &= x_1x_2+x_1x_3+\cdots+x_{n-1}x_n, \\ &\cdots\cdots\cdots\cdots\cdots \\ \sigma_n &= x_1x_2x_3\cdots x_n, \end{aligned}$$

where σ_k is the sum of all products of k of the variables $x_1, x_2, \cdots, x_n$. Any symmetric polynomial in n variables can be written in exactly one way as a polynomial in the elementary symmetric functions. For a polynomial $p(x)$ of degree n with roots $a_1, a_2, \cdots, a_n$,

$$\begin{aligned} p(x) &= (x-a_1)(x-a_2)(x-a_3)\cdots(x-a_n) \\ &= x^n-\sigma_1x^{n-1}+\sigma_2x^{n-2}-\sigma_3x^{n-3}+\cdots \\ &\qquad +(-1)^n\sigma_n, \end{aligned}$$

where $\sigma_1, \cdots, \sigma_n$ are the elementary symmetric functions of $a_1, \cdots, a_n$.

skew-symmetric determinants and matrices. See SKEW.

symmetric determinant. A determinant having all its *conjugate elements* equal, a determinant which is symmetric about the principal diagonal.

symmetric difference. See DIFFERENCE—difference of two sets.

symmetric distribution. See DISTRIBUTION.

symmetric dyadic. See DYAD.

symmetric form of the equations of a line in space. See LINE—equation of a straight line.

symmetric function. A function of two or more variables which remains unchanged under every interchange of two of the variables; $xy+xz+yz$ is a symmetric function of x, y, and z. Such a function is sometimes called **absolutely symmetric**; a function which remains unchanged under cyclic changes of the variables is **cyclosymmetric.** The word absolute is usually omitted, symmetry and cyclosymmetry being sufficient. The function

$$abc+a^2+b^2+c^2$$

has absolute symmetry, whereas

$$(a-b)(b-c)(c-a)$$

has only cyclosymmetry.

symmetric geometric configurations. (1) A geometric configuration (curve, surface, etc.) is said to be symmetric (have symmetry) with respect to a point, a line, or a plane, when for every point on the configuration there is another point of the configuration such that the pair is symmetric with respect to the point, line, or plane. The point is called the **center of symmetry**; the line, the **axis of symmetry;** and the plane, the **plane of symmetry.** See below, symmetric points. For a **plane curve,** the following are tests for symmetry: (1) In Cartesian coordinates, if its equation is unaltered when the variables are replaced by their negatives, it is symmetric with respect to the origin; if its equation is unaltered when y is replaced by $-y$, it is symmetric with respect to the x-axis (in this case the equation contains only even powers of y, if it is rational in y); if its equation is unaltered when x is replaced by $-x$, it is symmetric with respect to the y-axis (in this case the equation contains only even powers of x, if it is rational in x). If it is symmetric with respect to both axes, it is symmetric with respect to the origin, but the converse is not true. (2) In polar coordinates, (r, θ), a curve is symmetric with respect to the origin if its equation is unchanged when r is replaced by $-r$ ($r^2=\theta$ is symmetrical with respect to the origin); it is symmetric about the polar axis if its equation is unchanged when θ is replaced by $-\theta$ ($r=\cos\theta$ is symmetrical with respect to the polar axis); and it is symmetrical about the line $\theta=\frac{1}{2}\pi$ if its equation is unchanged when θ is replaced by $180°-\theta$

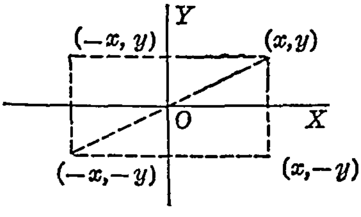
($r=\sin\theta$ is symmetrical about $\theta=\frac{1}{2}\pi$). The conditions for polar coordinates are sufficient, but not necessary. Similar tests for symmetry of other geometric configurations can be made. A plane figure has **two-fold symmetry** with respect to a point if after being revolved, in its plane, about the point through 180° it forms the same figure as before. If the angle through which it is revolved is 120°, it is said to have **three-fold symmetry**; if the angle is $180°/n$, it is said to have **n-fold symmetry** with respect to the point. A regular polygon of n sides has n-fold symmetry about its center. (2) Two geometric configurations are symmetric with respect to a point, line, or plane, if for each point of either configuration there is a point of the other configuration such that the pair of points is symmetric with respect to the point, line, or plane. One of the geometric configurations is then said to be the *reflection* of the other through the point, line, or plane.

symmetric group. See PERMUTATION—permutation group.

symmetric matrix. A matrix which is equal to its *transpose*; a square matrix which is symmetric about the principal diagonal. See ORTHOGONAL—orthogonal transformation.

symmetric pair of equations. A pair of equations which remains unchanged as a pair, although the equations may be interchanged, when the variables are interchanged. The equations $x^2+2x+3y-4=0$ and $y^2+2y+3x-4=0$ are a symmetric pair.

symmetric points. (1) Two points are said to be symmetric (have *symmetry*) with respect to a third point (called the **center of symmetry**) if the third point bisects the line joining the points. (2) Two points are said to possess symmetry with respect to a line or plane (called the **axis** or **plane of symmetry**) if the line, or plane, is the perpendicular bisector of the line segment



joining the two points. The pairs of points whose coordinates are (x, y) and $(-x, -y)$, or $(-x, y)$ and $(x, -y)$, are symmetric with respect to the origin; the points (x, y) and $(x, -y)$, or $(-x, y)$ and $(-x, -y)$, are symmetric with respect to the x-axis; the points (x, y) and $(-x, y)$, or $(x, -y)$ and $(-x, -y)$, are symmetric with respect to the y-axis.

symmetric relation. A relation which has the property that if a is related to b, then b is related in like manner to a. The equals relation of algebra is symmetric, since if $a=b$, then $b=a$. A relation is **asymmetric** if there are no pairs (a, b) such that a is related to b and b is related to a. The property of *being older than* is *asymmetric*; if a is older than b, then b is not older than a. A relation is *nonsymmetric* if there is at least one pair (a, b) such that a is related to b, but b is not related to a. The relation of *love* is *nonsymmetric*, since if a loves b, b may, or may not, love a.

symmetric spherical triangles. Spherical triangles whose corresponding sides and corresponding angles are equal, but appear in opposite order when viewed from the center of the sphere. The triangles are not superposable.

symmetric tensor. See TENSOR.

symmetric transformation. A transformation T defined on a **Hilbert space** H is symmetric if the inner products (Tx, y) and (x, Ty) are equal for every x and y in the domain of T. If, also, the domain of T is dense in H, then the second *adjoint* T^{**} of T is a symmetric transformation which is also *closed*. Any bounded symmetric transformation has an extension which is self-adjoint. A symmetric transformation whose domain (or range) is all of H is bounded and self-adjoint. For finite-dimensional spaces, a transformation T, which transforms vectors $x=(x_1, x_2, \cdots, x_n)$ into $Tx=(y_1, y_2, \cdots, y_n)$ with $y_i=\sum_j a_{ij}x_j$ for each i, is symmetric if and only if the matrix (a_{ij}) of its coefficients is a Hermitian matrix. See SELF—self-adjoint transformation.

symmetric trihedral angles. See TRIHEDRAL—trihedral angle.

SYM'ME-TRY, *adj.*, *n.* See various headings under SYMMETRIC.

axial symmetry. Symmetry with respect to a line. See SYMMETRIC—symmetric geometrical configurations. *Syn.* Line symmetry.

axis, center, and plane of symmetry. See SYMMETRIC—symmetric geometric configurations.

central symmetry. Symmetry with respect to a point. See SYMMETRIC—symmetric geometric configurations.

cyclosymmetry. See SYMMETRIC—symmetric function.

SYN-THET'IC, *adj.* **synthetic division.** Division of a polynomial in one variable, say x, by x minus a constant (positive or negative), making use of detached coefficients and a simplified arrangement of the work. Consider the division of $2x^2-5x+2$ by $x-2$. Using ordinary long division, the process would be written

$$\begin{array}{l|l} 2x^2-5x+2 & x-2 \\ \underline{2x^2-4x} & \overline{2x-1} \\ -x+2 & \\ \underline{-x+2} & \end{array}$$

Noting that the coefficient in the quotient is always the coefficient of the first term in the dividend, that it is useless to write down the $-x$, and that by changing the sign of -2 in the quotient one could add instead of subtract, the process can be put in the *synthetic division* form

$$\begin{array}{r|l} 2-5+2 & \underline{2} \\ \underline{4-2} & \\ 2-1+0 & \end{array}$$

The detached coefficients of the quotient, 2 and -1, are called the **partial remainders,** while the last term, here 0, is called the **remainder.**

synthetic geometry. The study of geometry by synthetic and geometric methods. See below, synthetic method of proof. Synthetic geometry usually refers to projective geometry. *Syn.* Pure geometry.

synthetic method of proof. A method of proof involving a combining of propositions into a whole or system; involving reasoning by advancing to a conclusion from principles established or assumed and propositions already proved; the opposite of *analysis.* *Syn.* Deductive method of proof.

synthetic substitution. Same as SYNTHETIC DIVISION. The latter is more commonly used.

SYS'TEM, *n.* (1) A set of quantities having some common property, such as the *system* of even integers, the *system* of lines passing through the origin, etc. (2) A set of principles concerned with a central objective, as, a *coordinate system*, a *system* of notation, etc.

coordinate system. See COORDINATE.

decimal system. See DECIMAL—decimal system.

dense system of numbers. See DENSE.

duodecimal system. See DUODECIMAL.

logarithmic system. Logarithms using a certain base, as the *Briggs system* (which uses 10 for a base), or the *natural system* (which uses $e=2.71828\cdots$).

metric system. See METRIC.

number system. A system of numbers distinguished from other numbers by some particular characteristic, as the number system with base 10, called the decimal system, or the binary system, with base 2.

system of circles. Sometimes used for family of circles. See CIRCLE—family of circles.

system of equations. A set of two or more equations, which are to be treated simultaneously, generally to be solved for values of the variables which satisfy all of the equations, if there are such values.

system of linear equations. See SIMULTANEOUS—simultaneous equations.

T

T, *n.* (*Statistics.*) The symbol usually assigned to $t=(\bar{x}-u)\sqrt{n}/s$, where $\bar{x}$ is the mean of a random sample of size n from a normal population with mean u; s is the estimate of the standard deviation of the normal population as estimated from the sample. Also known as STUDENT'S t. Its distribution function is

$$F(t_0)=\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi(n-1)}\,\Gamma\left(\frac{n-1}{2}\right)}\int_{-\infty}^{t_0}\left(1+\frac{t^2}{n-1}\right)^{-n/2}dt.$$

Applicable to the test of the hypothesis that a random sample of n observations is from a normal population with mean u, and with the variance unspecified. The parameter $n-1$ is often called the number of **degrees of freedom.** As $n \to \infty$, the distribution approaches normality. For n greater than 30, the normal distribution usually gives a sufficiently precise approximation. The "t" *test* and distribution was developed by "Student" (W. S. Gossett), although the distribution had been noted earlier by Helmert. In general, "t" is the ratio of a normally distributed variate with mean of zero to an independent estimate of the standard deviation of the variable, based on m degrees of freedom; "t"$^2 = \chi^2/m$, for m degrees of freedom. See CHI-SQUARE.

T score. A standardized score or variate in which the mean is 50 and the standard deviation is 10. Any variable may be converted to measurements in terms of T scores by dividing the deviation of the value from the mean by 10, and adding 50. Also may be obtained by multiplying the standard deviate score by 10 and adding 50.

TA'BLE, *n.* A systematic listing of results already worked out, which reduces the labor of computors and investigators or forms a basis for future predictions. See headings under ACCURACY, COMMUTATION, CONTINGENCY, CONVERSION, MORTALITY, and the tables in the appendix.

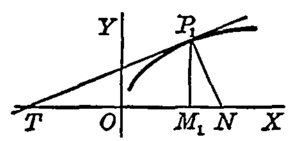
TAB'U-LAR, *adj.* **tabular differences.** The differences between successive values of a function, as recorded in a table. The tabular differences of a table of *logarithms* are the differences between successive mantissas, usually recorded in a column of their own. The tabular differences of a *trigonometric table* are the differences between successive recorded values of a trigonometric function.

TAC'NODE, *n.* Same as POINT OF OSCULATION. See OSCULATION.

TAC-POINT, *n.* A **tac-point** of a family of curves is a point where two different members of the family intersect and have a common tangent; a **tac-locus** is a set of tac-points. *E.g.*, for the family of circles of radius one which are tangent to the x-axis, the lines $y=1$ and $y=-1$ are tac-loci. See DISCRIMINANT—discriminant of a differential equation.

TAN'GEN-CY, *n.* **point of tangency.** The point in which a line tangent to a curve meets the curve, or the point in which a line or a plane tangent to a surface meets the surface. *Syn.* Point of contact.

TAN'GENT, *adj., n.* **length of a tangent.** The distance from the point of contact to the intersection of the tangent line with the x-axis. In the figure, the length of the tangent at P_1 is TP_1; the length of the **normal** at P_1 is NP_1; the **subtangent** at P_1 is TM_1; and the **subnormal** at P_1 is NM_1.

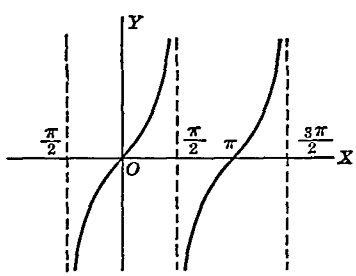


polar tangent. See POLAR—polar tangent.

tangent of an angle. See TRIGONOMETRIC—trigonometric functions.

tangent cone. See CONE—tangent cone of a quadric surface.

tangent curve. The graph of $y=\tan x$. The curve has a point of inflection at the origin, is asymptotic to the lines $x=-\frac{1}{2}\pi$ and $x=\frac{1}{2}\pi$, is convex toward the x-axis (except at the points of inflection), and duplicates itself in each successive interval of length π. See TRIGONOMETRIC—trigonometric functions.



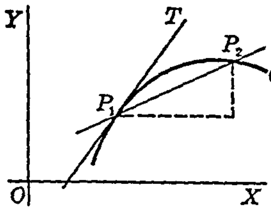
tangent formulas of spherical trigonometry. Same as HALF-ANGLE and HALF-SIDE FORMULAS. See TRIGONOMETRY—half-angle

and half-side formulas of spherical trigonometry.

tangent law, or law of tangents. A relation between two sides and the tangents of the sum and difference of the opposite angles of a plane triangle, which is adapted to calculations by logarithms. If A and B are two angles of a triangle, and a and b the sides opposite A and B, respectively, then the law is

$$\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

tangent lines and curves. The tangent line to a curve at point P (the **point of tangency** or **point of contact**) is the line in the limiting position, if this exists, of the secant line through a fixed point P on a curve C and a variable point P' on C as $P'\to P$ along C. This means that if there is a tangent line at a point P on a curve, this is the line L which passes through P and has the property that for any positive number ϵ there is a positive number δ such that, if Q is any point on the curve for which the distance from P to Q is less than δ, then the angle between the line L and the line through P and Q is less than ϵ. For the plane curve in the figure, the tangent line at P_1 is the limiting position of a secant (P_1P_2 in the figure) when two of its points of intersection with the curve (P_1 and P_2)



have moved into coincidence (at P_1). The tangent is the line P_1T. The equation of the tangent at a point on a curve is obtained by substituting the coordinates of the point and the slope of the curve at the point in the point-slope form of the equation of a line. The derivative y', evaluated at the point, is the slope of the tangent. **Two curves** are tangent at a point P if the two curves have the same tangent line at P. A curve or a line is **tangent to a surface** at a point P if the curve (or line) is tangent to a curve on the surface at the point P. See CONIC—tangent to a general conic.

tangent plane. The tangent plane to a surface at a point P is the plane which is such that each line in the plane which passes through P is tangent to the surface at P. If the first-order partial derivatives of $f(x, y, z)$ are continuous in a neighborhood of (x_0, y_0, z_0), then the direction numbers of the normal line of the plane tangent to the surface whose equation is $f(x, y, z)=0$, at the point (x_0, y_0, z_0), are the partial derivatives of $f(x, y, z)$ with respect to x, y, and z, respectively, evaluated at the point. Hence the equation of the tangent plane is

$$D_xf(x_0, y_0, z_0)(x-x_0) + D_yf(x_0, y_0, z_0)(y-y_0) + D_zf(x_0, y_0, z_0)(z-z_0)=0,$$

where D_x, D_y and D_z denote partial derivatives with respect to x, y, and z. See PLANE—equation of a plane, and PARTIAL—partial derivative. The tangent plane to a **cone** or **cylinder** at a point is the plane determined by the element through the point and the tangent to the directrix at its intersection with this element. The tangent plane to a **sphere** at a point P is the plane through P which touches the sphere only at P (the plane which is perpendicular to the radius which terminates at P). If the equation of a **general quadric surface** is $ax^2+by^2+cz^2+2dxy+2exz+2fyz+2gx+2hy+2kz+l=0$, then the equation of the tangent plane at the point (x_1, y_1, z_1) can be derived by replacing x^2 by xx_1, y^2 by yy_1, etc., $2xy$ by (xy_1+x_1y), etc., $2x$ by $(x+x_1)$, etc.

tangent surface of a space curve. The envelope of the family of *osculating planes* of the space curve; the totality of points on lines tangent to the space curve. See DEVELOPABLE—developable surface, and OSCULATING—osculating plane of a space curve at a point.

TAN-GEN'TIAL, *adj.* **tangential acceleration.** See ACCELERATION.

TAR'IFF, *n.* Duties, considered collectively. Sometimes used in the same sense as *duty*.

TAUBER. Tauberian theorem. A theorem which establishes some type of limit for a

specified class of functions, one of the assumptions being that the limit can be obtained by some stronger limit process. *E.g.*, this includes any theorem which establishes a sufficient condition for convergence of a series which is known to be summable by some (regular) method of summation. *Tauber's theorem* of this type states that if $f(x)=\sum_0^\infty a_n x^n$, where $\lim_{n\to\infty} na_n = 0$ and $f(x)\to S$ as $x\to 1$ (with $x<1$), then $\sum_0^\infty a_n$ converges and has the sum S. See ABEL—Abel's method of summation.

TAX, *n.* A charge levied for the support of the government.

direct tax. Tax levied upon the person who actually pays it, such as tax levied on real estate, personal property, etc.

indirect tax. A tax ultimately paid by a person other than the one upon whom it is levied, like taxes on industry which are paid by the consumer in the form of increased prices.

poll tax. Tax levied on individuals, usually on voters only.

TAYLOR. Taylor's formula. The formula in TAYLOR'S THEOREM.

Taylor's series. See below, Taylor's theorem.

Taylor's theorem. A theorem which defines a polynomial whose graph runs very close to that of a given function throughout a certain interval, and a **remainder** which supplies a numerical limit to the difference between the ordinates of the two curves; the approximate representation of a given function on a certain interval (in the neighborhood of a certain point) by a polynomial. *Tech.* For a function of one variable, say $f(x)$, *Taylor's theorem* can be written as

$$f(x)=f(a)+f'(a)(x-a)+f''(a)(x-a)^2/2! \\ +f'''(a)(x-a)^3/3!+\cdots \\ +f^{[n-1]}(a)(x-a)^{n-1}/(n-1)!+R_n,$$

where R_n is the **remainder,** or the remainder after n terms. The remainder has been put in several different forms, the usefulness of the particular form depending upon the particular type of function being expanded. Three of these forms in common use are: (1) **Lagrange's form,**

$$\frac{h^n}{n!}f^{[n]}(a+\theta h);$$

(2) **Cauchy's form,**

$$\frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^{[n]}(a+\theta h);$$

(3) **Schloemilch's form,**

$$\frac{h^n}{p(n-1)!}(1-\theta)^{n-p}f^{[n]}(a+\theta h).$$

In all of these, θ is some number between 0 and 1, and $h=x-a$. When $p=1$, or $p=n$, Schloemilch's form becomes Cauchy's form and Lagrange's form, respectively. If n be allowed to increase without limit in the polynomial obtained by *Taylor's theorem*, the result is called a **Taylor's series.** The sum of such a series represents the expanded function if, and only if, the limit of R_n as n becomes infinite is zero. If $a=0$ in a *Taylor's series* in one variable, the series is called a **Maclaurin's series.** The binomial expansion of $(x+a)^n$ is a Maclaurin's series, R_{n+1} being zero when n is an integer. In fact, if any function can be expressed as a power series, such as $c_0+c_1(x-a)+c_2(x-a)^2+\cdots+c_n(x-a)^n+\cdots$, that series is a *Taylor's series.* Obviously a function cannot be expanded in a *Taylor's series* which represents the function in the above sense unless it possesses derivatives of all orders on the interval under consideration. **For a function of two variables,** say $f(x, y)$, *Taylor's theorem* states that

$$f(x,y)=f(a,b) \\ +\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]f(a,b)+\cdots \\ +\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^{n-1}\frac{f(a,b)}{(n-1)!}+R_n,$$

where $f(a, b)$ following the brackets means that the partials within the brackets are to operate upon $f(x, y)$ at the point (a, b), and the brackets indicate that the expansion of the quantity within is to be a binomial expansion except that

$$\left(\frac{\partial}{\partial x}\right)^h\left(\frac{\partial}{\partial y}\right)^k$$

is to be replaced by

$$\frac{\partial^{h+k}}{\partial x^h \partial y^k},\quad\text{and}\quad\left(\frac{\partial}{\partial x}\right)^0\quad\text{and}\quad\left(\frac{\partial}{\partial y}\right)^0$$

are to be taken as unity; $|R_n|$ is equal to or less than the numerically greatest value of all the nth partial derivatives, multiplied by $(|x-a|+|y-b|)^n$. Explicitly,

$$R_n=\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^n\frac{f(x_n,y_n)}{n!},$$

for certain x_n and y_n such that

$$x_n=a+\theta(x-a),\ y_n=b+\theta(y-b),$$

where $0<\theta<1$. If an unlimited (infinite) number of terms of the form used above is taken [it being necessary that all the partial derivatives of $f(x, y)$ exist] the result is a **Taylor's series in two variables.** The series represents the function from which it was derived if, and only if, the remainder approaches zero as the number of terms becomes infinite. Similarly, *Taylor's theorem* and series can be extended to any number of variables. Taylor's theorem is also called TAYLOR'S FORMULA, and sometimes the EXTENDED or GENERALIZED mean value theorem, although the latter two are sometimes used for the second mean value theorem. See LAURENT—Laurent series.

TCHEBYCHEFF. Tchebycheff net of parametric curves on a surface. See PARAMETRIC—equidistant system of parametric curves on a surface.

Tchebycheff's differential equation. The differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0.$$

Tchebycheff's inequality. Let x be a random variable. Then for any non-negative real-valued function $f(x)$ and every $k>0$, the probability of $f(x)\geqq k$ is less than or equal to $E(f)/k$, where $E(f)$ is the *mean value* or *mathematical expectation* of $f(x)$. This general inequality may be specialized for particular cases, such as in the *Bienaymé-Tchebycheff inequality.*

Tchebycheff's polynomials. The polynomials defined by $T_0(x)=1$ and $T_n(x)=2^{1-n}\cos(n\arccos x)$ for $n\geqq 1$, or by

$$(1-t^2)/(1-2tx+t^2)=\sum_{n=0}^{\infty}T_n(x)(2t)^n.$$

For $n\geqq 2$, $T_{n+1}(x)-xT_n(x)+\frac{1}{4}T_{n-1}(x)=0$, while $T_2-xT_1+\frac{1}{4}T_0=-\frac{1}{4}$ and $T_1-xT_0=0$. $T_n(x)$ is a solution of Tchebycheff's differential equation. Also,

$$T_n(x)=2^{1-n}\frac{(x^2-1)^{1/2}}{1\cdot 3\cdots(2n-1)}\frac{d^n(x^2-1)^{n-1/2}}{dx^n}.$$

$T_n(x)$ is sometimes defined as 2^{n-1} times the above value. See JACOBI—Jacobi's polynomials.

Tchebycheff's theorem. (*Statistics.*) Let $x_1, x_2, x_3, \cdots$ be any random variables, and let m_n and σ_n be the mean and standard deviation of x_n. If $\sigma_n\to 0$ as $n\to\infty$, then the probability of $(|x_n-m_n|>\epsilon)$ tends to zero for any $\epsilon>0$, as $n\to\infty$. *E.g.*, if x_i is the mean of a random sample of i items from a population, the probability that the sample mean will diverge from the mean of the population by an amount in excess of ϵ tends to zero as the size of the sample increases. Also called the LAW OF LARGE NUMBERS.

TEM'PO-RAR'Y, *adj.* **temporary annuity.** See ANNUITY.

temporary life insurance. Same as TERM LIFE INSURANCE. See INSURANCE—life insurance.

TEN, *n.* **ten's place.** See PLACE—place value.

TEN'SION, *n.* Any force which tends to extend a body lengthwise, in contrast to a **compression** which is a force that tends to shorten or compress it. A weight hanging on a cord causes a *tension* in the cord, while a weight resting on a stool causes a *compression* in the legs of the stool.

modulus in tension. See HOOKE'S LAW and MODULUS—Young's modulus.

TEN'SOR, *adj., n.* An abstract object having a definitely specified system of components in every coordinate system under consideration and such that, under transformations of coordinates, the components of the object undergo a transformation of a certain nature. Explicitly, let

$$A^{pq\cdots t}_{jk\cdots m}$$

be one of a set of functions of the variables x^i $(i=1, 2, \cdots, n)$, where each index can take on the values $1, 2, \cdots n$ and the number of superscripts is r, the number of subscripts s. Then these n^{r+s} quantities are

the x-components of a tensor of order $r+s$, provided its components in any other system

$$x'^i\ (i=1, 2, \cdots, n)$$

are given by

$$A'^{pq\cdots t}_{jk\cdots m}=A^{ab\cdots d}_{ef\cdots h}\frac{\partial x'^p}{\partial x^a}\cdots\frac{\partial x'^t}{\partial x^d}\frac{\partial x^e}{\partial x'^j}\cdots\frac{\partial x^h}{\partial x'^m}$$

where the summation convention is to be applied to the indices $a, b, \ldots, d$ and $e, f, \cdots, h$. (See SUMMATION—summation convention.) Such a tensor is said to be *contravariant of order r* and *covariant of order s*. The superscripts are called **contravariant indices**, the subscripts **covariant indices**. See *contravariant tensor* and *covariant tensor* (below) for examples of tensors. When it is desired to distinguish between an abstract object of the above type whose domain of definition is a single point (in each coordinate system) and one whose domain of definition is a region, it is customary to call the former a **tensor** and the latter a **tensor field**. The above is also called an **absolute tensor field** to distinguish it from a *relative tensor field*. A **scalar field** or **invariant** is a tensor field which is contravariant and covariant of order zero (*i.e.*, it has only one component, and this has the same value in all coordinate systems). See below, relative tensor field of weight w.

addition and subtraction of tensors. The sum of two tensors $A^{i_1\cdots i_p}_{j_1\cdots j_q}$ and $B^{i_1\cdots i_p}_{j_1\cdots j_q}$, which have the same number of contravariant indices and the same number of covariant indices, is the tensor

$$T^{i_1\cdots i_p}_{j_1\cdots j_q}=A^{i_1\cdots i_p}_{j_1\cdots j_q}+B^{i_1\cdots i_p}_{j_1\cdots j_q},$$

and their difference is the tensor

$$S^{i_1\cdots i_p}_{j_1\cdots j_q}=A^{i_1\cdots i_p}_{j_1\cdots j_q}-B^{i_1\cdots i_p}_{j_1\cdots j_q}.$$

associated tensors. A tensor is said to be **associated with** the tensor $T^{i_1\cdots i_p}_{j_1\cdots j_q}$ if it can be obtained from $T^{i_1\cdots i_p}_{j_1\cdots j_q}$ by raising or lowering any number of the indices by means of a series of *inner multiplications* of the form $g^{i\sigma}T^{i_1\cdots i_p}_{j_1\cdots\sigma\cdots j_q}$ or $g_{j\sigma}T^{i_1\cdots\sigma\cdots i_p}_{j_1\cdots j_q}$, where g_{ij} is the *fundamental metric tensor* and g^{ij} is $1/g$ times the cofactor of g_{ji} in the determinant g which has g_{ij} in the ith row and jth column.

components of the stress tensor. See COMPONENT.

composition (or inner multiplication) of tensors. See INNER—inner product of tensors.

contraction of a tensor. See CONTRACTION.

contravariant tensor. A tensor which has only contravariant indices. If there are r indices, it is said to be a **contravariant tensor of order r.** If the variables are x^1, x^2, x^3, the differentials dx^1, dx^2, and dx^3 are the components of a contravariant tensor of order one (*i.e.*, a *contravariant vector*), since

$$dx'^i=\frac{\partial x'^i}{\partial x^j}dx^j$$

$$=\frac{\partial x'^i}{\partial x^1}dx^1+\frac{\partial x'^i}{\partial x^2}dx^2+\frac{\partial x'^i}{\partial x^3}dx^3,$$

for $i=1, 2$, or 3.

covariant tensor. A tensor which has only covariant indices. If there are s indices, it is said to be a **covariant tensor of order s.** The *gradient* of a function is a covariant tensor of order one (*i.e.*, a *covariant vector*). If the function is

$$f(x^1, x^2, x^3),$$

the components of the tensor are

$$\frac{\partial f}{\partial x^i}\quad(i=1, 2, 3),$$

and we have

$$\frac{\partial f}{\partial x'^i}=\frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial x'^i}$$

$$=\frac{\partial f}{\partial x^1}\frac{\partial x^1}{\partial x'^i}+\frac{\partial f}{\partial x^2}\frac{\partial x^2}{\partial x'^i}+\frac{\partial f}{\partial x^3}\frac{\partial x^3}{\partial x'^i}.$$

covariant and contravariant derivatives of a tensor. See COVARIANT and CONTRAVARIANT.

divergence of a tensor. See DIVERGENCE.

Einstein tensor. See RICCI—Ricci tensor.

fundamental metric tensor. See RIEMANN—Riemannian space.

mixed tensor. A tensor which has both contravariant and covariant indices.

multiple-point tensor field. A generalized tensor field whose components depend on the coordinates of two or more points. *E.g.*, the distance (say in the Euclidean plane) between two variable points in the plane is a two-point scalar field.

numerical tensor. A tensor which has the same components in all coordinate systems. The *Kronecker delta* δ^i_j and the

generalized Kronecker delta are numerical tensors.

product of tensors. The product of two tensors $A^{a_1\cdots a_n}_{i_1\cdots i_m}$ and $B^{b_1\cdots b_p}_{j_1\cdots j_q}$ is the tensor $C^{a_1\cdots a_n b_1\cdots b_p}_{i_1\cdots i_m j_1\cdots j_q} = A^{a_1\cdots a_n}_{i_1\cdots i_m} B^{b_1\cdots b_p}_{j_1\cdots j_q}$. This is also called the **outer product.** See INNER—inner product of tensors.

relative tensor field of weight w. Its definition differs from that of a tensor field by the presence of the *Jacobian* $\left|\frac{\partial x^i}{\partial \bar{x}^j}\right|$ to the wth power as a factor in the right-hand side of the transformation law of the components of the tensor field. A relative tensor field of weight 1 is called a **tensor density.** The *epsilon symbol* $\epsilon^{i_1 i_2 \cdots i_n}$ is a tensor density. The components of a *scalar field of weight one* (a *scalar density*) are related by

$$\bar{s}(\bar{x}^1, \cdots, \bar{x}^n) = \left|\frac{\partial x^i}{\partial \bar{x}^j}\right| s(x^1, \cdots, x^n).$$

If t_{ij} are the components of a covariant tensor field and if $t = |t_{ij}|$ is the n-rowed determinant with element t_{ij} in the ith row and jth column, then $\sqrt{t}$ is a *scalar density.*

Ricci tensor. See RICCI.

Riemann-Christoffel curvature tensor. See RIEMANN.

skew-symmetric tensor. When the interchange of two contravariant (or covariant) indices changes only the sign of each component, the tensor is said to be *skew-symmetric with respect to these indices.* A tensor is **skew-symmetric** if it is skew-symmetric with respect to every two contravariant and every two covariant indices.

strain tensor. See STRAIN.

symmetric tensor. When the relative position of two or more contravariant (or covariant) indices in the components of a tensor is immaterial, the tensor is said to be *symmetric with respect to these indices.* A tensor is **symmetric** if it is symmetric with respect to every two contravariant and every two covariant indices.

tensor analysis. The study of objects with components possessing characteristic laws of transformation under transformation of coordinates. The subject is intimately connected with the various Riemannian and non-Riemannian geometries, including the theory of surfaces in Euclidean and non-Euclidean spaces.

tensor density. See above, relative tensor field of weight w.

tensor field. See above, TENSOR.

TERM, *adj., n.* (1) A term of a **fraction** is the numerator or denominator of the fraction (see LOWEST—fraction in lowest terms). A term of a **proportion** is any one of the *extremes* or *means.* See PROPORTION. (2) A term of an **equation** or an **inequality** is the entire quantity on one (or the other) side of the sign of equality or inequality. *Member* is better usage. (3) For an expression which is written as the sum of several quantities, each of these quantities is called a term of the expression, *e.g.*, in $xy^2 + y \sin x - \frac{x+1}{y-1} - (x+y)$, the terms are xy^2, $y \sin x$, $-\frac{x+1}{y-1}$ and $-(x+y)$. In the **polynomial** $x^2 - 5x - 2$, the terms are x^2, $-5x$, and -2; in $x^2 + (x+2) - 5$, the terms are x^2, $(x+2)$, and -5. A **constant** (or **absolute**) term is a term which does not contain any of the variables. An **algebraic term** is a term containing only algebraic symbols and numbers; *e.g.*, $7x$, $x^2 + 3ay$, and $\sqrt{3x^2 + y}$. An algebraic term is **rational and integral** if the variable (or variables) do not appear under any radical sign, in any denominator, or with fractional or negative exponents. The terms $2xy$ and $2x^2y^2$ are rational and integral in x and y. A term which is not algebraic is said to be **transcendental.** Examples of transcendental terms are **trigonometric terms,** which contain only trigonometric functions and constants (sometimes a term is called trigonometric when it contains algebraic factors, also); **exponential terms,** which contain the variable only in exponents (sometimes terms are called exponential when they contain both exponential and algebraic factors); **logarithmic terms,** which contain the variable or variables affected by logarithms, such as $\log x$, $\log(x+1)$ (sometimes terms are called logarithmic when they contain algebraic factors also, as $x^2 \log x$). **Like** (or **similar**) terms are terms that contain the same unknowns, each unknown of the same kind being raised to the same power; $2x^2yz$ and $5x^2yz$ are *like terms.* (4) In *finance*, term means a period of time (see ANNUITY).

general term. A term containing parameters in such a way that the parameters can be given specific values so that the term itself reduces to any special term of some set under consideration. See BINOMIAL—binomial theorem. The general term in the general algebraic equation of the nth degree $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ would be written a_ix^{n-i} $(i = 0, \cdots, n)$.

term life insurance. See INSURANCE—life insurance.

TER'MI-NAL, *adj.* **terminal form of commutation columns.** See COMMUTATION—commutation tables.

terminal side of an angle. See ANGLE.

TER'MI-NAT'ING, *adj.* Coming to an end; limited; expressed in a finite number of figures or terms. *E.g.*, the decimal 3.147 is a **terminating decimal,** while the repeating decimal $7.414141\cdots$ is **nonterminating.** Infinite sequences and series are nonterminating.

terminating continued fraction. See FRACTION—continued fraction.

terminating plan of building and loan association. See BUILDING—building and loan association.

TER'NA-RY, *adj.* **ternary representation of numbers.** Writing numbers with the base 3 (see BASE—base of a system of numbers). *E.g.*, the number $38\frac{5}{27}$ in the decimal system would be 1102.012 when written with base 3.

TER-RES'TRI-AL, *adj.* **terrestrial triangle.** A spherical triangle on the earth's surface (considered as a sphere) having for its vertices the north pole and two points whose distance apart is being found.

TES'SER-AL, *adj.* **tesseral harmonic.** See HARMONIC—surface harmonic.

TET'RA-HE'DRAL, *adj.* **tetrahedral angle.** A polyhedral angle having four faces. See ANGLE—polyhedral angle.

tetrahedral surface. A surface admitting parametric representation of the form

$$x = A(u-a)^\alpha(v-a)^\beta,$$
$$y = B(u-b)^\alpha(v-b)^\beta,$$
$$z = C(u-c)^\alpha(v-c)^\beta,$$

where a, b, c, A, B, C, α, and β are constants.

TET'RA-HE'DRON, *n.* A four-faced polyhedron. *Syn.* Triangular pyramid. A **regular tetrahedron** is a tetrahedron having all of its faces equilateral triangles. See POLYHEDRON.

THE'O-REM, *n.* (1) A general conclusion proposed to be proved upon the basis of certain given hypotheses (assumptions). (2) A general conclusion which has been proved. See BINOMIAL—binomial theorem, FUNDAMENTAL—fundamental theorem of algebra, fundamental theorem of the integral calculus, IMPLICIT—implicit function theorem, MEAN—mean value theorems, etc.

THE'O-RY, *n.* The principles concerned with a certain concept, and the facts postulated and proved about it.

function theory. The theory of functions of a real variable, the theory of functions of a complex variable, etc.

group theory, or theory of groups. See GROUP.

linear theory. See ELASTICITY.

number theory, or theory of numbers (elementary). The study of integers and relations between them.

theory of equations. The study of methods of solving and the possibility of solving algebraic equations, and of the relation between roots and between roots and coefficients of equations.

THE'TA, *n.* The eighth letter $(\Theta, \theta, \vartheta)$ of the Greek alphabet.

theta functions. Let $q = e^{\pi i\tau}$, where τ is a constant complex number whose imaginary part is positive. The four theta functions (usually written without explicitly indicating the dependence on τ) are

$$\vartheta_1(z) = 2\sum_0^\infty (-1)^n q^{(n+1/2)^2} \sin(2n+1)z,$$

$$\vartheta_2(z) = 2\sum_0^\infty q^{(n+1/2)^2} \cos(2n+1)z,$$

$$\vartheta_3(z) = 1 + 2\sum_1^\infty q^{n^2} \cos 2nz,$$

$$\vartheta_4(z) = 1 + 2\sum_1^\infty (-1)^n q^{n^2} \cos 2nz.$$

Other notations are also used for these functions (*e.g.*, ϑ for ϑ_4 and $\vartheta_i(\pi z)$ for $\vartheta_i(z)$). It can be shown that $\vartheta_1(z) = -\vartheta_2(z+\frac{1}{2}\pi) = (-iq^{1/4}e^{iz})\vartheta_3(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = (-iq^{1/4}e^{iz})\vartheta_4(z+\frac{1}{2}\pi\tau)$. Each of the theta functions satisfies a relation analogous to

$$\vartheta_4(z+\pi) = \vartheta_4(z) = (-qe^{2iz})\vartheta_4(z+\pi\tau),$$

and are said to be *quasi doubly-periodic*. The theta functions are entire functions.

THOU'SAND, *n.* Ten hundred (1000).

THREE, *adj.*, *n.* **rule of three.** The rule that the product of the means of a proportion equals the product of the *extremes* (see PROPORTION). This rule enables one to find any one of the numbers of a proportion if the other three are given. If $3/x = 2/5$, then $2x = 15$ and $x = 7\frac{1}{2}$.

three-circle theorem. See HADAMARD.

three-dimensional geometry. The study of figures in three (as well as two) dimensions. See GEOMETRY—solid geometry, and DIMENSION.

three-point form of the equation of a plane. See PLANE—equation of a plane.

three-point problem: Given three points, A, B, C, the distance AB and BC being known, and a fourth point S, with the angles ASB and BSC known, to find the distance SB. This is the problem of finding the distance from a ship S to a point B on the shore.

TIME, *adj.*, *n.* Continuous existence as indicated by some sequence of events, such as the hours indicated by a clock or the rotation of the earth about its axis; the experience of duration or succession. **Mean solar time** (or **astronomical time**) is the average time between successive passages of the sun over the meridian of a place, the time that would be shown by a sun dial if the sun were always on the celestial equator (in the plane of the earth's equator) and moving at a uniform rate. **Apparent solar time** is the time indicated by the sun dial, which divides each day into 24 hours; the hour-angle of the apparent or true sun (see HOUR) plus 12 hours. The hours are not exactly the same length, owing to the inclination of the earth's axis to the plane of the ecliptic (the plane of the earth's orbit) and to the eccentricity (elliptic shape) of the earth's orbit. Also see SIDEREAL—sidereal time. **Standard time** is a uniform system of measuring time, originated for railroad use in the United States and Canada and now in common use. The American continent is divided into four belts, each extending through approximately 15° of longitude, designated as *Eastern*, *Central*, *Mountain*, and *Pacific*. The time in each belt is the mean solar time of its central meridian. For instance, 7 a.m. *Central Time* is 8 a.m. *Eastern Time*, 6 a.m. *Mountain Time*, and 5 a.m. *Pacific Time*. *Tech.* Standard time is the mean solar time of a standard meridian, a meridian whose longitude differs by a certain multiple of 15° from the longitude at Greenwich, 15° being equivalent to one hour.

time discount. See DISCOUNT—time discount.

time rate. See SPEED and VELOCITY.

time series. Data taken at time intervals, such as the temperature or the rainfall taken at a certain time each day for a succession of days.

TON, *n.* See DENOMINATE NUMBERS in the appendix.

TON-TINE', *adj.* **tontine annuity.** An annuity purchased by a group with the *benefit of survivorship* (*i.e.*, the share of each member who dies is divided among the others, the last survivor getting the entire annuity during the balance of his life).

tontine fund. A fund accumulated by investments of withheld annuity payments.

tontine insurance. Insurance in which all benefits except those due to death, including such as dividends and cash surrender values, are allowed to accumulate until the end of a certain period and then are divided among those who have carried this insurance throughout the period.

TOP'O-LOG'I-CAL, *adj.* **linear topological space.** See VECTOR—vector space.

topological group. An abstract group which is also a topological space and in which the group operations are continuous. Continuity of the group operations means that, for any elements x and y, (1) if W is a neighborhood of xy, then there exist neighborhoods U and V of x and y such that uv belongs to W if u and v belong to U

and V, respectively; (2) if V is any neighborhood of x^{-1} (the *inverse* of x), then there exists a neighborhood U of x such that u^{-1} belong to V if u belongs to U. The set of all real numbers is a topological group, as is also the group of all nonsingular square matrices of a certain order with multiplication as the group operation and a neighborhood of a matrix A as the set of all matrices B such that the *norm* of $A-B$ is less than some fixed number ϵ.

topological manifold. See MANIFOLD.

topological property. Any property of a geometrical figure A that holds as well for every figure into which A may be transformed by a topological transformation. Examples are the properties of *connectedness* and *compactness*, of subsets being *open* or *closed*, and of points being *limit points*.

topological space. A set T such that each of the members (**elements** or **points**) has associated with it a system of subsets, called **neighborhoods,** such that: (1) a point belongs to each of its neighborhoods; (2) for any two distinct points x and y of T there exist neighborhoods U and V of x and y, respectively, such that no point of T belongs to both U and V; (3) if neighborhoods U and V both contain the point x, then there exists a neighborhood W of x such that all points of W are in both U and V. This is called a **Hausdorff** (or T_2) **topological space.** If in place of (2) it is only required that there exist a neighborhood U of x not containing y, the space is a **Fréchet** (or T_1) **topological space**; if it is required that there exist either a neighborhood of x not containing y or a neighborhood of y not containing x, the space is a T_0 **topological space.** The most general **topological space** is obtained by omitting (2), this definition then being equivalent to T having associated with it a system of subsets (of T), called *open sets,* having the properties: (1) the sum of any number of open sets is open; (2) the intersection of a finite number of open sets is an open set. The plane is a topological space if neighborhoods are taken as the interiors of circles in the plane, or as the interiors of squares. Other examples are *metric spaces* and *Hilbert space,* if neighborhoods are defined as the interiors of spheres (*i.e.*, as sets of all points at less than some fixed distance from some given point).

topological transformation. A one-to-one correspondence between the points of two geometric figures A and B which is continuous in both directions; a one-to-one correspondence between the points of A and B such that open sets in A correspond to open sets in B, and conversely (or analogously for closed sets). If one figure can be transformed into another by a topological transformation, the two figures are said to be **topologically equivalent.** Continuous deformations are examples of topological transformations (see DEFORMATION). Any two "knots" formed by looping and interlacing a piece of string and then joining the ends together are topologically equivalent, but cannot necessarily be continuously deformed into each other. *Syn.* Homeomorphism.

TO-POL'O-GY, *n.* That branch of geometry which deals with the *topological properties* of figures. **Combinatorial topology** is the branch of topology which is the study of geometric forms by decomposing them into the simplest geometric figures (simplexes) which adjoin each other in a regular fashion (see COMPLEX—simplicial complex, and SURFACE). **Algebraic topology** includes the fields of topology which use algebraic methods (especially group theory) to a large extent (see HOMOLOGY—homology group). **Point-set topology** is the study of sets as accumulations of points (as contrasted to combinatorial methods of representing an object as a union of simpler objects) and describing sets in terms of topological properties such as being open, closed, compact, normal, regular, connected, etc.

topology of a space. The set of all *open subsets* of the space (the space must be a *topological space*). A set can be assigned a topology by specifying a family of subsets with the properties that any union and any finite intersection of members of the family is a member of the family. A given set of objects together with a topology is a topological space (see TOPOLOGICAL—topological space). For a normed linear space, the topology defined by the norm is sometimes called the **strong topology** to distinguish it from the weak topology (see WEAK—weak topology).

uniform topology. See UNIFORM.

TORQUE, *n.* See MOMENT—moment of force.

TOR'SION, *adj.*, *n.* **geodesic torsion.** See GEODESIC.

torsion coefficients of a group. If G is a commutative group with a finite set of generators, then G is a *Cartesian product* of infinite cyclic groups $F_1, F_2, \cdots, F_m$ and cyclic groups $H_1, H_2, \cdots, H_n$ of finite order. The number m and the orders $r_1, r_2, \cdots, r_n$ of $H_1, H_2, \cdots, H_n$ form a complete system of invariants. The numbers $r_1, r_2, \cdots, r_n$ are called **torsion coefficients** of G (G is said to be **torsion free** if $n=0$).

torsion of a space curve at a point. If P is a fixed point, and P' a variable point, on a directed space curve C, Δs the length of arc on C from P to P', and $\Delta\psi$ the angle between the positive directions of the binormals of C at P and P', then the **torsion** $1/\tau$ of C at P is defined, to within sign, by $1/\tau = \lim_{\Delta s \to 0} \pm \frac{\Delta\psi}{\Delta s}$. The sign of $1/\tau$ is chosen so that we have $d\gamma/ds = \beta/\tau$. See FRENET-SERRET FORMULAS. The torsion may be taken as a measure of the rate at which C is turning out of its osculating plane relative to the arc length s. We have

$$1/\tau = -\rho^2 \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix},$$

the primes denoting differentiation with respect to the arc length. The reciprocal of the torsion is called the **radius of torsion.** Some authors use the symbol τ rather than $1/\tau$ to denote the torsion.

TO'RUS, *n.* Same as ANCHOR RING.

TO'TAL, *adj.* **total curvature.** See CURVATURE—total curvature.

total differential. See DIFFERENTIAL.

TO'TAL-LY, *adj.* **totally bounded** and **totally disconnected.** See BOUNDED—bounded set of points, and DISCONNECTED—disconnected set.

TO'TIENT, *n.* **totient of an integer.** Same as *Euler's ϕ-function* of the integer (see EULER—Euler's ϕ-function); the number of *totitives* of the integer.

TOT'I-TIVE, *n.* **totitive of an integer.** An integer not greater than the given integer and relatively prime to it (having no factor in common with it except unity). Each integer less than a given prime is a totitive of the prime; 1, 3, 5, and 7 are the totitives of 8.

TRACE, *n.* **trace of a line in space.** (1) A point at which it pierces a coordinate plane. (2) Its projection in a coordinate plane; the intersection of a *projecting plane* of the line with the corresponding coordinate plane. When trace is used in the latter sense, the point of definition (1) is called the *piercing point.*

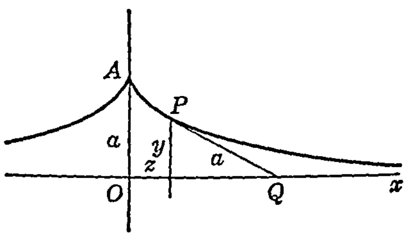
trace of a matrix. The sum of the elements of the principal diagonal (German "Spur").

traces of a surface. The curves in which it cuts the coordinate planes.

TRAC'ING, *n.* **curve tracing.** See CURVE—curve tracing.

TRAC'TRIX, *n.* The involute of a catenary; a curve the lengths of whose *tangents* are all equal; the path of one end (P in the figure) of a rod PQ of fixed length a attached to a point Q which moves along the x-axis from 0 to $\pm\infty$, the initial position of the rod being OA and the rod moving in such a way as to always be tangent to the path described by P. Its equation is

$$x = a \log \frac{(a \pm \sqrt{a^2 - y^2})}{y} \mp \sqrt{a^2 - y^2}.$$



TRA-JEC'TO-RY, *n.* (1) The path of a moving particle or a celestial body. (2) A curve which cuts all curves (or surfaces) of a given family at the same angle. An **orthogonal trajectory** is a curve which cuts all the members of a given family of curves (or surfaces) at right angles. (3) A curve or surface which fits some given law

such as passing through a given set of points.

TRAN'SCEN-DEN'TAL, *adj.* **transcendental curves.** Graphs of *transcendental functions.*

transcendental functions. Functions which cannot be expressed algebraically in terms of the variable (or variables) and constants. Contains terms involving trigonometric functions, logarithms, exponentials, etc. *Tech.* A transcendental function is any function which is not an *algebraic function.* An entire function is transcendental if it is not a polynomial. See FUNCTION—algebraic function.

transcendental number. See IRRATIONAL—irrational number.

TRANS-FI'NITE, *adj.* **transfinite induction.** The process of reasoning that, if some theorem is true for the first element of a well-ordered set S, and the theorem is true for an element a of S if it is true for each element preceding a, then the theorem is true for every element of S. This principle follows from the property of well-ordered sets that each nonnull subset has a first element. Thus the set of all elements for which the theorem is not true has a first element if it has any. See ORDERED—well-ordered set, and ZORN—Zorn's lemma.

transfinite number. A cardinal or ordinal number which is not an integer. See CARDINAL.

TRANS-FORM', *n.* **transform of an element of a group.** The transform of an element A by an element X is the element $B=X^{-1}AX$. The set of all transforms of A by elements of the group is the set of **conjugates** of A and is a **conjugate set** (or **class**) of elements of the group. The set of different subgroups obtained by transforming a given subgroup by all the elements of the group is a **conjugate set of subgroups**; any two of these subgroups are **conjugate** to each other. See GROUP, and INVARIANT—invariant subgroup.

transform of a matrix. A matrix B related to a given matrix A by $B=P^{-1}AP$, where P is a nonsingular matrix.

TRANS-FOR-MA'TION, *adj., n.* A passage from one figure or expression to another, as: (1) the changing of one algebraic expression to another one having different form (*e.g.*, see below, congruent transformation); (2) the changing of an equation or algebraic expression by substituting for the variables their values in terms of another set of variables; (3) a *correspondence* or *mapping* of one space on another or on the same space [*e.g.*, see LINEAR—linear transformation (2)].

affine transformation. See AFFINE.

adjoint transformation. See ADJOINT.

collineatory transformation. (1) A nonsingular linear transformation of $(n-1)$-dimensional Euclidean space of the form $y_i=\sum_{j=1}^{n} a_{ij}x_j$ $(i=1, 2, \cdots, n)$ in homogeneous coordinates; a transformation which takes collinear points into collinear points. (2) A transformation of the form $P^{-1}AP$ of a matrix A by a nonsingular matrix P; A and B are then said to be **similar** and are **transforms** of each other. These two concepts are closely related. For let the coordinates of two points x and y be related by $y_i=\sum_{j=1}^{n} a_{ij}x_j$ $(i=1, 2, \cdots, n)$ or symbolically by $y=Ax$, where x is thought of as a one-column matrix (vector) with elements (components) $x_1, x_2, \cdots, x_n$ (and likewise for y). If P is the matrix of a nonsingular linear transformation, with $y=Py'$ and $x=Px'$, then $Py'=APx'$, or $y'=P^{-1}APx'=Bx'$, in the new frame of reference introduced by the linear transformation defined by P. See EQUIVALENT—equivalent matrices. *Syn.* Collineation.

congruent transformation. A transformation of the form $B=P^TAP$ of a matrix A by a nonsingular matrix P, where P^T is the transpose of P. B is said to be **congruent** to A. Let the quadratic form $Q=\sum_{i,j=1}^{n} a_{ij}x_ix_j$ be written symbolically as $Q=(x)A\{x\}$, where (x) is the one-row matrix $(x_1, \cdots, x_n)$, $\{x\}$ is the similar one-column matrix, and multiplication is matrix multiplication. If

$$x_i=\sum_{j=1}^{n} p_{ij}y_j \quad (i=1, 2, \cdots, n),$$

or symbolically $\{x\}=P\{y\}$, then

$$(x)=(y)P^T$$

and $Q=(y)\cdot P^TAP\cdot\{y\}$. Thus the matrix A is transformed into a congruent matrix under a linear transformation of the variables. Every symmetric matrix is congruent to a diagonal matrix, and hence every quadratic form can be changed to a form of type $\Sigma k_ix_i^2$ by a linear transformation. See DISCRIMINANT—discriminant of a quadratic form, and ORTHOGONAL—orthogonal transformation.

conjunctive transformation. See EQUIVALENT—equivalent matrices. A conjunctive transformation is related to *Hermitian forms* in the same way that a *congruent transformation* is related to *quadratic forms* except that P^T is replaced by the Hermitian conjugate of P. See above, congruent transformation. Every Hermitian matrix can be made diagonal by a conjunctive transformation, and hence every Hermitian form can be reduced to the form $\sum_{i=1}^{n} a_iz_i\bar{z}_i$ by a linear transformation, where a_i is real for each i.

division transformation. Same as LONG DIVISION, but rarely used.

equiangular transformation. Same as ISOGONAL TRANSFORMATION.

Euler's transformation. See EULER—Euler's transformation.

factoring of a transformation. See FACTORIZATION—factorization of a transformation.

Hermitian transformation. See HERMITIAN—Hermitian transformation.

homogeneous transformation. A transformation whose equations are algebraic and whose terms are all of the same degree. Rotation of axes, reflection in the axes, stretching, and shrinking are homogeneous transformations.

homothetic transformation. See SIMILITUDE—transformation of similitude.

identical transformation. A transformation which makes no change in a configuration (or function); a transformation such as $x'=x$, $y'=y$. It is trivial in itself but is important as the product of a transformation and its inverse. *Syn.* Identity transformation. See GROUP.

inverse transformation. The transformation which exactly undoes the effect of a given transformation. The transformation $x'=1/x$ is its own inverse, because two reciprocations of a quantity return it to its original value. The transformations $x'=\sin x$ and $x=\arcsin x'$ are inverses of each other if the closed intervals $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ and $[-1, +1]$ are the *domain* and *range* of $\sin x$, and are the *range* and *domain* of arc $\sin x$; then $\sin(\arcsin x')\equiv x'$ and $\arcsin(\sin x)\equiv x$. *Tech.* If T is a one-to-one transformation of a set X onto a set Y, then the **inverse** of T is the transformation T^{-1} which maps a point y of Y onto a point x of X if T maps x onto y. A transformation has an inverse if and only if it is one-to-one. However, one speaks of the **inverse image** of a set S in the range of a transformation T as the set of all points which map into points of S; this inverse image is denoted by $T^{-1}(S)$. If T is single valued, then T^{-1} is a one-to-one mapping of the subsets of the range of T onto the class of inverse images.

isogonal transformation. See ISOGONAL—isogonal transformation.

linear transformation. See LINEAR—linear transformation.

matrix of a linear transformation. See MATRIX—matrix of a linear transformation.

normal transformation. See NORMAL.

orthogonal transformation. See ORTHOGONAL.

product of two transformations. The transformation resulting from the successive application of the two given transformations. Such a product may not be commutative, *i.e.*, the product may depend upon the order in which the transformations are applied. *E.g.*, the transformation $x=x'+a$ and $x=(x')^2$ are not commutative, since replacing x by $x'+a$ and the new x (x' with the prime dropped) by $(x')^2$ gives $x=(x')^2+a$ as the product transformation, while reversing the order gives $x=(x'+a)^2$.

rational transformation. The replacement of the variables of an equation, or function, by other variables which are each rational functions of the first. The transformations $x'=x+2$, $y'=y+3$ and $x'=x^2$, $y'=y^2$ are rational transformations.

reducible transformation. See REDUCIBLE.

simple shear transformation. A transformation which represents a shearing motion for which a coordinate axis in the plane, or a coordinate plane in space, does not move; in a plane, it is a transformation

of the form $x'=x$, $y'=lx+y$, or $x'=ly+x$, $y'=y$.

symmetric transformation. See SYMMETRIC.

topological transformation. See TOPOLOGICAL.

transformation of coordinates. Changing the coordinates of a point to another set which refer to a new system of coordinates, either of the same type or of another type. Examples are affine transformations, linear transformations, translation of axes, rotation of axes, and transformations between Cartesian and polar or spherical coordinates.

transformation group. A set of transformations which form a group. *Syn.* Group of transformations. See GROUP, and above, product of two transformations, and inverse transformation.

transformation of similitude. See SIMILITUDE.

unitary transformation. See UNITARY—unitary transformation.

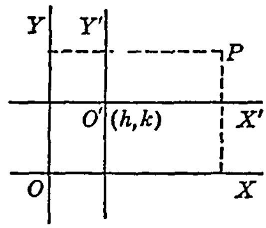
TRANS'IT, *n.* (*Surveying.*) An instrument for measuring angles. Consists essentially of a small telescope which rotates horizontally and vertically, the angles through which it rotates being indicated on graduated scales.

TRAN'SI-TIVE, *adj.* **transitive relation.** A relation which has the property that if A bears the relation to B and B bears the same relation to C, then A bears the relation to C. Equality in arithmetic is transitive, since if $A=B$ and $B=C$, then $A=C$. A relation which is not transitive is **intransitive** or **nontransitive** according as there does not, or does, exist a set of objects A, B, C such that A bears the relation to B, B bears the relation to C, and A bears the relation to C. The relation of *being the father of* is *intransitive*, since if A is the father of B, and B is the father of C, then A is not the father of C. The relation of *being a friend of* is *nontransitive*, since if A is a friend of B, and B is a friend of C, then A may or may not be a friend of C.

TRANS-LA'TION, *adj., n.* **translation of axes.** Changing the coordinates of points to coordinates referred to new axes parallel to the old. Used to change the form of equations so as to aid in the study of their loci; *e.g.*, one may desire to translate the axes so that the new origin is on the curve, which rids the equation of the constant term, or to translate the axes until they are coincident with the axes of symmetry when these are parallel to the axes, as in the case of conics, thus getting rid of the first degree terms. **Translation formulas** are formulas expressing a translation of axes analytically. In the *plane,* these formulas are

$$x=x'+h, \quad y=y'+k,$$

where h and k are the coordinates of the origin of the x', y' system with reference to the x, y system; *i.e.*, when $x'=y'=0$,



$x=h$ and $y=k$. In *space,* if the new origin has the coordinates (h, k, l) with respect to the old axes, and a point has coordinates (x', y', z') with respect to the new axes, and coordinates (x, y, z) with respect to the old axes, then $x=x'+h$, $y=y'+k$, $z=z'+l$.

translation and rotation. A transformation which both translates and rotates the axes. Used, for instance, in the study of the general quadratic in x and y to remove the xy and x and y terms. The transformation formulas are

$$x=x'\cos\theta-y'\sin\theta+h,$$
$$y=x'\sin\theta+y'\cos\theta+k,$$

where h and k are the coordinates of the new origin relative to the old and θ is the angle through which the positive x-axis is rotated to be parallel to the positive x'-axis.

translation surface. See SURFACE—surface of translation.

TRANS'POR-TA'TION, *adj., n.* **Hitchcock transportation problem.** The linear programming problem of minimizing the total cost of moving ships between ports.

Thus if there are a_i ships in port A_i ($i=1, 2, \cdots, n$) and it is desired to deliver a total of b_j of the ships to port B_j ($j=1, 2, \cdots, m$) with $\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j$, and the cost of moving one ship from A_i to B_j is c_{ij}, then we want to choose nonnegative integers x_{ij} that will minimize $\sum_{i,j=1}^{n,m} c_{ij}x_{ij}$, subject to the constraints $\sum_{j=1}^{m} x_{ij}=a_i$, $\sum_{i=1}^{n} x_{ij}=b_j$. See PROGRAMMING—linear programming.

TRANS-POSE′, *n.*, *v.* (*Algebra.*) To move a term from one side of an equation to the other and change its sign. This is equivalent to subtracting the term from both sides. The equation $x+2=0$ becomes $x=-2$ after *transposing* the 2.

transpose of a matrix. The matrix resulting from interchanging the rows and columns in the given matrix.

TRANS′PO-SI′TION, *n.* (1) The act of transposing terms from one side of an equation to the other. See TRANSPOSE. (2) The interchange of two objects; a cyclic permutation of two objects. See PERMUTATION—cyclic permutation.

TRANS-VER′SAL, *adj.* (1) A line intersecting a system of lines. See ANGLE—angles made by a transversal. (2) See TRANSVERSALITY—transversality condition.

TRANS′VER-SAL′I-TY, *adj.* **transversality condition** (*Calculus of Variations*). A condition generalizing the fact that the shortest line segment joining a point (x_1, y_1) to a curve C must be orthogonal to C at the point (x_2, y_2) where the segment meets C. For a curve C with parametric equations $x=X(t)$, $y=Y(t)$, the transversality condition is

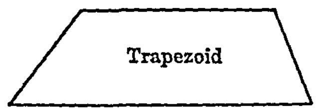
$$(f-y'f_{y'})X_t+f_{y'}Y_t=0,$$

which must be satisfied at the point (x_2, y_2) if the function $y=y(x)$ minimizes the integral $I=\int_{x_1}^{x_2} f(x, y, y')\,dx$, where (x_1, y_1) is fixed and (x_2, y_2) is constrained only to lie on C. A curve which satisfies the transversality condition relative to a curve C and an integral $I=\int_{x_1}^{x_2} f(x, y, y')\,dx$ and which minimizes this integral for (x_2, y_2) on C is called a **transversal.** See FOCAL—focal point.

TRANS-VERSE′, *adj.* **transverse axis of a hyperbola.** See HYPERBOLA.

TRA-PE′ZI-UM, *n.* A quadrilateral, none of whose sides are parallel.

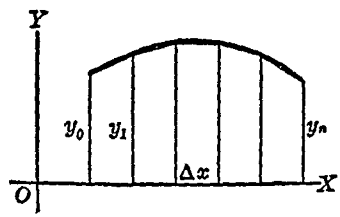
TRAP′E-ZOID, *adj.*, *n.* A quadrilateral which has two parallel sides. It is sometimes required that the other sides be nonparallel. The parallel sides are the bases



of the trapezoid and the **altitude** is the perpendicular distance between the bases. An **isosceles trapezoid** is a trapezoid in which the nonparallel sides are equal. The area of a trapezoid is the product of its altitude and one-half the sum of the bases, written

$$A=h\,\frac{(b_1+b_2)}{2}.$$

trapezoid rule. A rule for approximating the area between an arc of a curve, a straight-line segment, and perpendiculars from the extremities of the curve to the line segment. The rule is as follows: Divide



the line segment into equal segments and draw perpendiculars at each division point to intersect the curve, then connect all intersections of the curve with the perpendiculars (including the bounding perpendiculars), in successive pairs, by straight lines. The given area is then approximated

by the sum of the areas of the successive trapezoids. The total area of the trapezoids is one-half the sum of the bounding perpendiculars (first and last) plus the sum of all the intermediate perpendiculars, multiplied by the common width of the trapezoids. *Syn.* TRAPEZOID FORMULA.

$$\text{area} = \left[\tfrac{1}{2}(y_0 + y_n) + \sum_{i=1}^{n-1} y_i\right]\Delta x.$$

TRE'FOIL, *n.* See MULTIFOIL.

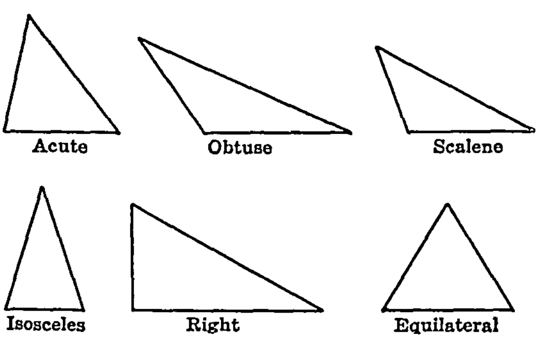
TREND, *adj., n.* (*Statistics.*) The general drift, tendency, or bent of a set of data; such, for instance, as the price of steel over a long period of time. Particular data will generally fluctuate from the *trend.* See RESIDUAL, and LINE—line of best fit.

trend line. See LINE.

secular trend. In a time series, the part of the variation which is the result of slowly changing, long lasting forces. Usually characterized by a monotonically increasing or decreasing function of time.

TRI'AL, *n.* **uniformity trial.** The *replication* of an experiment under the same controlled condition. Two or more random samples from the same population can be used for uniformity trials. Also called **dummy treatments,** since the controllable conditions of the trial are the same as in the other trials.

TRI'AN'GLE, *n.* (1) The figure formed by connecting three points (the **vertices**) not in a straight line by straight line segments.



(2) A part (region) of a plane bounded by straight-line segments joining three points. Six kinds of triangles are illustrated. As indicated in the above figures, an **acute triangle** is a triangle whose interior angles are all acute; an **obtuse triangle** is a triangle that contains an obtuse interior angle; a **scalene triangle** is a triangle with no two sides equal; an **isosceles triangle** is a triangle with two equal sides (the third side is called the **base** and the angle opposite it the **vertex**); a **right triangle** is a triangle one of whose angles is a right angle (the side opposite the right angle is called the **hypotenuse** and the other two sides the **legs** of the right triangle); an **equilateral triangle** is a triangle with all three sides equal (it must then also be equiangular, *i.e.,* have its three interior angles equal). An **oblique triangle** is a triangle which contains no right angles. The **altitude** of a triangle is the perpendicular distance from a vertex to the opposite side, which has been designated as the **base.** The **area** of a triangle is one-half the product of the base and the corresponding altitude. The area is equal to one-half the determinant whose first column consists of the abscissas of the vertices, the second of the ordinates (in the same order), and the third entirely of ones (this is positive if the points are taken around the triangle in counterclockwise order).

astronomical triangle. The spherical triangle on the celestial sphere which has for its vertices the nearer celestial pole, the zenith, and the celestial body under consideration. See HOUR—hour angle and hour circle.

excenter, incenter, and orthocenter of a triangle. See EXCENTER, INCENTER, and ORTHOCENTER.

Pascal's triangle. See PASCAL.

pedal triangle. See PEDAL.

polar triangle. See POLAR—polar triangle.

solution of a triangle. See SOLUTION.

spherical triangle. See SPHERICAL.

terrestrial triangle. See TERRESTRIAL.

triangle of plane sailing. The right spherical triangle (treated as a plane triangle) which has for legs the difference in latitude and the departure of two places, and for its hypotenuse the rhumb line between the two places.

TRI-AN'GU-LAR, *adj.* Like a triangle; having three sides.

triangular number. See NUMBER—triangular numbers.

triangular prism. A prism with triangular bases.

triangular pyramid. A pyramid whose base is a triangle. *Syn.* Tetrahedron.

TRI-AN′GU-LA′TION, *n.* A triangulation of a topological space T is a *homeomorphism* of T onto a polyhedron consisting of the points belonging to simplexes of a simplicial complex (see COMPLEX—simplicial complex). A space is said to be **triangulable** if it is homeomorphic to a simplicial complex. *E.g.*, the surface of an ordinary sphere is triangulable since it is homeomorphic to the surface of an inscribed regular tetrahedron, the homeomorphism consisting of projections of points of the sphere onto the tetrahedron along radii (and of points of the tetrahedron onto the sphere along radii). The surface of a regular tetrahedron is a simplicial complex whose simplexes are triangles. This mapping of the tetrahedron onto the sphere divides the sphere into four spherical triangles corresponding to the four faces of the tetrahedron.

TRI′DENT, *n.* **trident of Newton.** The cubic curve defined by the equation $xy=ax^3+bx^2+cx+d$ $(a\neq 0)$. It cuts the x-axis in 1 or 3 points and is asymptotic to the y-axis if $d\neq 0$. If $d=0$, the equation factors into $x=0$ (the y-axis) and $y=ax^2+bx+c$ (a parabola).

TRIG′O-NO-MET′RIC, *adj.* **inverse trigonometric function.** The multiple-valued function whose values for a given value of its argument are the numbers (or angles) whose trigonometric function is the given argument. The inverse sine of x is the function whose values for a given value of x are the numbers (or angles) whose sine is x. The inverse functions of an angle A are written either $\sin^{-1} A$, $\cos^{-1} A$, $\tan^{-1} A$, etc., or arc sin A, arc cos A, arc tan A, etc. See VALUE—principal value of an inverse trigonometric function, and ARCSINE, ARCCOSINE, etc. *Syn.* Antitrigonometric function. The **inverse trigonometric curves** are the graphs in rectangular coordinates of the inverse trigonometric functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc.; the graphs of the equations $y=\sin^{-1} x$, $y=\cos^{-1} x$, $y=\tan^{-1} x$, etc. Since these equations may be written $x=\sin y$, etc., their graphs are simply graphs of the trigonometric functions with the positive x-axis and y-axis interchanged, or, what is the same, the graphs of the trigonometric functions reflected in the line $y=x$.

trigonometric cofunctions. Trigonometric functions which are equal when the arguments are complementary. The *sine* and *cosine* are cofunctions, as are also the *tangent* and *cotangent*, and the *secant* and *cosecant*.

trigonometric curves. Graphs of the trigonometric functions in rectangular coordinates. See SINE—sine curve, COSINE—cosine curve, COTANGENT—cotangent curve, TANGENT—tangent curve, SECANT—secant curve, and COSECANT—cosecant curve. The term *trigonometric curve* is also applied to the graphs of any function involving only trigonometric functions, such as $\sin 2x+\sin x$ or $\sin x+\tan x$.

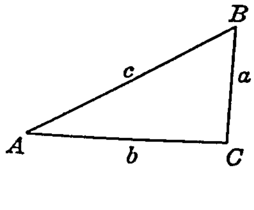
trigonometric equation. An equation that states a relation between trigonometric functions of the unknown angles (or numbers), such as $\cos\beta-\sin\beta=0$.

trigonometric form (representation) of a complex number. Same as POLAR FORM. See POLAR.

trigonometric functions. For **acute angles,** the trigonometric functions of the angles are certain ratios of the sides of a right triangle containing the angle. If A is an angle in a right triangle with hypotenuse denoted by c, side opposite A by a and side adjacent by b, the trigonometric functions of A are

$$\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{b}{a}, \frac{c}{b}, \text{ and } \frac{c}{a}.$$

They are named, respectively, sine A, cosine A, tangent A, cotangent A, secant A, and cosecant A, and written $\sin A$, $\cos A$, $\tan A$, ctn A (or $\cot A$), $\sec A$, and $\csc A$. Other trigonometric functions not so commonly used are versed sine of $A=1-\cos$

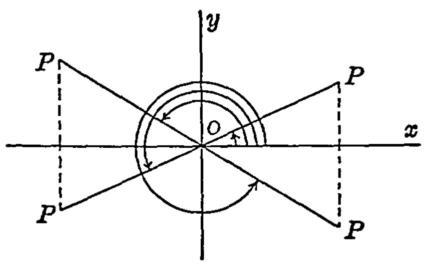


A=versine of A, written vers A; coversed sine of $A=1-\sin A$=versed cosine of A, written covers A; exsecant of $A=\sec A$ —1, written exsec A; haversine of $A=\frac{1}{2}$ vers A, written hav A. Now let A be **any positive or negative angle** (less than or greater than a right angle) described about the origin of a system of rectangular coordinates by a line OP with one end at the origin and the other, P, in its final position, having the coordinates (b, a), where b is negative if P is in the second or third quadrants and a is negative if P is in the third or fourth quadrants. If $OP=r$, then

$$\sin A=\frac{a}{r},\quad \cos A=\frac{b}{r},\quad \tan A=\frac{a}{b},$$
$$\operatorname{ctn} A=\frac{b}{a},\quad \sec A=\frac{r}{b},\quad \csc A=\frac{r}{a},$$

or

$$\begin{aligned}
&\sin A=(\text{ordinate})/r,\\
&\cos A=(\text{abscissa})/r,\\
&\tan A=(\text{ordinate})/(\text{abscissa}),\\
&\operatorname{ctn} A=(\text{abscissa})/(\text{ordinate}),\\
&\sec A=r/(\text{abscissa}),\\
&\csc A=r/(\text{ordinate}),
\end{aligned}$$



and the other functions are defined in terms of these as in the case of an acute angle. Although each of the above six functions has the same algebraic sign in two quadrants, if two functions which are not reciprocals are given, the quadrant in which the angle lies is uniquely determined; *e.g.*, if the sine of an angle is positive and the cosine negative, the angle lies in the second quadrant. The **variation** of the trigonometric functions as the angle varies from 0° to 360° is usually described by stating the values for the angles 0°, 90°, 180°, and 270°, since each function has its greatest or least values (or becomes infinite) at certain of these points. For the sine they are 0, 1, 0, -1; for the cosine 1, 0, -1, 0; for the tangent 0, ∞, 0, ∞; for the cotangent, $-\infty$, 0, $-\infty$, 0; for the secant, 1, ∞, -1, $-\infty$; for the cosecant, $-\infty$, 1, ∞, -1. The signs ∞ and $-\infty$ here mean that the function increases or decreases, respectively, without limit as the angle approaches the given angle in a counterclockwise direction. The opposite signs (on ∞) would result if the direction were clockwise. The following are **fundamental relations** between the trigonometric functions:

$$\sin x=\frac{1}{\csc x},\quad \cos x=\frac{1}{\sec x},$$
$$\tan x=\frac{1}{\operatorname{ctn} x},\quad \tan x=\frac{\sin x}{\cos x},$$
$$\sin^2 x+\cos^2 x=1,\quad \tan^2 x+1=\sec^2 x,$$
$$\operatorname{ctn}^2 x+1=\csc^2 x.$$

These can all be derived directly from the definitions of the functions and the *Pythagorean theorem*, and are called the fundamental identities, or relations, of trigonometry. The latter three are called Pythagorean identities. In most fields of mathematics, it is customary to speak of trigonometric functions of **numbers** rather than angles. A trigonometric function of a number x has a value equal to that of the given trigonometric function of an angle whose radian measure is equal to x. The sine and cosine of a number x can also be defined by means of the series:

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots,$$
$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots.$$

Then the other trigonometric functions can be defined by the above fundamental relations. For a **complex number** z, the sine and cosine of z may be defined in terms of the exponential function,

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i},\quad \cos z=\frac{e^{iz}+e^{-iz}}{2},$$

or by series,

$$\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots,$$
$$\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots.$$

The other trigonometric functions of z are defined in terms of these in the usual way.

trigonometric identities. See TRIGONOMETRY—identities of plane trigonometry.

trigonometric integral. An integral in which the integrand is a trigonometric function.

trigonometric series. A series of the form

$$a_0+(a_1\cos x+b_1\sin x)+(a_2\cos 2x+b_2\sin 2x)+\cdots=a_0+\Sigma(a_n\cos nx+b_n\sin nx),$$

where the a's and b's are constants. See FOURIER—Fourier series.

trigonometric substitutions. Substitutions used to rationalize quadratic surds of the forms

$$\sqrt{a^2-x^2},\quad \sqrt{x^2+a^2},\quad \sqrt{x^2-a^2}.$$

The substitutions $x=a\sin u$, $x=a\tan u$, and $x=a\sec u$ reduce these surds to $a\cos u$, $a\sec u$, $a\tan u$, respectively. The quadratic surd $\sqrt{x^2+px+q}$ can always be put into one of the above forms by completing the square. See INTEGRATION—change of variables in integration.

TRIG'O-NOM'E-TRY, *n.* The name "trigonometry" is derived from two Greek words, combining to mean measurement of triangles. While the solution of triangles forms an important part of modern trigonometry, it is by no means the only part or even the most important part. In the development of methods for the solution of triangles by computation, certain **trigonometric functions** occur (see TRIGONOMETRIC). The study of the properties of these functions and their applications to various mathematical problems, including the solution of triangles, constitute the subject matter of trigonometry. Trigonometry has applications in surveying, navigation, construction work and many branches of science. It is particularly essential for most branches of mathematics and physics. In **plane trigonometry,** the solution of plane triangles is considered; **spherical trigonometry** treats of the solution of spherical triangles. See various headings under TRIGONOMETRY, TRIGONOMETRIC, and SPHERICAL.

half-angle formulas of plane trigonometry. (1) Formulas for the solution of plane triangles which give relations between the sides and one of the angles of a triangle; used in place of the *cosine law* because they are better adapted to calculation by logarithms. The half-angle formulas are

$$\tan\tfrac{1}{2}A=r/(s-a),$$
$$\tan\tfrac{1}{2}B=r/(s-b),$$
$$\tan\tfrac{1}{2}C=r/(s-c),$$

where A, B, C are the angles of the triangle, a, b, c the sides opposite A, B, C, respectively, $s=\frac{1}{2}(a+b+c)$, and

$$r=\sqrt{(s-a)(s-b)(s-c)/s}.$$

(2) See below, identities of trigonometry.

half-angle and half-side formulas of spherical trigonometry. The **half-angle formulas** are formulas giving the tangents of half of an angle of a spherical triangle in terms of functions of the sides. If α, β, γ are the angles, a, b, c, respectively, the opposite sides, and $s=\frac{1}{2}(a+b+c)$, then

$$\tan\tfrac{1}{2}\alpha=\frac{r}{\sin(s-a)},$$

where

$$r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

Formulas for $\tan\frac{1}{2}\beta$ and $\tan\frac{1}{2}\gamma$ are obtained from this formula by a cyclic change between a, b, c. The **half-side formulas** give the tangents of one-half of each of the sides in terms of the angles. If α, β, γ are the angles, a, b, c, respectively, the sides opposite these angles, and $S=\frac{1}{2}(\alpha+\beta+\gamma)$, then

$$\tan\tfrac{1}{2}a=R\cos(S-\alpha),$$

where

$$R=\sqrt{\frac{-\cos S}{\cos(S-\alpha)\cos(S-\beta)\cos(S-\gamma)}}.$$

Formulas for $\tan\frac{1}{2}b$ and $\frac{1}{2}c$ are obtained by a cyclic change between α, β and γ.

identities of plane trigonometry. Equations which express relations among trigonometric functions that are valid for all values of the unknowns (variables) for which the functions involved are defined. The *fundamental relations* among the trigonometric functions are the simplest trigonometric identities (see TRIGONOMETRIC—trigonometric functions). The following are some more trigonometric identities: The **reduction formulas** are identities expressing the values of the trigonometric functions of

angles greater than 90° in terms of functions of angles less than 90°. The formulas for the sine, cosine, and tangent are:

$$\begin{aligned}\sin(90°\pm A)&=\cos A,\\ \sin(180°\pm A)&=\mp\sin A,\\ \sin(270°\pm A)&=-\cos A,\\ \cos(90°\pm A)&=\mp\sin A,\\ \cos(180°\pm A)&=-\cos A,\\ \cos(270°\pm A)&=\pm\sin A,\\ \tan(90°\pm A)&=\mp\cot A,\\ \tan(180°\pm A)&=\pm\tan A,\\ \tan(270°\pm A)&=\mp\cot A,\end{aligned}$$

where, in each formula, either the upper signs, or the lower, are to be used throughout. The **Pythagorean identities** are:

$$\sin^2 x+\cos^2 x=1,$$
$$\tan^2 x+1=\sec^2 x,$$
$$1+\cot^2 x=\csc^2 x.$$

The **addition and subtraction formulas (identities)** are formulas expressing the sine, cosine, tangent, etc., of the sum or difference of two angles in terms of functions of the angles. They answer the need created by the fact that functions of sums and differences are not distributive; *i.e.*, $\sin(x\pm y)\neq\sin x\pm\sin y$. The most important of these formulas are

$$\sin(x\pm y)=\sin x\cos y\pm\cos x\sin y,$$
$$\cos(x\pm y)=\cos x\cos y\mp\sin x\sin y,$$

and

$$\tan(x\pm y)=\frac{\tan x\pm\tan y}{1\mp\tan x\tan y}.$$

(The upper signs, and the lower signs, are to be taken together throughout.) The **double-angle formulas (identities)** are formulas expressing the sine, cosine, tangent, etc., of twice an angle in terms of functions of the angle. These are easily obtained by replacing y by x in the above *addition formulas*. The most important are

$$\sin 2x=2\sin x\cos x,$$
$$\cos 2x=\cos^2 x-\sin^2 x,$$

and

$$\tan 2x=2\tan x/(1-\tan^2 x).$$

The **half-angle formulas (identities)** are formulas expressing trigonometric functions of half an angle in terms of functions of the angle. They are easily derivable from the *double angle formulas* (with the double angle, $2x$, replaced by A and x replaced by $\frac{1}{2}A$). The most important half-angle formulas are

$$\sin\tfrac{1}{2}A=\sqrt{(1-\cos A)/2},$$
$$\cos\tfrac{1}{2}A=\sqrt{(1+\cos A)/2},$$

and

$$\tan\tfrac{1}{2}A=\sin A/(1+\cos A),$$
$$=(1-\cos A)/\sin A.$$

The **product formulas** (identities) are the formulas

$$\sin x\cos y=\tfrac{1}{2}[\sin(x+y)+\sin(x-y)],$$
$$\cos x\sin y=\tfrac{1}{2}[\sin(x+y)-\sin(x-y)],$$
$$\cos x\cos y=\tfrac{1}{2}[\cos(x+y)+\cos(x-y)],$$
$$\sin x\sin y=\tfrac{1}{2}[\cos(x-y)-\cos(x+y)].$$

These can be easily derived from the *addition* and *subtraction formulas*.

spherical trigonometry. See SPHERICAL.

TRI-HE'DRAL, *adj., n.* A figure formed by three noncoplanar lines which intersect in a point. A trihedral formed by three directed lines is a **directed trihedral.** The trihedral formed by the three axes of a system of Cartesian coordinates in space is a directed trihedral, called the **coordinate trihedral.** If the three directed lines of a directed trihedral are also ordered (*i.e.*, designated as the first, second, and third in some way), then the directed trihedral is **left-handed** if, when the thumb of the left hand extends along and in the positive direction of the first line, the fingers fold in the direction in which the second line could be rotated through an angle of 90° (or less than 180° if the axes are oblique) about the first line to coincide with the third; it is **right-handed** if, when the thumb of the right hand extends along and in the positive direction of the first line, the fingers fold in the direction in which the second line could be rotated through an angle of 90° (or less than 180° if the axes are oblique) about the first line to coincide with the third. A **trirectangular trihedral** is a trihedral formed by three mutually perpendicular lines. If the lines forming a directed trihedral are L_1, L_2, L_3, a necessary and sufficient condition that the trihedral be trirectangular is that the value of the determinant whose rows are l_1, m_1, n_1; l_2, m_2, n_2; and l_3, m_3, n_3 (in these orders) be equal to ± 1, where l_1, m_1, n_1; l_2, m_2, n_2; and l_3, m_3, n_3 are, respectively, the direction cosines of

the lines L_1, L_2, and L_3, the subscripts denoting the order in which the lines are taken. The positive sign occurs when the trihedral is **right handed,** the negative sign when it is **left handed,** if the coordinate axes form a right-handed system.

moving trihedral of space curves and surfaces. The *moving trihedral* (or *trihedral*) of a directed space curve C is the configuration consisting of the tangent, principal normal, and binormal of C at a variable point of C. The moving trihedral of a **surface** relative to a directed curve on the surface is defined as follows: For a point P of a directed curve C on a surface S, let α be the unit vector from P in the positive direction of the tangent to C at P, let γ be the unit vector from P in the positive direction of the normal to S at P, and let β be the unit vector from P in the tangent plane to S at P, and such that α, β, γ have the same mutual orientation as the x, y, z axes. Then axes directed along the vectors α, β, γ form the moving trihedral of S relative to C. The **rotations of the moving trihedral** of a surface are a certain set of six particular functions determining the orientation of the *moving trihedral* of the surface, but not its position in space.

trihedral angle. A polyhedral angle having three faces. Two trihedral angles are **symmetric** if they have their face angles equal in pairs but arranged in opposite order. Such trihedral angles are not superposable.

TRIL'LION, *n.* (1) In the U. S. and France, the number represented by one followed by 12 zeros. (2) In England, the number represented by one followed by 18 zeros.

TRI-NO'MI-AL, *n.* A polynomial of three terms, such as x^2-3x+2.

perfect trinomial square. See SQUARE—perfect trinomial square.

TRI'PLE, *adj., n.* Threefold; consisting of three.

number triple. Written (a, b, c). Same as the components of a space vector. When the vector is localized (has its initial point at the origin) the number triple denotes the rectangular coordinates of the extreme point of the vector. The number triple (a, b, c) can also be used to represent any object which can be determined in some specified way when three real numbers are given; *e.g.*, the point with spherical coordinates (a, b, c) or the circle with radius a and center (b, c).

triple integral. See INTEGRAL—iterated integral, multiple integral.

triple of conjugate harmonic functions. Three functions, $x(u, v)$, $y(u, v)$, $z(u, v)$, harmonic in a common domain D and there satisfying the relations $E=G$, $F=0$. Such functions give conformal maps of D on minimal surfaces.

triple root of an equation. A root which occurs three times. See MULTIPLE—multiple root.

triple (scalar) product of three vectors. The scalar $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$, written $(\mathbf{ABC})$ or $[\mathbf{ABC}]$. The dot indicates *scalar multiplication* and the cross *vector multiplication.* If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are written in the forms $\mathbf{A}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{B}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, $\mathbf{C}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$, the triple scalar product is the determinant of the coefficients of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. It is clear from this determinant that any cyclic change between the vectors in a triple product does not alter its value. The numerical value of the product is equal to the volume of the parallelepiped of which the vectors of the triple product are coterminal sides.

TRI'PLY, *adv.* Containing a property three times; repeating an operation three times.

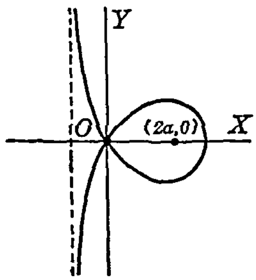
triply orthogonal system of surfaces. See ORTHOGONAL.

TRI'REC-TAN'GU-LAR, *adj.* **trirectangular spherical triangle.** A spherical triangle having three right angles.

TRI-SEC'TION, *n.* The process of dividing into three equal parts.

trisection of an angle. The problem of trisecting any angle with straightedge and compasses, alone. It was proved impossible by P. L. Wantzel in 1847. Any angle can be trisected, however, in several ways, for instance, by the use of a *protractor,* the *limaçon* of Pascal, the *conchoid* of Nicomedes, or the *trisectrix* of Maclaurin. See TRISECTRIX.

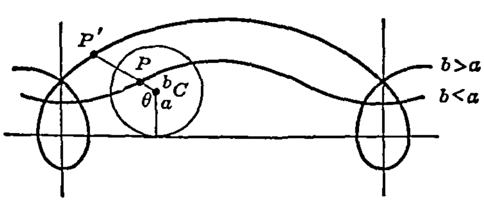
TRI-SEC′TRIX, *n.* The locus of the equation $x^3+xy^2+ay^2-3ax^2=0$. The curve is symmetric with respect to the x-axis, passes through the origin, and has the line $x=-a$ as an asymptote. It is of interest in connection with the problem of trisecting a given angle, for if a line having an angle of inclination $3A$ is drawn through the point $(2a, 0)$, then the line passing through the origin and the point of intersection of this line with the trisectrix has an angle of inclination A. Also called the trisectrix of Maclaurin.



TRIV′I-AL, *adj.* **trivial solutions of a set of homogeneous linear equations.** Zero values for all the variables, *trivial* because they are a solution of any system of homogeneous equations. Solutions in which at least one of the variables has a value different from zero are called **nontrivial solutions.** See CONSISTENCY—consistency of linear equations.

TRO′CHOID, *n.* The plane locus of a point on the radius of a circle, or on the radius produced, as the circle rolls (in a plane) on a fixed straight line. If a is the radius of the rolling circle, b the distance from the center of this circle to the point describing the curve, and θ the angle (in radians) subtended by the arc which has contacted the fixed line in getting to the point under consideration, then the parametric equations of the *trochoid* are

$$x=a\theta-b\sin\theta \quad \text{and} \quad y=a-b\cos\theta.$$



When b is greater than a, the curve has a loop between every two arches, nodes at $\theta=\theta_1+n\pi$, where $0<\theta_1<\pi$ and $a\theta_1-b\sin\theta_1=0$; it is then called a **prolate cycloid.** If b is less than a, the curve never touches the base line; it is then called a **curtate cycloid** (the names prolate and curtate are sometimes interchanged). As b approaches zero, the curve tends to smooth out nearer to the straight line described by the center of the circle. When $b=a$ the curve is a **cycloid.**

TROY, *adj.* **troy weight.** A system of weights having a pound of 12 ounces as its basic unit. It is used mostly for weighing fine metals. See DENOMINATE NUMBERS in the appendix.

TRUN′CAT-ED, *adj.* See CONE, DISTRIBUTION, PRISM, PYRAMID.

TSCHEBYSCHEFF. See TCHEBYCHEFF.

TURN′ING, *adj.* **turning point.** A point on a curve at which the ordinates of the curve cease increasing and begin decreasing, or vice versa; a maximum or minimum point.

TWIST′ED, *adj.* **twisted curve.** See CURVE.

TWO, *adj.* **two-dimensional geometry.** The study of figures in a plane. See GEOMETRY—plane geometry.

two-point form of the equation of a straight line. See LINE—equation of a straight line.

TYCHONOFF. Tychonoff space. See REGULAR—regular space.

Tychonoff theorem. See PRODUCT—Cartesian product.

TYDAC. An imaginary computing machine having representative characteristics of existing machines. TYDAC is an acronym for *Typical Digital Automatic Computer.*

TYPE, *n.* **problem of type.** The problem of determining the type of a given simply connected Riemann surface. See below, type of a Riemann surface.

type of a Riemann surface. Any simply connected Riemann surface can be mapped conformally on exactly one of the following: the interior of the unit circle; the finite plane (the punctured complex plane, excluding the point at infinity); the closed complex plane (including the point at infinity). In these three cases the surface is said to be respectively of **hyperbolic, parabolic,** or **elliptic type.**

U

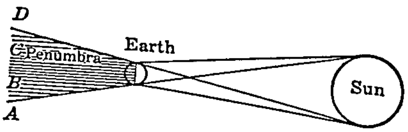
UL'TRA-FIL'TER, *n.* See FILTER.

UM-BIL'IC, *n.* Same as UMBILICAL POINT.

UM-BIL'I-CAL, *adj.* **umbilical geodesic of a quadric surface.** A geodesic lying on the surface S and passing through an umbilical point of S.

umbilical point on a surface. A point of a surface S which is either a *circular point* or a *planar point* of S. A point of S is an umbilical point of S if, and only if, its first and second fundamental quadratic forms are proportional. The normal curvature of S is the same in all directions on S at an umbilical point of S. All points on a sphere or plane are umbilical points. The points where an ellipsoid of revolution cuts the axis of revolution are umbilical points. *Syn.* Umbilic.

UM'BRA, *n.* [*pl.* **umbrae**]. The part of the shadow of an object from which all direct light is excluded. For the sun and the earth, the part of the shadow in the cone tangent to the earth and sun is in complete shadow, and is known as the **umbra,** while the outer region (the **penumbra**) shades from full illumination at A and D, through partial illumination at B and C, to complete shadow at the midpoint.



UN-BI'ASED, or **UN-BI'ASSED,** *adj.* **unbiased estimate.** See ESTIMATE and BIASED.

unbiased test of hypothesis. See HYPOTHESIS—test of hypothesis.

UN-BOUND'ED, *adj.* **unbounded function.** A function which is not bounded; a function such that for any number M there is a value of the function whose numerical value is larger than M. *Tech.*, f is unbounded on the set S if for any number M there is a point x_m of S such that $|f(x_m)| > M$. The function $1/x$ is unbounded on the interval $0 < x \leqq 1$; $\tan x$ is unbounded on the interval $0 \leqq x < \frac{1}{2}\pi$.

UN'CON-DI'TION-AL, *adj.* **unconditional inequality.** See INEQUALITY.

UN'DE-FINED', *adj.* **undefined term.** A term used without specific mathematical definition, whatever meaning it has being purely psychological; a term which satisfies certain axioms, but is not otherwise defined.

UN'DE-TER'MINED, *adj.* **undetermined coefficients.** Parameters (unknowns) inserted in terms (usually in algebraic polynomials) to be determined so as to make the terms take certain desired forms. *E.g.*, if it is desired to factor x^2-3x+2, the factors may be taken to be $x+a$ and $x+b$, where a and b are to be determined so as to make the product of these two factors equal to the original expression; *i.e.*,

$$x^2+(a+b)x+ab \equiv x^2-3x+2,$$

whence $a+b=-3$ and $ab=2$, from which $a=-1$, $b=-2$, or $a=-2$, $b=-1$. See PARTIAL—partial fractions.

U-NI-CUR'SAL, *adj.* **unicursal curve.** A curve whose equation, $f(x, y)=0$, and some parametric equations $x=\theta(t)$, $y=\phi(t)$ have the same loci, $\theta(t)$ and $\phi(t)$ being rational functions of t.

U'NI-FORM, *adj.* **uniform acceleration.** Acceleration in which there are equal changes in the velocity in equal intervals of time. *Syn.* Constant acceleration.

uniform circular motion. Motion around a circle with constant speed.

uniform continuity. For a function defined on an interval of the real numbers, uniform continuity is continuity such that it is possible to divide the interval into sub-

intervals such that the greatest numerical difference between the largest and the smallest values of the function (oscillation of the function) in every interval is as small as desired. The *sine curve* is uniformly continuous in any interval, while the *tangent curve* is not uniformly continuous in the interval $0 \leqq x < \frac{1}{2}\pi$, although it is continuous in this interval. *Tech.* A function f is uniformly continuous in a given interval if for any positive number ϵ there exists a number δ such that $|f(x_1)-f(x_2)| < \epsilon$ whenever $|x_1-x_2| < \delta$, where x_1 and x_2 are any two numbers in the interval. If the interval is closed (if the end points are included), then any function which is continuous in the interval is also *uniformly continuous.* Let F be a function defined on a set S, where S may be a set of real numbers, of points in the plane or space, or of points in a metric space; let the range of F be a set of real numbers, of points in the plane or space, or of points in a metric space. Then F is *uniformly continuous* on S if for any positive number ϵ there exists a number δ such that the numerical distance between $F(x_1)$ and $F(x_2)$ is less than ϵ whenever the numerical distance between x_1 and x_2 is less than δ. If S is compact, then F is uniformly continuous if it is continuous at each point of S.

uniform convergence of a series. See CONVERGENCE.

uniform convergence of a set of functions. Convergence such that the difference between each function and its limit can be made less than the same arbitrary positive number for a common interval of values of the argument. *Tech.* If the set of functions is such that each function f_i has a limit L_i, as x approaches x_0, then they *converge uniformly*, as x approaches x_0, if for any $\epsilon > 0$ there can be found a δ such that

$$|f_i(x) - L_i| < \epsilon$$

for all i, when $|x-x_0| < \delta$. See ASCOLI'S THEOREM.

uniform scale. See LOGARITHMIC—logarithmic coordinate paper.

uniform speed and velocity. See CONSTANT—constant speed and velocity.

uniform topology. The topology of a topological space T is said to be a **uniform topology** if there is a family F of subsets of the Cartesian product $T \times T$ such that a subset A of T is *open* if and only if, for any x of A, there is a member V of F for which the set of all y with (x, y) in V is a subset of A—in addition, the family F must have the properties: (i) each member of F contains each (x, x) for x in T; (ii) for each V of F, V^{-1} belongs to $F(V^{-1}$ is the set of all (x, y) with (y, x) in $V)$; (iii) for each V of F there is a V^* of F which has the property that V contains all (x, z) for which there is a y such that (x, y) and (y, z) belong to V^*; (iv) the intersection of two members of F is a member of F; (v) a subset of $T \times T$ is a member of F if it contains a member of F. A family of sets which satisfies conditions (i)–(v) is said to be a **uniformity**, or a **uniform structure** for T. A family F of subsets of $T \times T$ which satisfies (i), (ii), and (iii) is sometimes said to be a uniformity (it can be shown that the set of all finite intersections of members of such a family is a *base* for a uniformity which satisfies (i)–(v), a base B of a uniformity U being a subset of U which has the property that each member of U contains a member of B). A topological space with a uniform topology is *metrizable* if and only if it is a Hausdorff topological space and its uniformity has a countable base. If T is a metric space, then T has a uniformity which is the family of all subsets V of $T \times T$ for which there is a positive number ϵ such that V contains all (x, y) for which $d(x, y) < \epsilon$.

U'NI-FORM'I-TY, *adj.* **uniformity trial.** See TRIAL.

U'NI-LAT'ER-AL, *adj.* **unilateral surface.** Same as ONE-SIDED SURFACE. See SURFACE.

U'NI-MOD'U-LAR, *adj.* **unimodular matrix.** A square matrix whose determinant is equal to 1.

UN'ION, *n.* **union of sets.** Same as the sum of the sets. See SUM.

U-NIQUE', *adj.* Leading to one and only one result; consisting of one, and only one. The product of two integers is unique; the square root of an integer is not.

unique factorization theorem. See FACTORIZATION.

U-NIQUE'LY, *adv.* **uniquely defined.** A concept so defined that it is the only concept that fits that definition.

U'NIT, *n.* A standard of measurement such as an inch, a foot, a centimeter, a pound, or a dollar; a single one of a number, used as the basis of counting or calculating. See COMPLEX—complex unit, FORCE—unit of force, and MATRIX.

unit circle and **unit sphere.** A circle (or sphere) whose radius is one unit (one inch. one foot, etc., depending upon what system of measurement is being used). The circle (sphere) of unit radius having its center at the origin of coordinates is spoken of as **the unit circle (the unit sphere).**

unit fraction. See FRACTION.

unit square and **unit cube.** A square (cube) with its sides equal to unity (one inch, one foot, etc., depending upon what system of measurement is being used).

U'NI-TAR'Y, *adj.* Undivided; relating to a unit or units.

unitary analysis. See ANALYSIS.

unitary matrix. See MATRIX.

unitary space. Same as HERMITIAN VECTOR SPACE. See VECTOR—vector space, and HILBERT—Hilbert space.

unitary transformation. (1) A linear transformation whose adjoint is also its inverse. For **finite-dimensional spaces,** a linear transformation, which transforms $x=(x_1, x_2, \cdots, x_n)$ into $Tx=(y_1, y_2, \cdots, y_n)$ for which

$$y_i=\sum_{j=1}^{n} a_{ij}x_j \quad (i=1, 2, \cdots, n),$$

is unitary if and only if the matrix (a_{ij}) is unitary, or if and only if it leaves the Hermitian form

$$x_1\bar{x}_1+x_2\bar{x}_2+\cdots+x_n\bar{x}_n$$

invariant. If (x, y) denotes the inner product of elements of a **Hilbert space** H, then a transformation T of H into H is unitary if $(Tx, Ty)=(x, y)$ for each x and y, or if T is an *isometric mapping* of H into H $[(Tx, Tx)=(x, x)$ for each $x]$. A unitary transformation is also a normal transformation. (2) A unitary transformation of a **matrix** A is a transformation of the form $P^{-1}AP$, where P is a unitary matrix. The concepts of unitary transformation of a finite-dimensional space and of a matrix are related in the same way as for *orthogonal transformations* except that the transpose A^T is replaced by the Hermitian conjugate of A. A Hermitian matrix can be reduced to diagonal form by a unitary transformation; hence every Hermitian form can be reduced to the type $\sum_{i=1}^{\infty} p_i x_i \bar{x}_i$ by a unitary transformation such as described above. See ORTHOGONAL—orthogonal transformation, and SPECTRAL—spectral theorem.

UNITED STATES RULE. (*Mathematics of Finance.*) A rule for evaluating a debt upon which partial payments have been made. The rule is: Apply each payment first to the interest, any surplus being deducted from the principal. If a payment is less than all interest due, the balance of the interest must not be added to the principal, and cannot draw interest. This is the legal rule.

U'NI-TY, *n.* Same as ONE or 1.

root of unity. Any complex number z such that $z^n=1$ for some positive integer n (z is called an nth root of unity). The nth roots of unity are the numbers $\cos\left(\frac{k}{n}\cdot 360°\right) + i\sin\left(\frac{k}{n}\cdot 360°\right)$ for $k=0, 1, 2, \cdots, n-1$ (see DE MOIVRE—De Moivre's theorem). The set of all nth roots of unity is a *group* with multiplication as the group operation; they are n in number and equally spaced around the unit circle in the complex plane. A *primitive* nth root of unity is an nth root of unity which is not a root of unity of a lower order than n. Primitive roots are always imaginary except in the cases $n=1$ and $n=2$. The primitive square root of unity is -1; the primitive cube roots are

$$\frac{-1\pm\sqrt{3}i}{2};$$

the primitive fourth roots are $\pm i$.

UNIVAC. An automatic digital computing machine manufactured by the Sperry Rand Corporation. UNIVAC is an acronym for *Universal Automatic Computer.*

U'NI-VER'SAL, *adj.* **universal quantifier.** See QUANTIFIER.

UN-KNOWN′, *adj.*, *n.* **unknown quantity.** A letter or literal expression whose numerical value is implicit in certain given conditions by means of which this value is to be found. It is used chiefly in connection with equations. In the equation $x+2=4x+5$, x is the *unknown*. In word problems, the quantity to be found is called the *unknown*.

UP′PER, *adj.* **upper bound of a set of numbers.** See BOUND.

upper limit of integration. See INTEGRAL—definite integral.

URYSOHN. Urysohn's lemma. If P and Q are two nonintersecting closed sets (in a normal topological space T), there exists a real function $f(p)$ defined and continuous in T and such that $0 \leqq f(p) \leqq 1$ for all p, with $f(p)=0$ for p in P, and $f(p)=1$ for p in Q. See METRIC—metric space.

Urysohn's theorem. See METRIC—metric space.

V

VAL′U-A′TION, *n.* Act of finding or determining the value of.

valuation of bonds. See BOND.

VAL′UE, *n.* **absolute value.** See ABSOLUTE—absolute value of a real number, MODULUS—modulus of a complex number, VECTOR—absolute value of a vector.

accumulated value of an annuity. See ACCUMULATED.

assessed value. See ASSESSED.

book value. The book value of a **bond** is the purchase price minus the amount accumulated for amortization of the premium, or plus the amount of the accumulation of the discount, according as the bond has been bought at a premium or discount. When a bond is purchased at a premium, each dividend provides both for interest on the investor's principal and for a return of part of the premium, the *book value* at a given time being the purchase price minus the accumulation of the parts of the dividends used for amortization of the premium (the book value at maturity is equal to the face value). When the yield rate is greater than the dividend rate, the bond sells at a discount. The income is then greater than the dividend for any period, so the difference is added to the purchase price of the bond to bring it back to par value at maturity. These increasing values are the *book values* of the bond. The book value of a **debt** is equal to the difference between the face value of the debt and the sinking fund set up to pay the debt; if the debt is being amortized, the book value is the amount which, with interest, would equal the amount of the debt at the time it is due. The book value of **depreciating assets** (such as equipment) is the difference between the cost price and the accumulated depreciation charges at the date under consideration.

line value of a trigonometric function. A line whose length is numerically equal to the value of the function, usually taken as the line which is the numerator in the definition of the function with the line in the denominator drawn of unit length. The vertex of the angle is frequently placed at the center of a unit circle.

market value. See MARKET.

mean value theorems. See headings under MEAN.

numerical value. Same as ABSOLUTE VALUE.

par value. See PAR.

permissible value. See PERMISSIBLE.

place value. See PLACE.

present value. A sum of money which, with accrued interest, will equal a specified sum at some specified future time, or equal several sums at different times, as in the case of the present value (cost) of an annuity. The present value of a sum of money, A_n, due in n years at interest rate i is, at simple interest,

$$P=A_n/(1+ni).$$

At compound interest, it is

$$P=A_n/(1+i)^n=A_n(1+i)^{-n}.$$

The present value of an **ordinary annuity** is

$$A=R\frac{(1+i)^n-1}{i(1+i)^n}=\frac{R}{i}[1-(1+i)^{-n}],$$

where A is the present value, R the periodic payment, n the number of periods, and i the rate of interest per period. See TABLE

VI in the appendix. The present value of an **annuity due** is

$$A = R\frac{(1+i)^n - 1}{i(1+i)^{n-1}}$$
$$= \frac{R}{i}\left[1 + i - (1+i)^{-n+1}\right].$$

principal value of an inverse trigonometric function. For the arc-sine, arc-cosine, and arc-tangent, the numerically smallest value, the positive angle being taken when there are values numerically equal but opposite in sign. The principal value of arc-cotangent is usually taken in the interval $0 < \text{arc-cot}\, x < \pi$, but sometimes in $-\pi/2 < \text{arc-cot}\, x \leqq \pi/2$. There is no uniformity in definitions of the principal values of arc-sec x and arc-csc x, the most usual choices being $0 \leqq y \leqq \pi$, or $-\pi \leqq y < -\frac{1}{2}\pi$ and $0 \leqq y < \frac{1}{2}\pi$, for $y = \text{arc-sec}\, x$, and $-\frac{1}{2}\pi \leqq y \leqq \frac{1}{2}\pi$, or $-\pi < y \leqq -\frac{1}{2}\pi$ and $0 < y \leqq \frac{1}{2}\pi$, for $y = \text{arc-csc}\, x$. For all inverse trigonometric functions, the principal values of all positive numbers for which the particular function has real values then lie in the first quadrant. In the case of negative numbers, the principal value of the arc-sine is between $-\frac{1}{2}\pi$ and 0; the arc-cosine between $\frac{1}{2}\pi$ and π; and the arc-tangent between $-\frac{1}{2}\pi$ and 0. *E.g.*, the principal value of $\sin^{-1}\frac{1}{2}$, $\sin^{-1}(-\frac{1}{2})$, $\cos^{-1}(-\frac{1}{2})$ and $\tan^{-1}(-1)$ are, respectively, $\frac{1}{6}\pi$, $-\frac{1}{6}\pi$, $\frac{2}{3}\pi$, and $-\frac{1}{4}\pi$.

scrap or salvage value. See SCRAP.

surrender value of an insurance policy. See SURRENDER—surrender value of an insurance policy.

value of an expression. The result that would be obtained if the indicated operations were carried out. The value of $\sqrt{9}$ is 3; the value of

$$\int_a^b 2x\,dx \quad \text{is} \quad b^2 - a^2.$$

The value of the polynomial $x^2 - 5x - 7$ is -1 when $x = 6$. See EVALUATE.

value of an insurance policy. The difference between the expectation of the future benefit and the expectation of the future net premiums; the difference between the accumulated premiums and the accumulated losses. *Syn.* Terminal reserve.

wearing value. Same as REPLACEMENT COST.

VANDERMONDE DETERMINANT. See DETERMINANT.

VAN'ISH, *v.* To become zero; to take on the value zero.

VAN'ISH-ING, *adj.* Approaching zero as a limit; taking on the value zero.

VAR'I-A-BIL'I-TY, *n.* (*Statistics.*) Same as DISPERSION.

measures of variability. The *range, quartile deviation, average deviation,* and *standard deviation* are common usages.

VAR'I-A-BLE, *n.* A quantity which can take on any of the numbers of some set, such as the set of real numbers, the rational numbers, all numbers between two given numbers, or all numbers. The set of values which a variable may assume is sometimes stated explicitly and sometimes only implied. See FUNCTION.

chance, random, or stochastic variable. See CHANCE—chance variable.

change of variable in differentiation and integration. See CHAIN—chain rule, and INTEGRATION—change of variables in integration.

dependent and independent variables. See FUNCTION.

separation of variables. See DIFFERENTIAL—differential equation with variables separable.

VAR'I-ANCE, *adj., n.* The square of the standard deviation, denoted by σ^2. See DEVIATION—standard deviation.

analysis of variance. The statistical analysis of the variance of a random variable to determine if certain factors associated with the variable contribute to that variance. Also a method of analyzing differences between means of sets of samples, differentiated on the basis of a factor whose influence on means of the group is to be investigated. For instance, the analysis by the Latin square. Here the arrangements of data will form a square with one source of variability marking the r rows and one other source associated with the $k = r$ columns. The third source is assigned at random to the various k^2 combinations of the first two factors with the restriction that each value of the third source of varia-

tion can appear but once in each row and column. The analysis of variance yields: (1) a measure of the variance due to each of the sources of variation; (2) an interaction which must be used as the error term. This interaction cannot be tested for significance. The side of the square should usually be between 4 and 10. See BLOCK—randomized blocks, and VARIANCE—analysis of variance.

variance ratio. See DISTRIBUTION—F distribution.

variance-ratio transformation. The transformation $z=\frac{1}{2}\log_e(s_1^2/s_2^2)$, where s_i^2 are independent estimates of the variance of a normal distribution. However, common usage is to use the equivalent form $F=e^{2z}=s_1^2/s_2^2$. See DISTRIBUTION—F distribution, and FISHER—Fisher's z distribution.

VA'RI-ATE, *n.* **normalized variate.** (*Statistics.*) The transformation which converts the variate x into a normally distributed variate is a **normal transformation,** the new variate being called a **normalized variate.** *E.g.*, if the variate x has a Poisson distribution, $\sqrt{x}$ is approximately normally distributed. The sample proportions of samples from a binomial distribution will be approximately normally distributed if the *arc-sine of the square root* of the percentage is used. A distribution may be normalized if the variate value is converted to the standard deviation score which corresponds to the percentile position of the score to be converted. In the above, a *variate* is a random variable which is a numerical-valued variable defined on a given sample space.

VAR'I-A'TION, *n.* (*Calculus of Variations.*) A *variation* $\delta y(x)$ of a function $y(x)$ is any function $\delta y(x)$ which is added to $y(x)$ to give a new function $y(x)+\delta y(x)$. The name *calculus of variations* was adopted as a result of this notation, introduced by Lagrange in about 1760 when comparing the value of an integral along an arc with its value along a neighboring arc. The **first variation of an integral**

$$I=\int_a^b f(x,y,y')\,dx$$

is

$$\delta I=\frac{d}{d\epsilon}\int_a^b f(x,y+\epsilon\phi,y'+\epsilon\phi')\,dx]_{\epsilon=0},$$

if it exists for suitably restricted variations $\phi(x)$ of $y(x)$. If $\phi(a)=\phi(b)=0$, then $\delta I=\int_a^b \phi\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]dx$. A function $y(x)$ is said to make I **stationary,** or I is said to have a **stationary value,** if the first variation of I is zero for all *admitted* variations $\phi(x)$ of $y(x)$ such that $\phi(a)=\phi(b)=0$, a variation $\phi(x)$ being *admitted* if it satisfies certain conditions (*e.g.*, that it be rectifiable, or that it have a continuous derivative). A necessary condition that $y(x)$ make I have a (relative) maximum or minimum is that it give a stationary value of I. The **nth variation** $\delta^n I$ is given by

$$\delta^n I=\frac{d^n}{d\epsilon^n}\int_a^b f(x,y+\epsilon\phi,y'+\epsilon\phi')\,dx]_{\epsilon=0}.$$

See CALCULUS—calculus of variations.

calculus of variations. See CALCULUS.

coefficient of variation. (*Statistics.*) The quantity $V=100\sigma/\bar{x}$, where σ is the *standard deviation* of the variable x, and $\bar{x}$ is the *mean.*

combined variation. One quantity varying as some combination of other quantities, such as z varying directly as y and inversely as x.

direct variation. When two variables are so related that their ratio remains constant, one of them is said to *vary directly* as the other, or they are said to *vary proportionately*; *i.e.*, when $y/x=c$, or $y=cx$, where c is a constant, y is said to *vary directly* as x. This is sometimes written: $y\propto x$.

fundamental lemma of the calculus of variations. See FUNDAMENTAL.

inverse variation. When the ratio of one variable to the reciprocal of the other is constant (*i.e.*, when the product of the two variables is constant) one of them is said to *vary inversely* as the other, *i.e.*, if $y=c/x$, or $xy=c$, y is said to *vary inversely* as x, or x to *vary inversely* as y.

joint variation. When one variable *varies directly* as the product of two variables, the one variable is said to *vary jointly* as the other two, *i.e.*, when $x=kyz$, x varies jointly as y and z. When $x=kyz/w$, x is said to vary jointly as y and z and inversely as w.

variation of a function in an interval (a, b). The least upper bound of the sum of the oscillations in the closed subintervals

(a, x_1), $(x_1, x_2), \cdots, (x_n, b)$ $(a<x_1<x_2<\cdots<b)$ of (a, b), for all possible such subdivisions. A function with finite variation in (a, b) is said to be of **bounded variation** or **limited variation** in (a, b).

variation of a function on a surface. On a surface S: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, the rate of change of a given function $f(u, v)$ at a point P of S varies with the direction from P on S. It vanishes in the direction of the tangent to the curve $f=$ const., and has its greatest absolute value in the direction on S perpendicular to $f=$const.; this latter value is given by

$$\left|\frac{df}{ds}\right|=\frac{\left[E\left(\frac{\partial f}{\partial v}\right)^2-2F\frac{\partial f}{\partial u}\frac{\partial f}{\partial v}+G\left(\frac{\partial f}{\partial u}\right)^2\right]^{1/2}}{[EG-F^2]^{1/2}}$$

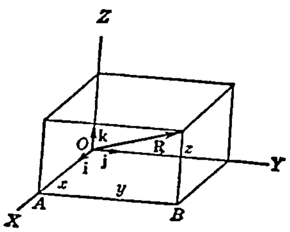
See SURFACE—fundamental coefficients of a surface, and GRADIENT—gradient of a function.

variation of parameters. See DIFFERENTIAL—linear differential equations.

variation of sign in an algebraic equation. A change of sign between two successive terms when arranged in descending order. The equation $x-2=0$ has one *variation of sign*, while $x^3-x^2+2x-1=0$ has three. See DESCARTES—Descartes' rule of signs.

variation of sign in an ordered set of numbers. A change of sign between two successive numbers; *e.g.*, the sequence 1, 2, -3, 4, -5, has three variations of sign.

VEC'TOR, *adj., n.* A quantity which has magnitude and direction. Its direction is defined relative to some fixed directed line when it is in a given plane, and relative to three fixed directed lines when in space. Any set of vectors whose *vector sum* is a given vector are said to be **components** of this vector, although the component of a vector in a given direction is understood to be the projection of the vector onto a line in this direction. If the unit vectors in the



directions of the x, y and z axes are denoted by i, j, and k, then the components parallel to the axes are of the form xi, yj, and zk, and the vector can be written $xi+yj+zk$, or (x, y, z). The vector $R=xi+yj+zk$ is shown in the figure. See ACCELERATION, FORCE—force vector, VELOCITY, and below, vector space.

absolute value of a vector. The numerical length of the vector (without regard to direction); the square root of the sum of the squares of its components along the axes. The absolute value of $2i+3j+4k$ is $\sqrt{29}$; in general, the absolute value of $ai+bj+ck$ is $\sqrt{a^2+b^2+c^2}$. *Syn.* Numerical value.

addition and multiplication of vectors. See SUM—sum of vectors, MULTIPLICATION—multiplication of vectors.

contravariant vector field. A contravariant tensor field of order one. See TENSOR—contravariant tensor.

covariant vector field. A covariant tensor field of order one. The gradient of a scalar field is a covariant vector field, while $t_i(x^1, x^2, \cdots, x^n)$ is locally the gradient of some scalar field if the conditions

$$\frac{\partial t_i}{\partial x^j}=\frac{\partial t_j}{\partial x^i}$$

are satisfied for all i and j in a region where the partial derivatives of t_i exist and are continuous. See TENSOR—covariant tensor.

derivative of a vector. See DERIVATIVE—derivative of a vector.

dominant vector. See DOMINANT.

irrotational vector. See IRROTATIONAL.

orthogonal vectors. See ORTHOGONAL.

parallel (contravariant) vector field. See PARALLEL—parallel displacement of a vector along a curve.

position vector. A vector from the origin to a point under consideration. If the point has Cartesian coordinates x, y, and z, the *position vector* is $R=xi+yj+zk$ (see the illustration under VECTOR).

radius vector. See POLAR—polar coordinates, SPHERICAL—spherical coordinates.

reciprocal systems of vectors. See RECIPROCAL.

solenoidal vector. See SOLENOIDAL.

vector analysis. The study of vectors, relations between vectors, and their applications.

vector function of a vector. A vector $\mathbf{F}' = \mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z'$ is said to be a function of $F = ix + jy + kz$ when x', y', and z' are functions of x, y, and z. $\mathbf{F}'$ is said to be a **linear function** of F when $F'(A+B) = F'(A) + F'(B)$ for all A and B, or when $x' = a_1x + b_1y + c_1z$, $y' = a_2x + b_2y + c_2z$, and $z' = a_3x + b_3y + c_3z$, where a_i, b_i, and c_i are constants.

vector potential. See POTENTIAL.

vector product of vectors. See MULTIPLICATION—multiplication of vectors.

vector space. (1) A space of vectors, as ordinary vectors in three dimensions, or (in general) a space of elements called *vectors* described by n components $(x_1, x_2, \cdots, x_n)$. The space is a **real vector space** if the components are real numbers. The sum of vectors $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ is defined as $x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$ and ax as $(ax_1, ax_2, \cdots)$. The **scalar product** (or **inner product** or **dot product**) of x and y is $\sum_{i=1}^{n} x_iy_i$, and the length or **norm** of the vector x is $\left[\sum_{i=1}^{n} x_i^2\right]^{1/2}$. If the components are complex numbers, then the **(Hermitian) scalar product** of x and y is defined as $\sum_{i=1}^{n} x_i\bar{y}_i$ and the length or **norm** of x as $\left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$. The space is then a **Hermitian vector space** (or **unitary space**). The vectors may have an infinite number of components, as for Hilbert space. (2) A set V of elements, called *vectors*, such that any vectors x and y determine a unique vector $x + y$ and any vector x and scalar a have a product ax in V, with the properties: (1) V is an Abelian group with addition as the group operation; (2) $a(x+y) = ax + ay$ and $(a+b)x = ax + bx$; (3) $(ab)x = a(bx)$ and $1 \cdot x = x$; properties (2) and (3) hold for any vectors x and y and any scalars a and b. The scalars may be real numbers, complex numbers, or elements of some other field. *Syn.* Linear space, linear vector space. A vector space is a **linear topological space** (or **topological vector space**) if it is a topological group and scalar multiplication is continuous (*i.e.*, for any neighborhood W of $a \cdot x$ there is a neighborhood U of a and a neighborhood V of x such that $b \cdot y$ is in W if b and y belong to U and V, respectively. A vector space is a **normed vector space** (or **normed linear space**) if there is a real number $\|x\|$ (called the **norm** of x) associated with each "vector" x and $\|x\| > 0$ if $x \neq 0$, $\|ax\| = |a|\ \|x\|$, $\|x+y\| \leqq \|x\| + \|y\|$. A normed vector space is also a linear topological space. See BANACH SPACE and HILBERT SPACE for examples of normed vector spaces. Also see BASIS, and ORTHOGONAL—orthogonal vectors.

VEC-TO'RI-AL, *adj.* **vectorial angle.** See POLAR—polar coordinates in the plane.

VE-LOC'I-TY, *n.* Directed speed. The velocity of an object at time t is the limit of the **average velocity** as the time interval (Δt) approaches zero (see below, average velocity). Velocity is sometimes called **instantaneous velocity** to distinguish it from average velocity. If an object is moving along a straight line, its velocity is spoken of as **linear** (or **rectilinear**) **velocity** and is equal in magnitude to the speed. If an object is moving along a curve, its velocity is sometimes called **curvilinear velocity** (its direction is that of the tangent to the curve, its magnitude the speed of the object). Velocity is said to be **absolute,** or **relative,** according as it is computed relative to a coordinate system considered stationary, or relative to a moving coordinate system. If the coordinates of a moving point are x, y, and z, its *position vector* is $R = xi + yj + zk$, and its *vector velocity* is $dR/dt = (dx/dt)i + (dy/dt)j + (dz/dt)k$. See SPEED.

angular velocity. (1) If a particle is moving in a plane, its *angular velocity* about a point in the plane is the rate of change of the angle between a fixed line and the line joining the moving particle to the fixed point. (As far as angular velocity is concerned, the particle may as well be considered as moving on a circle.) (2) If a rigid body is rotating about an axis, its *angular velocity* is (is represented by) a vector directed along the axis in the direction a right-hand screw would advance if subject to the given rotation and having magnitude equal to the angular speed of rotation about the axis (*i.e.*, the number of degrees or radians through which the body rotates per unit of time).

average velocity. The difference between the position vectors of a body at the ends of a given time interval, divided by the length of the interval. If $R=x(t)i+y(t)j+z(t)k$ is the *position vector* of a particle at time t, the average velocity over the time interval Δt is

$$\frac{x(t+\Delta t)-x(t)}{\Delta t}i+\frac{y(t+\Delta t)-y(t)}{\Delta t}j+\frac{z(t+\Delta t)-z(t)}{\Delta t}k,$$

or $\dfrac{R(t+\Delta t)-R(t)}{\Delta t}$, which is the resultant of the average velocities along the x, y, and z axes. See VECTOR.

constant (or uniform) velocity. See CONSTANT—constant speed and velocity.

VERSED, *adj.* **versed sine and versed cosine.** See TRIGONOMETRIC—trigonometric functions of an acute angle.

VER-SI-E′RA, *n.* See WITCH.

VER′TEX, *n.* [*pl.* **vertices**]. See ANGLE,—polyhedral angle, CONE, CONICAL—conical surface, ELLIPSE, HYPERBOLA, PARABOLA, PARABOLOID, PENCIL—pencil of lines through a point, POLYGON, POLYHEDRON, PYRAMID, and TRIANGLE.

opposite vertex to a side (or vertex) in a triangle or polygon. The vertex separated from the side (or vertex) by the same number of sides in each direction around the triangle or polygon, *if* there is such a vertex.

VER′TI-CAL, *adj.* **vertical angles.** Two angles such that each side of one is a prolongation through the vertex of a side of the other.

vertical line. (1) A line perpendicular to a horizontal line. The horizontal line is usually thought of as being directed from left to right and the vertical line as being directed upward, when they are coordinate axes. (2) A line perpendicular to the plane of the horizon. (3) A line from the observer to his zenith, *i.e.*, the plumb line.

VI′BRAT-ING, *p.* **equation of vibrating string.** The equation

$$\frac{\partial^2 y}{\partial t^2}=\frac{T}{\rho}\frac{\partial^2 y}{\partial x^2},$$

where x is the direction in which the string is stretched, y denotes displacement, and t is the time variable; T is the tension in the string and ρ is the density (mass per unit length) of the string. The boundary conditions are usually taken to be of the form $y=f(x)$ at $t=0$ and $\partial y/\partial t=0$ at $t=0$. It is assumed that the string is perfectly flexible, that T is constant, and that T is large enough that gravity can be neglected in comparison with it.

VI-BRA′TION, *n.* A periodic motion; a motion which is approximately periodic. *Syn.* Oscillation. See OSCILLATION.

VIN′CU-LUM, *n.* See AGGREGATION.

VI′TAL, *adj.* **vital statistics.** Statistics relating to the length of life and the number of persons dying during certain years, the kind of statistics from which mortality tables are constructed.

VITALI. Vitali covering. Let S be a set in n-dimensional Euclidean space and J be a class of sets such that, for each point x of S, there is a positive number $\alpha(x)$ and a sequence of sets $U_1, U_2, \cdots$, which belong to J, each of which contains x, and which have the properties that their *diameters* approach zero and for each integer n there is a *cube* C_n such that $m(U_n)\geqq\alpha(x)m(C_n)$ and C_n contains U_n [$m(U_n)$ and $m(C_n)$ are the *measures* of U_n and C_n]. Such a class of sets is said to **cover S in the sense of Vitali.**

Vitali covering theorem. Let S be a set in n-dimensional Euclidean space. The Vitali covering theorem states that, if a class J of closed sets is a *Vitali covering* of S, then there is a finite or denumerably infinite sequence of pairwise disjoint sets belonging to J whose union contains all of S except a set of *measure zero.*

Vitali set. A set of real numbers such that no two numbers of the set have a difference which is a rational number and each real number is equal to a rational number plus a member of the set. Such a set can be formed by choosing exactly one

element from each *coset* of the rational numbers, considered as a subgroup of the additive group of real numbers. A Vitali set is nonmeasurable and an intersection of a Vitali set with an interval either is of measure zero or is nonmeasurable. See SIERPINSKI—Sierpinski set.

VOLT, *n.* A unit of measure of electromotive force. (1) The **absolute volt** is the steady potential difference which must exist across a conductor which carries a steady current of one absolute ampere and which dissipates thermal energy at the rate of one watt. The absolute volt has been the legal standard of potential difference since 1950. (2) The **International volt,** the legal standard prior to 1950, is the steady potential difference which must be maintained across a conductor which has a resistance of one International ohm and which carries a steady current of one International ampere.

1 Int. volt = 1.000330 Abs. volts.

VOLTERRA. Volterra's integral equations. *Volterra's integral equation of the first kind* is the equation

$$f(x)=\int_a^x K(x,t)y(t)\,dt,$$

and *Volterra's integral equation of the second kind* is $y(x)=f(x)+\lambda\int_a^x K(x,t)y(t)\,dt$, in which $f(x)$ and $K(x,t)$ are two given functions, and $y(x)$ is the unknown function. The function $K(x,t)$ is called the **kernel** or **nucleus** of the equation. Volterra's equation of the second kind is said to be **homogeneous** if $f(x)\equiv 0$. For an example, see ABEL—Abel's problem.

Volterra's reciprocal functions. Two functions $K(x,y)$ and $k(x,y;\lambda)$ for which

$$K(x,y)+k(x,y;\lambda)=\lambda\int_a^b k(x,t;\lambda)K(t,y)\,dt.$$

If the *Fredholm determinant* $D(\lambda)\neq 0$ and $K(x,y)$ is continuous in x and y, then $k(x,y;\lambda)=-D(x,y;\lambda)/[\lambda D(\lambda)]$, where $D(x,y;\lambda)$ is the *first Fredholm minor.* If $g(x)$ is a solution of

$$g(x)=f(x)+\lambda\int_a^b K(x,t)g(t)\,dt,$$

then $f(x)$ is a solution of

$$f(x)=g(x)+\lambda\int_a^b k(x,t;\lambda)f(t)\,dt,$$

and conversely. The function $k(x,y;\lambda)$ is called the **resolvent kernel.** The above is sometimes modified by letting $\lambda=1$. See KERNEL—iterated kernels.

Volterra's solution of Volterra's integral equations. If $f(x)$ and $K(x,t)$ are continuous functions of x in $a\leqq x\leqq b$ and of x and t in $a\leqq t\leqq x\leqq b$, respectively, then the Volterra integral equation of the second kind,

$$y(x)=f(x)+\lambda\int_a^x K(x,t)y(t)\,dt,$$

has a unique continuous solution given by the formula

$$y(x)=f(x)+\int_a^x k(x,t;\lambda)f(t)\,dt,$$

where $k(x,y;\lambda)$ is the *resolvent kernel* of the given *kernel* $K(x,y)$ and is a continuous function of x and t for $a\leqq t\leqq x\leqq b$. *Volterra's integral equation of the first kind,* $f(x)=\lambda\int_a^x K(x,t)y(t)\,dt$, can be reduced to an equation of the second kind by differentiation, giving

$$f'(x)=\lambda K(x,x)y(x)+\lambda\int_a^x \frac{\partial K(x,t)}{\partial x}y(t)\,dt,$$

it being assumed that $\frac{\partial K(x,t)}{\partial x}$ exists and is continuous. The above is sometimes modified by letting $\lambda=1$.

VOL'UME, *n.* **coefficient of volume expansion.** See COEFFICIENT.

differential (or element) of volume. See ELEMENT—element of integration.

volume of a solid. The greatest lower bound of the sum of the volumes of nonoverlapping cubes which together completely cover the solid, the volume of a cube being defined as the cube of the length of its sides (see the specific solids for formulas). See MEASURABLE—measurable set.

volumes of similar solids. See SOLID—similar solids.

VUL'GAR, *adj.* **vulgar fraction.** See FRACTION.

W

W. W-surface, *n.* Same as WEINGARTEN SURFACE. See SURFACE.

WALK, *n,* random walks. See RANDOM.

WALLIS. Wallis' formulas. Formulas giving the values of the definite integrals from 0 to $\frac{1}{2}\pi$ of each of the functions $\sin^m x$, $\cos^m x$, and $\sin^m x \cos^n x$, for m and n any positive integers.

Wallis' product for π. The infinite product

$$\frac{\pi}{2}=\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2k}{2k-1}\frac{2k}{2k+1}\cdots.$$

WARING'S PROBLEM. The problem, proposed by Waring, of showing that, for any integer n, there is an integer $K(n)$ such that any integer can be represented as the sum of not more than $K(n)$ numbers, each of which is an nth power of an integer. In particular, any integer can be represented as the sum of not more than 4 squares, and as the sum of not more than 9 cubes. This problem was solved by Hilbert (1909).

WAST'ING, *adj.* **wasting assets.** Same as DEPRECIATING ASSETS.

WATT, *n.* A metric unit of measure of power; the power required to keep a current of one ampere flowing under a potential drop of one volt; about $\frac{1}{736}$ of one horsepower (English and American). The **international watt** is defined in terms of the *international ampere* and the *international volt* and differs slightly from the **absolute watt,** which is equivalent to 10^7 ergs (one joule) of work per second.

watt-hour. A unit of measure of electric energy; the work done by one watt acting for one hour. It equals $36\cdot10^9$ ergs.

WAVE, *adj.* **wave equation.** The partial differential equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}.$$

In the theory of sound, this equation is satisfied by the velocity potential (in a perfect gas); in the theory of elastic vibrations, it is satisfied by each component of the displacement; and, in the theory of electric or electromagnetic waves, it is satisfied by each component of the electric or magnetic (force) vector. The constant c is the velocity of propagation of the periodic disturbance.

wave length of motion as represented by trigonometric functions. The period of the trigonometric function. See PERIOD—period of a function.

WEAK, *adj.* **weak compactness.** A set S contained in a normed linear space N is *weakly compact* provided each sequence of elements of S has a subsequence which *converges weakly* to a point of S. A Banach space has the property that each bounded closed subset is weakly compact if and only if the space is reflexive.

weak completeness. A normed linear space N is *weakly complete* provided each weakly convergent sequence of elements of N is weakly convergent to an element of N. A weakly complete normed linear space is complete (and is a Banach space). A reflexive Banach space is weakly complete, but the space l^1 [of sequences $x=(x_1, x_2, \cdots)$ for which $\|x\|=\sum^{\infty}|x_i|$ is finite] is weakly complete and not reflexive.

weak convergence. A sequence of elements $x_1, x_2, \cdots$ of a normed linear space N is *weakly convergent* (or is a *weakly fundamental sequence*) if $\lim f(x_n)$ exists for each continuous linear functional defined on N. If $\lim f(x_n)=f(x)$ for each f, then the sequence is said to **converge weakly to x** (and x is a **weak limit** of the sequence). A continuous linear functional f is a **weak* limit** (or **w*-limit**) of a sequence $f_1, f_2, \cdots$ of continuous linear functionals if $\lim f_i(x)=f(x)$ for each x of N. Weak* is read as **weak-star.** See below, weak topology.

weak topology. The **weak topology** of a normed linear space N is generated by the set of neighborhoods defined as follows: For each positive number ϵ, element x_0 of N, and finite set $f_1, f_2, \cdots, f_n$ of continuous linear functionals defined on N, the set U of all x for which $|f_k(x)-f_k(x_0)|<\epsilon$ for each k is a neighborhood of x_0. Open sets of the topology are then unions of such neighborhoods. The **weak* topology** (or **w*-topology**) of the *first conjugate space* N^* of a normed linear space N is generated by the set of neighborhoods defined as follows: For each positive number ϵ, element f_0 of N^*, and finite set $x_1, x_2, \cdots, x_n$ of elements of N, the set V of all f for which $|f(x_k)-f_0(x_k)|<\epsilon$ for each k is a neighborhood of

f_0. The unit sphere of N^* (the set of all f with $\|f\| \leqq 1$) is compact in the weak* topology. For a reflexive Banach space, the weak* topology of B^* and the weak topology of B^* are identical.

WEAR'ING, *adj.* **wearing value of equipment.** The difference between the purchase price and the scrap value. Also called **original wearing value.** The difference between the book value and scrap value is the **remaining wearing value.**

WEDDLE'S RULE. An alternative to Simpson's rule for approximating the area bounded by a curve $y=f(x)$, the x-axis, and the ordinates $x=a$ and $x=b$. The interval (a, b) is divided into $6n$ equal parts, and the formula is

$$A=\frac{b-a}{20n}[y_a+5y_1+y_2+6y_3+y_4+5y_5 +y_6+\cdots+5y_{6n-1}+y_{6n}].$$

See NEWTON—Newton's three-eighths rule, and SIMPSON'S RULE.

WEIERSTRASS. equations of Weierstrass. Integral equations for the coordinate functions of all real minimal surfaces in isothermic representation:

$$x=R\int(1-u^2)\zeta(u)\,du,$$

$$y=R\int i(1+v^2)\zeta(u)\,du,$$

$$z=R\int 2u\zeta(u)\,du,$$

where R denotes the real part of any function. *E.g.*, the right helicoid is obtained by setting $\zeta(u)=ik/2u^2$, where k is a real constant. The equations of Weierstrass can be obtained from those of Enneper by letting u and v, and ζ and ψ, be conjugate imaginaries. See ENNEPER—equations of Enneper. The functions x, y, z are harmonic in accordance with a theorem of Weierstrass that a necessary and sufficient condition for a surface given in isothermic representation to be a minimal surface is that the coordinate functions be harmonic. See SURFACE—surface of Scherk, surface of Henneberg, and surface of Enneper.

theorem of Weierstrass. See above, equations of Weierstrass.

Weierstrass approximation theorem. A continuous function may be approximated over a closed interval by a polynomial, to any assigned degree of accuracy. *Tech.* For every function $f(x)$ continuous on $[a, b]$ and any positive number ϵ there exists a polynomial $P(x)$ such that

$$|f(x)-P(x)|<\epsilon$$

for every x on the closed interval $[a, b]$. This theorem has been generalized by Stone. Let T be a compact topological space and S be a set of continuous real-valued functions defined on T. Then each continuous real-valued function defined on T can be uniformly approximated by a member of S if S has the following properties: (1) if f and g are members of S and a is a real number, then af, $f+g$, and $f\times g$ are members of S; (2) if x and y are distinct points of T and a and b and real numbers, there is a member f of S for which $f(x)=a$ and $f(y)=b$.

Weierstrass' elliptic (or P) functions. See ELLIPTIC.

Weierstrass' M-test for uniform convergence: If $|f_1(x)|$, $|f_2(x)|$, $|f_3(x)|, \cdots$ are bounded for x on the interval (a, b) by the corresponding terms of the sequence $M_1, M_2, M_3, \cdots$, and ΣM_n converges, then $\Sigma f_n(x)$ converges uniformly on (a, b). *E.g.*, the terms of the sequence $x, x^2, x^3, \cdots$ are bounded for x on the interval $(0, \frac{1}{2})$ by the corresponding terms of $\frac{1}{2}, (\frac{1}{2})^2, (\frac{1}{2})^3, \cdots$, and $\Sigma(\frac{1}{2})^n$ converges. Hence Σx^n converges uniformly on $(0, \frac{1}{2})$.

Weierstrass' necessary condition (*Calculus of Variations*). A condition that must be satisfied if the function y is to minimize $\int_{x_1}^{x_2} f(x, y, y')\,dx$; namely, the condition $E(x, y, y', Y')\geqq 0$ for all admissible $(x, y, Y')\neq(x, y, y')$, where

$$E=f(x, y, Y')-f(x, y, y')-(Y'-y')f_{y'}(x, y, y').$$

See CALCULUS—calculus of variations, EULER—Euler's equation, LEGENDRE—Legendre's necessary condition.

WEDGE, *n.* **elliptic wedge.** The surface generated by a straight line intersecting a given line, remaining parallel to a plane, and intersecting an ellipse with its plane

parallel to the given line but not containing it.

spherical wedge. See SPHERICAL.

WEIGHT, *n.* (1) The gravitational pull on a body. See POUND. (2) See AVERAGE.

apothecaries' weight. The system of weights used by druggists. The pound and the ounce are the same as in troy weight, but the subdivisions of the ounce are different. See DENOMINATE NUMBERS in the appendix.

avoirdupois weight. The system of weights which uses a pound of 16 ounces as its basic unit. See the appendix.

beta weight. See CORRELATION—normal correlation.

pound of weight. See POUND.

troy weight. The system of weights using a pound consisting of 12 ounces. Used mostly for weighing metals. See DENOMINATE NUMBERS in the appendix.

WEIGHT'ED, *adj.* **weighted mean.** Same as WEIGHTED ARITHMETIC AVERAGE. See AVERAGE.

WHIRLWIND. An automatic digital computing machine at the Mass. Inst. of Tech.

WHOLE, *adj.* **whole-life insurance.** See INSURANCE—life insurance.

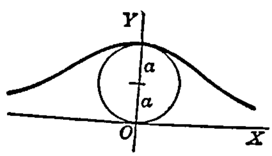
WIDTH, *n.* Same as BREADTH.

WILSON'S THEOREM. The number $[(n-1)!+1]$ is divisible by n if, and only if, n is a prime. *E.g.*, $4!+1=25$ is divisible by 5, but $5!+1=121$ is not divisible by 6.

WIND'ING, *adj.* **winding number.** The number of times a closed curve in the plane passes around a designated point of the plane in the counterclockwise (positive) direction. *Tech.* Let C be a closed curve in the plane which is the image of a circle for a continuous transformation. This means that C has parametric equations of type $x=u(t)$, $y=v(t)$ for $0\leqq t\leqq 1$, where u and v are continuous functions with $u(0)=u(1)$ and $v(0)=v(1)$; or (equivalently) an equation of type $w\equiv f(z)$ for $|z|=1$, where z and w are complex numbers and f is a continuous function. If P is chosen as a point which is not on C, the numbers $\{t_i\}$ satisfy $t_0=0<t_1<t_2<\cdots<t_n=1$, and Q_i is the point for which the parameter t is equal to t_i $(i=1, 2, \cdots, n)$, then there is a positive number E such that the quantity $\frac{1}{2\pi}\sum_1^n\theta_i$ has a value $n(C,P)$ which is independent of the choice of the numbers $\{t_i\}$ provided $(t_i-t_{i-1})<E$ for each i, where θ_i is the angle (in radian measure) from the line PQ_{i-1} to the line PQ_i for each i. This number $n(C,P)$ is an integer and is called the **winding number** of C relative to P; it is also called the **index** of P relative to C. The winding number of a curve is not changed by a continuous deformation of the curve for which the curve does not pass through the point P. *E.g.*, if $p(z)$ is a polynomial of degree n with $p(0)\neq 0$, and C_K is the image of the circle $|z|=K$ for the mapping $w=p(z)$, then for sufficiently large K the winding number of C_K relative to the origin is n, and for sufficiently small K the winding number of C_K relative to the origin is 0. Since these curves can be continuously deformed into each other (by letting K vary continuously), there must be a value of K for which the curve C_K passes through the origin and therefore a value of z for which $p(z)=0$. This gives a proof of the *fundamental theorem of algebra*. If P is the complex number a and the curve C is defined by $w=f(z)$, where f is *piecewise differentiable*, then

$$n(C, a)=\frac{1}{2\pi i}\int_C\frac{dz}{z-a}.$$

WITCH, *n.* A plane cubic curve defined by drawing a circle of radius a, tangent to the x-axis at the origin, then drawing a line through the origin and forming a right triangle with its hypotenuse on this line and one leg parallel to the x-axis, the other parallel to the y-axis, and passing, respectively, through the points of intersection of this line with the circle and the line $y=2a$. The witch is then the locus of the intersection of the legs of all such tri-



angles. Its equation in rectangular coordinates is $x^2y=4a^2(2a-y)$. The *witch* is usually called the **witch of Agnesi,** after Donna Maria Gaetana Agnesi, who discussed the curve. It is also called the **versiera.**

WORD, *n.* In a digital computing machine, the content of one storage position; an ordered set of digits considered as a unit, regardless of whether they designate a number, an address code, an instruction to the machine, or a combination of these.

WORK, *n.* The work W done by the constant force F in producing a displacement s is the product of the component of the force F in the direction of displacement by the scalar s. Thus, $W=Fs\cos\theta$, where θ is the angle between the direction of the force F and the displacement s. If the force F is variable, the usual procedures of integral calculus lead to the definition $W=\int_{t_0}^{t} F\cdot\frac{dr}{dt}\,dt$, of the work done in the interval of time $t-t_0$. The expression $F\cdot\frac{dr}{dt}$ is the scalar product of the force vector F and the vector velocity $v=\frac{dr}{dt}$. The **element** (or **differential**) **of work,** $F\cdot\frac{dr}{dt}\,dt$, is equal to fds, where f is the component of the force in the direction of motion and $s(t)$ is the distance passed over between time 0 and time t.

WRON'SKI-AN, *n.* The *Wronskian* of n functions $u_1, u_2, \cdots, u_n$ is the determinant of order n which has these functions as the elements of the first row, and their kth derivatives as the elements of the $(k+1)$st row $(k=1, 2, \cdots, n-1)$. If the Wronskian of n functions is not identically zero, the functions are *linearly independent*, while n functions are *linearly dependent* on an interval (a, b) if their Wronskian is identically zero on (a, b) (it being assumed that their first $n-1$ derivatives are continuous) and they are solutions of a differential equation of the form

$$p_0\frac{d^ny}{dx^n}+p_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+p_{n-1}\frac{dy}{dx}+p_n\,y=0,$$

where the functions p_i are continuous on (a, b) and p_0 is not zero at any point of (a, b). Indeed, the n solutions of this differential equation are linearly dependent and their Wronskian is identically zero on (a, b), if their Wronskian vanishes at a single point of (a, b).

X

X. (1) The letter most commonly used to denote an **unknown number** or **variable.** (2) Used to denote one of the axes in a system of Cartesian coordinates. See CARTESIAN.

x-axis. See CARTESIAN—Cartesian coordinates.

Y

Y. **y-axis.** See CARTESIAN—Cartesian coordinates.

YARD, *n.* A unit of English linear measure equal to 3 feet; the distance between two lines on a specially prepared and carefully preserved bar, at a temperature of 62°F. See DENOMINATE NUMBERS in the appendix.

YATES. Yates correction for continuity. (*Statistics.*) The computed χ_c^2 for a 2×2 table, or for the test of an observed proportion with one degree of freedom, is biased, since χ^2 is continuous and the χ_c^2 for the one-degree-of-freedom case 2×2 table is discrete. The following formula for χ^2 contains a correction which results in a reasonably close approximation to the continuous distribution of χ^2, even when the expected number of cases in a cell of the 2×2 table is less than 5:

$$\text{Cor. }\chi_c^2=\sum_{i=1}^{4}\frac{(|x_i-m_i|-\frac{1}{2})^2}{m_i},$$

where x_i is the observed frequency and m_i is the expected frequency in the ith cell. See CHI-SQUARE.

YAW, *adj.* **yaw angle.** In exterior ballistics, the angle between the direction of the axis of a shell and the direction of its velocity vector.

YEAR, *n.* The year is the longest natural unit for measuring time. Several different

kinds of years are in use, but all of them depend on the revolution of the earth about the sun. The **sidereal year** is the time during which the earth makes one complete revolution around the sun with respect to the stars. (Its length in mean solar days is 365 days, 6 hours, 9 minutes, 9.5 seconds). The **tropical year** (also called the **astronomical, equinoctial, natural,** or **solar year**) is the time required for the earth (or apparently the sun) to pass from the vernal equinox back to the vernal equinox (its length is 365 days, 5 hours, 48 minutes, 46 seconds). Because of the precession of the equinoxes, this is 20 minutes, 23.5 seconds shorter than the sidereal year. The tropical year is the basis of practically all ancient and modern calendars. The **anomalistic year** is the time required for the earth to pass from some point in its elliptic orbit back to the same point again and is 365 days, 6 hours, 13 minutes, 53 seconds in length, differing from the other two because of the fact that the major axis of the earth's orbit is slowly moving at the rate of 11″ per year. The **civil year** (also called the **calendar** or **legal year**) is 365 days (except on **leap years,** when it is 366 days). A **commercial year** is 360 days, as used in computing simple interest.

YIELD, *n.* (*Finance.*) The rate per cent which gives a certain profit. *Syn.* Rate per cent yield, yield rate, investor's (investment) rate. Thus the **yield of a bond** is the effective rate of interest which a person realizes on an investment in this type of bonds (the **approximate yield** is the average interest per period divided by the average capital invested).

YOUNG'S INEQUALITY. Let the function $y=f(x)$ be continuous and strictly increasing for $x\geq 0$, with $f(0)=0$, let $x=f(y)$ be the inverse function, and let $a\geq 0$ and $b\geq 0$ be numbers in the ranges of x and y, respectively. Then Young's inequality is

$$ab\leq\int_0^a f(x)\,dx+\int_0^b g(y)\,dy,$$

the sign of equality holding if and only if $b=f(a)$. The result, which becomes intuitively clear when a figure is drawn showing the three cases $b<f(a)$, $b=f(a)$, and $b>f(a)$, has many applications in the theory of inequalities.

YOUNG'S MODULUS. See MODULUS.

Z

Z. Fisher's z. See FISHER.

z-axis. See CARTESIAN—Cartesian coordinates.

ZE'NITH, *n.* **zenith distance of a star.** The angular distance from the zenith to the star, measured along the great circle through the zenith, the nadir, and the star. It is the complement of the altitude. See HOUR—hour angle and hour circle.

zenith of an observer. The point on the celestial sphere directly above the observer; the point where a plumb line, extended upward, would pierce the celestial sphere.

ZENO. Zeno's paradox of Achilles and the tortoise. A tortoise has a head start on Achilles equal to the distance from a to b and both start running, Achilles after the tortoise. Although Achilles runs faster than the tortoise, he would never catch up with the tortoise, since, while Achilles goes from a to b, the tortoise goes from b to c, and while Achilles goes from b to c, the tortoise goes from c to d, etc., this process never ending. The explanation of the fallacy is that motion is measured by space intervals per unit of time, not by numbers of points. If Achilles takes time $t_1, t_2, t_3 \cdots$ to go from a to b, b to c, c to d, $\cdots$, then Achilles will catch the tortoise in time equal to $\sum_{i=1}^{n} t_i$, if this sum is finite. If the tortoise travels 10 feet per second and Achilles 20 feet and the tortoise starts 10 feet in advance, Achilles will catch him at the end of the first second, since then $t_1=\frac{1}{2}$, $t_2=\frac{1}{4}$, $\cdots$, $t_n=1/2^n$, $\cdots$ and $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1$. However, if Achilles always runs faster than the tortoise, but the tortoise gradually increases his speed so that $t_1=1$, $t_2=\frac{1}{2}$, $t_3=\frac{1}{3}$, $\cdots$, $t_n=1/n$, $\cdots$, then $\sum_{i=1}^{n} t_i$ becomes large beyond all bounds as n increases and Achilles will never catch the tortoise.

ZERMELO. **Zermelo's axiom.** See CHOICE—axiom of choice, and ZORN—Zorn's lemma.

ZE'RO, *adj., n.* (1) The cardinal number denoting the absence of any of the units that have been under consideration. (2) The ordinal number denoting the initial point, or origin. *Tech.* The quantity which when added to another quantity does not alter the former; if $a+b=a$, b is called *zero.* See ANGLE—zero angle, and EXPONENT.

division by zero. See DIVISION.

division of zero. The quotient of zero and any other number is zero; $0/k=0$ for all k not zero, since $0=k\times 0$. (We can't say $0/0=0$, for $0/0$ would also be 1, or 5, or any number, since $0=0\times 1=0\times 5$, etc.) See DIVISION.

factorial zero. Defined as equal to unity. See FACTORIAL.

multiplication by zero. The product of zero and any other number is zero, *i.e.*, $0\times k=k\times 0=0$ for all k. See MULTIPLICATION—multiplication of two integers.

zero of a function. A value of the argument for which the function is zero. A **real zero** is a real number for which the function is zero. If the function $f(x)$ has only real number values for real number values of x (*e.g.*, if $f(x)$ is a **polynomial** with real numbers as coefficients), then the real zeros of $f(x)$ are the values of x for which the curve $y=f(x)$ touches the x-axis. See ROOT—root of an equation. If z_0 is a zero (also called **zero point**) of an **analytic function** $f(z)$ of a complex variable, then there is a positive integer k such that $f(z)\equiv(z-z_0)^k\phi(z)$, where $\phi(z)$ is analytic at z_0 and $\phi(z_0)\neq 0$. The integer k is called the **order** of the zero.

zero-sum game. See GAME.

ZE'TA, *adj., n.* The sixth letter of the Greek alphabet, written ζ; capital, Z.

Riemann zeta function. The zeta function $\zeta(z)=\zeta(x+iy)$ is defined for $x>1$ by the series

$$\zeta(z)=\sum_{n=1}^{\infty} n^{-z}=\sum_{n=1}^{\infty} e^{-z\log n},$$

where $\log n$ is real. The function can be defined by analytic continuation for all finite z. It is a meromorphic function having a simple pole at $z=1$.

ZON'AL, *adj.* **zonal harmonic.** See HARMONIC—zonal harmonic.

ZONE, *n.* A portion of a sphere bounded by the two intersections of two parallel planes with the sphere. One of the planes may be a tangent plane, in which case one of the circular intersections is a point and the zone is said to be a **zone of one base.** A **base** of a zone is an intersection with the sphere of one of the planes forming the zone. The **altitude** of a zone is the perpendicular distance between the planes cutting the zone out of the sphere. The **area** of a zone is equal to the product of its altitude and the perimeter of a great circle of the sphere, *i.e.*, $2\pi rh$, where r is the radius of the sphere and h the altitude of the zone.

zone of a surface of revolution. The portion of the surface contained between two planes normal to the axis of revolution.

ZORN. **Zorn's lemma.** The maximal principle: If T is *partially ordered* and each *simply ordered* subset has an upper bound in T, then T contains at least one *maximal element* (an element x such that there is no y of T with $x<y$). Other alternative forms of this principle are: (1) A *partially ordered* set contains at least one *maximal simply ordered subset* (a simply ordered subset S such that there is no y not in S for which $x<y$ for each x of S). (2) If a collection A of sets has the property that for each *nest* in A there is a member of A which contains each member of the nest, then there is a *maximal member* of A (a member of A which is not contained in any other member of A). (3) A collection of sets which is of *finite character* has a *maximal member.* (4) Any set can be well-ordered (see ORDERED—well-ordered set). (5) The axiom of choice (see CHOICE). If the finite axiom of choice is assumed, all of the above principles are logically equivalent.

APPENDIX

TABLE I

COMMON LOGARITHMS*

To get a natural logarithm, multiply the common logarithm by 2.302585 ($=\log_e 10$).
(See LOGARITHM.)

N	0	1	2	3	4	5	6	7	8	9
100	00 000	00 043	00 087	00 130	00 173	00 217	00 260	00 303	00 346	00 389
101	432	475	518	561	604	647	689	732	775	817
102	860	903	945	988	01 030	01 072	01 115	01 157	01 199	01 242
103	01 284	01 326	01 368	01 410	452	494	536	578	620	662
104	703	745	787	828	870	912	953	995	02 036	02 078
105	02 119	02 160	02 202	02 243	02 284	02 325	02 366	02 407	02 449	02 490
106	531	572	612	653	694	735	776	816	857	898
107	938	979	03 019	03 060	03 100	03 141	03 181	03 222	03 262	03 302
108	03 342	03 383	423	463	503	543	583	623	663	703
109	743	782	822	862	902	941	981	04 021	04 060	04 100
110	04 139	04 179	04 218	04 258	04 297	04 336	04 376	04 415	04 454	04 493
111	532	571	610	650	689	727	766	805	844	883
112	922	961	999	05 038	05 077	05 115	05 154	05 192	05 231	05 269
113	05 308	05 346	05 385	423	461	500	538	576	614	652
114	690	729	767	805	843	881	918	956	994	06 032
115	06 070	06 108	06 145	06 183	06 221	06 258	06 296	06 333	06 371	06 408
116	446	483	521	558	595	633	670	707	744	781
117	819	856	893	930	967	07 004	07 041	07 078	07 115	07 151
118	07 188	07 225	07 262	07 298	07 335	372	408	445	482	518
119	555	591	628	664	700	737	773	809	846	882
120	07 918	07 954	07 990	08 027	08 063	08 099	08 135	08 171	08 207	08 243
121	08 279	08 314	08 350	386	422	458	493	529	565	600
122	636	672	707	743	778	814	849	884	920	955
123	991	09 026	09 061	09 096	09 132	09 167	09 202	09 237	09 272	09 307
124	09 342	377	412	447	482	517	552	587	621	656
125	09 691	09 726	09 760	09 795	09 830	09 864	09 899	09 934	09 968	10 003
126	10 037	10 072	10 106	10 140	10 175	10 209	10 243	10 278	10 312	346
127	380	415	449	483	517	551	585	619	653	687
128	721	755	789	823	857	890	924	958	992	11 025
129	11 059	11 093	11 126	11 160	11 193	11 227	11 261	11 294	11 327	361
130	11 394	11 428	11 461	11 494	11 528	11 561	11 594	11 628	11 661	11 694
131	727	760	793	826	860	893	926	959	992	12 024
132	12 057	12 090	12 123	12 156	12 189	12 222	12 254	12 287	12 320	352
133	385	418	450	483	516	548	581	613	646	678
134	710	743	775	808	840	872	905	937	969	13 001
135	13 033	13 066	13 098	13 130	13 162	13 194	13 226	13 258	13 290	13 322
136	354	386	418	450	481	513	545	577	609	640
137	672	704	735	767	799	830	862	893	925	956
138	988	14 019	14 051	14 082	14 114	14 145	14 176	14 208	14 239	14 270
139	14 301	333	364	395	426	457	489	520	551	582
140	14 613	14 644	14 675	14 706	14 737	14 768	14 799	14 829	14 860	14 891
141	922	953	983	15 014	15 045	15 076	15 106	15 137	15 168	15 198
142	15 229	15 259	15 290	320	351	381	412	442	473	503
143	534	564	594	625	655	685	715	746	776	806
144	836	866	897	927	957	987	16 017	16 047	16 077	16 107
145	16 137	16 167	16 197	16 227	16 256	16 286	16 316	16 346	16 376	16 406
146	435	465	495	524	554	584	613	643	673	702
147	732	761	791	820	850	879	909	938	967	997
148	17 026	17 056	17 085	17 114	17 143	17 173	17 202	17 231	17 260	17 289
149	319	348	377	406	435	464	493	522	551	580
150	17 609	17 638	17 667	17 696	17 725	17 754	17 782	17 811	17 840	17 869

Prop. Pts.

	41	42	43
1	4.1	4.2	4.3
2	8.2	8.4	8.6
3	12.3	12.6	12.9
4	16.4	16.8	17.2
5	20.5	21.0	21.5
6	24.6	25.2	25.8
7	28.7	29.4	30.1
8	32.8	33.6	34.4
9	36.9	37.8	38.7

	38	39	40
1	3.8	3.9	4.0
2	7.6	7.8	8.0
3	11.4	11.7	12.0
4	15.2	15.6	16.0
5	19.0	19.5	20.0
6	22.8	23.4	24.0
7	26.6	27.3	28.0
8	30.4	31.2	32.0
9	34.2	35.1	36.0

	35	36	37
1	3.5	3.6	3.7
2	7.0	7.2	7.4
3	10.5	10.8	11.1
4	14.0	14.4	14.8
5	17.5	18.0	18.5
6	21.0	21.6	22.2
7	24.5	25.2	25.9
8	28.0	28.8	29.6
9	31.5	32.4	33.3

	32	33	34
1	3.2	3.3	3.4
2	6.4	6.6	6.8
3	9.6	9.9	10.2
4	12.8	13.2	13.6
5	16.0	16.5	17.0
6	19.2	19.8	20.4
7	22.4	23.1	23.8
8	25.6	26.4	27.2
9	28.8	29.7	30.6

	29	30	31
1	2.9	3.0	3.1
2	5.8	6.0	6.2
3	8.7	9.0	9.3
4	11.6	12.0	12.4
5	14.5	15.0	15.5
6	17.4	18.0	18.6
7	20.3	21.0	21.7
8	23.2	24.0	24.8
9	26.1	27.0	27.9

*Adapted from Mackie and Hoyle's Elementary College Mathematics, by permission of the publishers, Ginn and Company.

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
150	17 609	17 638	17 667	17 696	17 725	17 754	17 782	17 811	17 840	17 869
151	898	926	955	984	18 013	18 041	18 070	18 099	18 127	18 156
152	18 184	18 213	18 241	18 270	298	327	355	384	412	441
153	469	498	526	554	583	611	639	667	696	724
154	752	780	808	837	865	893	921	949	977	19 005
155	19 033	19 061	19 089	19 117	19 145	19 173	19 201	19 229	19 257	19 285
156	312	340	368	396	424	451	479	507	535	562
157	590	618	645	673	700	728	756	783	811	838
158	866	893	921	948	976	20 003	20 030	20 058	20 085	20 112
159	20 140	20 167	20 194	20 222	20 249	276	303	330	358	385
160	20 412	20 439	20 466	20 493	20 520	20 548	20 575	20 602	20 629	20 656
161	683	710	737	763	790	817	844	871	898	925
162	952	978	21 005	21 032	21 059	21 085	21 112	21 139	21 165	21 192
163	21 219	21 245	272	299	325	352	378	405	431	458
164	484	511	537	564	590	617	643	669	696	722
165	21 748	21 775	21 801	21 827	21 854	21 880	21 906	21 932	21 958	21 985
166	22 011	22 037	22 063	22 089	22 115	22 141	22 167	22 194	22 220	22 246
167	272	298	324	350	376	401	427	453	479	505
168	531	557	583	608	634	660	686	712	737	763
169	789	814	840	866	891	917	943	968	994	23 019
170	23 045	23 070	23 096	23 121	23 147	23 172	23 198	23 223	23 249	23 274
171	300	325	350	376	401	426	452	477	502	528
172	553	578	603	629	654	679	704	729	754	779
173	805	830	855	880	905	930	955	980	24 005	24 030
174	24 055	24 080	24 105	24 130	24 155	24 180	24 204	24 229	254	279
175	24 304	24 329	24 353	24 378	24 403	24 428	24 452	24 477	24 502	24 527
176	551	576	601	625	650	674	699	724	748	773
177	797	822	846	871	895	920	944	969	993	25 018
178	25 042	25 066	25 091	25 115	25 139	25 164	25 188	25 212	25 237	261
179	285	310	334	358	382	406	431	455	479	503
180	25 527	25 551	25 575	25 600	25 624	25 648	25 672	25 696	25 720	25 744
181	768	792	816	840	864	888	912	935	959	983
182	26 007	26 031	26 055	26 079	26 102	26 126	26 150	26 174	26 198	26 221
183	245	269	293	316	340	364	387	411	435	458
184	482	505	529	553	576	600	623	647	670	694
185	26 717	26 741	26 764	26 788	26 811	26 834	26 858	26 881	26 905	26 928
186	951	975	998	27 021	27 045	27 068	27 091	27 114	27 138	27 161
187	27 184	27 207	27 231	254	277	300	323	346	370	393
188	416	439	462	485	508	531	554	577	600	623
189	646	669	692	715	738	761	784	807	830	852
190	27 875	27 898	27 921	27 944	27 967	27 989	28 012	28 035	28 058	28 081
191	28 103	28 126	28 149	28 171	28 194	28 217	240	262	285	307
192	330	353	375	398	421	443	466	488	511	533
193	556	578	601	623	646	668	691	713	735	758
194	780	803	825	847	870	892	914	937	959	981
195	29 003	29 026	29 048	29 070	29 092	29 115	29 137	29 159	29 181	29 203
196	226	248	270	292	314	336	358	380	403	425
197	447	469	491	513	535	557	579	601	623	645
198	667	688	710	732	754	776	798	820	842	863
199	885	907	929	951	973	994	30 016	30 038	30 060	30 081
200	30 103	30 125	30 146	30 168	30 190	30 211	30 233	30 255	30 276	30 298

Prop. Pts.

	28	29
1	2.8	2.9
2	5.6	5.8
3	8.4	8.7
4	11.2	11.6
5	14.0	14.5
6	16.8	17.4
7	19.6	20.3
8	22.4	23.2
9	25.2	26.1

	26	27
1	2.6	2.7
2	5.2	5.4
3	7.8	8.1
4	10.4	10.8
5	13.0	13.5
6	15.6	16.2
7	18.2	18.9
8	20.8	21.6
9	23.4	24.3

	24	25
1	2.4	2.5
2	4.8	5.0
3	7.2	7.5
4	9.6	10.0
5	12.0	12.5
6	14.4	15.0
7	16.8	17.5
8	19.2	20.0
9	21.6	22.5

	22	23
1	2.2	2.3
2	4.4	4.6
3	6.6	6.9
4	8.8	9.2
5	11.0	11.5
6	13.2	13.8
7	15.4	16.1
8	17.6	18.4
9	19.8	20.7

	21
1	2.1
2	4.2
3	6.3
4	8.4
5	10.5
6	12.6
7	14.7
8	16.8
9	18.9

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
200	30 103	30 125	30 146	30 168	30 190	30 211	30 233	30 255	30 276	30 298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	31 006	31 027	31 048	31 069	31 091	31 112	31 133	31 154
205	31 175	31 197	31 218	31 239	31 260	31 281	31 302	31 323	31 345	31 366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	32 035	32 056	32 077	32 098	32 118	32 139	32 160	32 181	32 201
210	32 222	32 243	32 263	32 284	32 305	32 325	32 346	32 366	32 387	32 408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	33 001	33 021
214	33 041	33 062	33 082	33 102	33 122	33 143	33 163	33 183	203	224
215	33 244	33 264	33 284	33 304	33 325	33 345	33 365	33 385	33 405	33 425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	34 005	34 025
219	34 044	34 064	34 084	34 104	34 124	34 143	34 163	34 183	203	223
220	34 242	34 262	34 282	34 301	34 321	34 341	34 361	34 380	34 400	34 420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	35 005
224	35 025	35 044	35 064	35 083	35 102	35 122	35 141	35 160	35 180	199
225	35 218	35 238	35 257	35 276	35 295	35 315	35 334	35 353	35 372	35 392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	36 003	36 021	36 040	36 059	36 078	36 097	36 116	36 135	36 154
230	36 173	36 192	36 211	36 229	36 248	36 267	36 286	36 305	36 324	36 342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	37 014	37 033	37 051	37 070	37 088
235	37 107	37 125	37 144	37 162	37 181	37 199	37 218	37 236	37 254	37 273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	38 003
240	38 021	38 039	38 057	38 075	38 093	38 112	38 130	38 148	38 166	38 184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	38 917	38 934	38 952	38 970	38 987	39 005	39 023	39 041	39 058	39 076
246	39 094	39 111	39 129	39 146	39 164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	39 794	39 811	39 829	39 846	39 863	39 881	39 898	39 915	39 933	39 950

Prop. Pts.

	21	22	20	19	18	17
1	2.1	2.2	2.0	1.9	1.8	1.7
2	4.2	4.4	4.0	3.8	3.6	3.4
3	6.3	6.6	6.0	5.7	5.4	5.1
4	8.4	8.8	8.0	7.6	7.2	6.8
5	10.5	11.0	10.0	9.5	9.0	8.5
6	12.6	13.2	12.0	11.4	10.8	10.2
7	14.7	15.4	14.0	13.3	12.6	11.9
8	16.8	17.6	16.0	15.2	14.4	13.6
9	18.9	19.8	18.0	17.1	16.2	15.3

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
250	39 794	39 811	39 829	39 846	39 863	39 881	39 898	39 915	39 933	39 950
251	967	985	40 002	40 019	40 037	40 054	40 071	40 088	40 106	40 123
252	40 140	40 157	175	192	209	226	243	261	278	295
253	312	329	346	364	381	398	415	432	449	466
254	483	500	518	535	552	569	586	603	620	637
255	40 654	40 671	40 688	40 705	40 722	40 739	40 756	40 773	40 790	40 807
256	824	841	858	875	892	909	926	943	960	976
257	993	41 010	41 027	41 044	41 061	41 078	41 095	41 111	41 128	41 145
258	41 162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481
260	41 497	41 514	41 531	41 547	41 564	41 581	41 597	41 614	41 631	41 647
261	664	681	697	714	731	747	764	780	797	814
262	830	847	863	880	896	913	929	946	963	979
263	996	42 012	42 029	42 045	42 062	42 078	42 095	42 111	42 127	42 144
264	42 160	177	193	210	226	243	259	275	292	308
265	42 325	42 341	42 357	42 374	42 390	42 406	42 423	42 439	42 455	42 472
266	488	504	521	537	553	570	586	602	619	635
267	651	667	684	700	716	732	749	765	781	797
268	813	830	846	862	878	894	911	927	943	959
269	975	991	43 008	43 024	43 040	43 056	43 072	43 088	43 104	43 120
270	43 136	43 152	43 169	43 185	43 201	43 217	43 233	43 249	43 265	43 281
271	297	313	329	345	361	377	393	409	425	441
272	457	473	489	505	521	537	553	569	584	600
273	616	632	648	664	680	696	712	727	743	759
274	775	791	807	823	838	854	870	886	902	917
275	43 933	43 949	43 965	43 981	43 996	44 012	44 028	44 044	44 059	44 075
276	44 091	44 107	44 122	44 138	44 154	170	185	201	217	232
277	248	264	279	295	311	326	342	358	373	389
278	404	420	436	451	467	483	498	514	529	545
279	560	576	592	607	623	638	654	669	685	700
280	44 716	44 731	44 747	44 762	44 778	44 793	44 809	44 824	44 840	44 855
281	871	886	902	917	932	948	963	979	994	45 010
282	45 025	45 040	45 056	45 071	45 086	45 102	45 117	45 133	45 148	163
283	179	194	209	225	240	255	271	286	301	317
284	332	347	362	378	393	408	423	439	454	469
285	45 484	45 500	45 515	45 530	45 545	45 561	45 576	45 591	45 606	45 621
286	637	652	667	682	697	712	728	743	758	773
287	788	803	818	834	849	864	879	894	909	924
288	939	954	969	984	46 000	46 015	46 030	46 045	46 060	46 075
289	46 090	46 105	46 120	46 135	150	165	180	195	210	225
290	46 240	46 255	46 270	46 285	46 300	46 315	46 330	46 345	46 359	46 374
291	389	404	419	434	449	464	479	494	509	523
292	538	553	568	583	598	613	627	642	657	672
293	687	702	716	731	746	761	776	790	805	820
294	835	850	864	879	894	909	923	938	953	967
295	46 982	46 997	47 012	47 026	47 041	47 056	47 070	47 085	47 100	47 114
296	47 129	47 144	159	173	188	202	217	232	246	261
297	276	290	305	319	334	349	363	378	392	407
298	422	436	451	465	480	494	509	524	538	553
299	567	582	596	611	625	640	654	669	683	698
300	47 712	47 727	47 741	47 756	47 770	47 784	47 799	47 813	47 828	47 842

Prop. Pts.

	18	17	16	15	14
1	1.8	1.7	1.6	1.5	1.4
2	3.6	3.4	3.2	3.0	2.8
3	5.4	5.1	4.8	4.5	4.2
4	7.2	6.8	6.4	6.0	5.6
5	9.0	8.5	8.0	7.5	7.0
6	10.8	10.2	9.6	9.0	8.4
7	12.6	11.9	11.2	10.5	9.8
8	14.4	13.6	12.8	12.0	11.2
9	16.2	15.3	14.4	13.5	12.6

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
300	47 712	47 727	47 741	47 756	47 770	47 784	47 799	47 813	47 828	47 842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	48 015	48 029	48 044	48 058	48 073	48 087	48 101	48 116	48 130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	48 430	48 444	48 458	48 473	48 487	48 501	48 515	48 530	48 544	48 558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	49 010	49 024	49 038	49 052	49 066	49 080	49 094	49 108	49 122
310	49 136	49 150	49 164	49 178	49 192	49 206	49 220	49 234	49 248	49 262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	49 831	49 845	49 859	49 872	49 886	49 900	49 914	49 927	49 941	49 955
316	969	982	996	50 010	50 024	50 037	50 051	50 065	50 079	50 092
317	50 106	50 120	50 133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	50 515	50 529	50 542	50 556	50 569	50 583	50 596	50 610	50 623	50 637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	51 001	51 014	51 028	51 041
324	51 055	51 068	51 081	51 095	51 108	51 121	135	148	162	175
325	51 188	51 202	51 215	51 228	51 242	51 255	51 268	51 282	51 295	51 308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	51 851	51 865	51 878	51 891	51 904	51 917	51 930	51 943	51 957	51 970
331	983	996	52 009	52 022	52 035	52 048	52 061	52 075	52 088	52 101
332	52 114	52 127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	52 504	52 517	52 530	52 543	52 556	52 569	52 582	52 595	52 608	52 621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	53 007
339	53 020	53 033	53 046	53 058	53 071	53 084	53 097	53 110	53 122	135
340	53 148	53 161	53 173	53 186	53 199	53 212	53 224	53 237	53 250	53 263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	53 782	53 794	53 807	53 820	53 832	53 845	53 857	53 870	53 882	53 895
346	908	920	933	945	958	970	983	995	54 008	54 020
347	54 033	54 045	54 058	54 070	54 083	54 095	54 108	54 120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	54 407	54 419	54 432	54 444	54 456	54 469	54 481	54 494	54 506	54 518

Prop. Pts.

	15	14	13	12
1	1.5	1.4	1.3	1.2
2	3.0	2.8	2.6	2.4
3	4.5	4.2	3.9	3.6
4	6.0	5.6	5.2	4.8
5	7.5	7.0	6.5	6.0
6	9.0	8.4	7.8	7.2
7	10.5	9.8	9.1	8.4
8	12.0	11.2	10.4	9.6
9	13.5	12.6	11.7	10.8

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
350	54 407	54 419	54 432	54 444	54 456	54 469	54 481	54 494	54 506	54 518
351	531	543	555	568	580	593	605	617	630	642
352	654	667	679	691	704	716	728	741	753	765
353	777	790	802	814	827	839	851	864	876	888
354	900	913	925	937	949	962	974	986	998	55 011
355	55 023	55 035	55 047	55 060	55 072	55 084	55 096	55 108	55 121	55 133
356	145	157	169	182	194	206	218	230	242	255
357	267	279	291	303	315	328	340	352	364	376
358	388	400	413	425	437	449	461	473	485	497
359	509	522	534	546	558	570	582	594	606	618
360	55 630	55 642	55 654	55 666	55 678	55 691	55 703	55 715	55 727	55 739
361	751	763	775	787	799	811	823	835	847	859
362	871	883	895	907	919	931	943	955	967	979
363	991	56 003	56 015	56 027	56 038	56 050	56 062	56 074	56 086	56 098
364	56 110	122	134	146	158	170	182	194	205	217
365	56 229	56 241	56 253	56 265	56 277	56 289	56 301	56 312	56 324	56 336
366	348	360	372	384	396	407	419	431	443	455
367	467	478	490	502	514	526	538	549	561	573
368	585	597	608	620	632	644	656	667	679	691
369	703	714	726	738	750	761	773	785	797	808
370	56 820	56 832	56 844	56 855	56 867	56 879	56 891	56 902	56 914	56 926
371	937	949	961	972	984	996	57 008	57 019	57 031	57 043
372	57 054	57 066	57 078	57 089	57 101	57 113	124	136	148	159
373	171	183	194	206	217	229	241	252	264	276
374	287	299	310	322	334	345	357	368	380	392
375	57 403	57 415	57 426	57 438	57 449	57 461	57 473	57 484	57 496	57 507
376	519	530	542	553	565	576	588	600	611	623
377	634	646	657	669	680	692	703	715	726	738
378	749	761	772	784	795	807	818	830	841	852
379	864	875	887	898	910	921	933	944	955	967
380	57 978	57 990	58 001	58 013	58 024	58 035	58 047	58 058	58 070	58 081
381	58 092	58 104	115	127	138	149	161	172	184	195
382	206	218	229	240	252	263	274	286	297	309
383	320	331	343	354	365	377	388	399	410	422
384	433	444	456	467	478	490	501	512	524	535
385	58 546	58 557	58 569	58 580	58 591	58 602	58 614	58 625	58 636	58 647
386	659	670	681	692	704	715	726	737	749	760
387	771	782	794	805	816	827	838	850	861	872
388	883	894	906	917	928	939	950	961	973	984
389	995	59 006	59 017	59 028	59 040	59 051	59 062	59 073	59 084	59 095
390	59 106	59 118	59 129	59 140	59 151	59 162	59 173	59 184	59 195	59 207
391	218	229	240	251	262	273	284	295	306	318
392	329	340	351	362	373	384	395	406	417	428
393	439	450	461	472	483	494	506	517	528	539
394	550	561	572	583	594	605	616	627	638	649
395	59 660	59 671	59 682	59 693	59 704	59 715	59 726	59 737	59 748	59 759
396	770	780	791	802	813	824	835	846	857	868
397	879	890	901	912	923	934	945	956	966	977
398	988	999	60 010	60 021	60 032	60 043	60 054	60 065	60 076	60 086
399	60 097	60 108	119	130	141	152	163	173	184	195
400	60 206	60 217	60 228	60 239	60 249	60 260	60 271	60 282	60 293	60 304

Prop. Pts.

	13	12	11	10
1	1.3	1.2	1.1	1.0
2	2.6	2.4	2 2	2.0
3	3.9	3.6	3.3	3 0
4	5.2	4.8	4.4	4.0
5	6.5	6.0	5.5	5.0
6	7.8	7.2	6.6	6.0
7	9.1	8.4	7.7	7.0
8	10.4	9.6	8.8	8.0
9	11.7	10.8	9 9	9.0

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
400	60 206	60 217	60 228	60 239	60 249	60 260	60 271	60 282	60 293	60 304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	60 746	60 756	60 767	60 778	60 788	60 799	60 810	60 821	60 831	60 842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	61 002	61 013	61 023	61 034	61 045	61 055
408	61 066	61 077	61 087	61 098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268
410	61 278	61 289	61 300	61 310	61 321	61 331	61 342	61 352	61 363	61 374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
414	700	711	721	731	742	752	763	773	784	794
415	61 805	61 815	61 826	61 836	61 847	61 857	61 868	61 878	61 888	61 899
416	909	920	930	941	951	962	972	982	993	62 003
417	62 014	62 024	62 034	62 045	62 055	62 066	62 076	62 086	62 097	107
418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
420	62 325	62 335	62 346	62 356	62 366	62 377	62 387	62 397	62 408	62 418
421	428	439	449	459	469	480	490	500	511	521
422	531	542	552	562	572	583	593	603	613	624
423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	62 839	62 849	62 859	62 870	62 880	62 890	62 900	62 910	62 921	62 931
426	941	951	961	972	982	992	63 002	63 012	63 022	63 033
427	63 043	63 053	63 063	63 073	63 083	63 094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
429	246	256	266	276	286	296	306	317	327	337
430	63 347	63 357	63 367	63 377	63 387	63 397	63 407	63 417	63 428	63 438
431	448	458	468	478	488	498	508	518	528	538
432	548	558	568	579	589	599	609	619	629	639
433	649	659	669	679	689	699	709	719	729	739
434	749	759	769	779	789	799	809	819	829	839
435	63 849	63 859	63 869	63 879	63 889	63 899	63 909	63 919	63 929	63 939
436	949	959	969	979	988	998	64 008	64 018	64 028	64 038
437	64 048	64 058	64 068	64 078	64 088	64 098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
439	246	256	266	276	286	296	306	316	326	335
440	64 345	64 355	64 365	64 375	64 385	64 395	64 404	64 414	64 424	64 434
441	444	454	464	473	483	493	503	513	523	532
442	542	552	562	572	582	591	601	611	621	631
443	640	650	660	670	680	689	699	709	719	729
444	738	748	758	768	777	787	797	807	816	826
445	64 836	64 846	64 856	64 865	64 875	64 885	64 895	64 904	64 914	64 924
446	933	943	953	963	972	982	992	65 002	65 011	65 021
447	65 031	65 040	65 050	65 060	65 070	65 079	65 089	099	108	118
448	128	137	147	157	167	176	186	196	205	215
449	225	234	244	254	263	273	283	292	302	312
450	65 321	65 331	65 341	65 350	65 360	65 369	65 379	65 389	65 398	65 408

Prop. Pts.

	11
1	1.1
2	2.2
3	3.3
4	4.4
5	5.5
6	6.6
7	7.7
8	8.8
9	9.9

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

	9
1	.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
450	65 321	65 331	65 341	65 350	65 360	65 369	65 379	65 389	65 398	65 408
451	418	427	437	447	456	466	475	485	495	504
452	514	523	533	543	552	562	571	581	591	600
453	610	619	629	639	648	658	667	677	686	696
454	706	715	725	734	744	753	763	772	782	792
455	65 801	65 811	65 820	65 830	65 839	65 849	65 858	65 868	65 877	65 887
456	896	906	916	925	935	944	954	963	973	982
457	992	66 001	66 011	66 020	66 030	66 039	66 049	66 058	66 068	66 077
458	66 087	096	106	115	124	134	143	153	162	172
459	181	191	200	210	219	229	238	247	257	266
460	66 276	66 285	66 295	66 304	66 314	66 323	66 332	66 342	66 351	66 361
461	370	380	389	398	408	417	427	436	445	455
462	464	474	483	492	502	511	521	530	539	549
463	558	567	577	586	596	605	614	624	633	642
464	652	661	671	680	689	699	708	717	727	736
465	66 745	66 755	66 764	66 773	66 783	66 792	66 801	66 811	66 820	66 829
466	839	848	857	867	876	885	894	904	913	922
467	932	941	950	960	969	978	987	997	67 006	67 015
468	67 025	67 034	67 043	67 052	67 062	67 071	67 080	67 089	099	108
469	117	127	136	145	154	164	173	182	191	201
470	67 210	67 219	67 228	67 237	67 247	67 256	67 265	67 274	67 284	67 293
471	302	311	321	330	339	348	357	367	376	385
472	394	403	413	422	431	440	449	459	468	477
473	486	495	504	514	523	532	541	550	560	569
474	578	587	596	605	614	624	633	642	651	660
475	67 669	67 679	67 688	67 697	67 706	67 715	67 724	67 733	67 742	67 752
476	761	770	779	788	797	806	815	825	834	843
477	852	861	870	879	888	897	906	916	925	934
478	943	952	961	970	979	988	997	68 006	68 015	68 024
479	68 034	68 043	68 052	68 061	68 070	68 079	68 088	097	106	115
480	68 124	68 133	68 142	68 151	68 160	68 169	68 178	68 187	68 196	68 205
481	215	224	233	242	251	260	269	278	287	296
482	305	314	323	332	341	350	359	368	377	386
483	395	404	413	422	431	440	449	458	467	476
484	485	494	502	511	520	529	538	547	556	565
485	68 574	68 583	68 592	68 601	68 610	68 619	68 628	68 637	68 646	68 655
486	664	673	681	690	699	708	717	726	735	744
487	753	762	771	780	789	797	806	815	824	833
488	842	851	860	869	878	886	895	904	913	922
489	931	940	949	958	966	975	984	993	69 002	69 011
490	69 020	69 028	69 037	69 046	69 055	69 064	69 073	69 082	69 090	69 099
491	108	117	126	135	144	152	161	170	179	188
492	197	205	214	223	232	241	249	258	267	276
493	285	294	302	311	320	329	338	346	355	364
494	373	381	390	399	408	417	425	434	443	452
495	69 461	69 469	69 478	69 487	69 496	69 504	69 513	69 522	69 531	69 539
496	548	557	566	574	583	592	601	609	618	627
497	636	644	653	662	671	679	688	697	705	714
498	723	732	740	749	758	767	775	784	793	801
499	810	819	827	836	845	854	862	871	880	888
500	69 897	69 906	69 914	69 923	69 932	69 940	69 949	69 958	69 966	69 975

Prop. Pts.

	10	9	8
1	1.0	0.9	0.8
2	2.0	1.8	1.6
3	3.0	2.7	2.4
4	4.0	3.6	3.2
5	5.0	4.5	4.0
6	6.0	5.4	4.8
7	7.0	6.3	5.6
8	8.0	7.2	6.4
9	9.0	8.1	7.2

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
500	69 897	69 906	69 914	69 923	69 932	69 940	69 949	69 958	69 966	69 975
501	984	992	70 001	70 010	70 018	70 027	70 036	70 044	70 053	70 062
502	70 070	70 079	088	096	105	114	122	131	140	148
503	157	165	174	183	191	200	209	217	226	234
504	243	252	260	269	278	286	295	303	312	321
505	70 329	70 338	70 346	70 355	70 364	70 372	70 381	70 389	70 398	70 406
506	415	424	432	441	449	458	467	475	484	492
507	501	509	518	526	535	544	552	561	569	578
508	586	595	603	612	621	629	638	646	655	663
509	672	680	689	697	706	714	723	731	740	749
510	70 757	70 766	70 774	70 783	70 791	70 800	70 808	70 817	70 825	70 834
511	842	851	859	868	876	885	893	902	910	919
512	927	935	944	952	961	969	978	986	995	71 003
513	71 012	71 020	71 029	71 037	71 046	71 054	71 063	71 071	71 079	088
514	096	105	113	122	130	139	147	155	164	172
515	71 181	71 189	71 198	71 206	71 214	71 223	71 231	71 240	71 248	71 257
516	265	273	282	290	299	307	315	324	332	341
517	349	357	366	374	383	391	399	408	416	425
518	433	441	450	458	466	475	483	492	500	508
519	517	525	533	542	550	559	567	575	584	592
520	71 600	71 609	71 617	71 625	71 634	71 642	71 650	71 659	71 667	71 675
521	684	692	700	709	717	725	734	742	750	759
522	767	775	784	792	800	809	817	825	834	842
523	850	858	867	875	883	892	900	908	917	925
524	933	941	950	958	966	975	983	991	999	72 008
525	72 016	72 024	72 032	72 041	72 049	72 057	72 066	72 074	72 082	72 090
526	099	107	115	123	132	140	148	156	165	173
527	181	189	198	206	214	222	230	239	247	255
528	263	272	280	288	296	304	313	321	329	337
529	346	354	362	370	378	387	395	403	411	419
530	72 428	72 436	72 444	72 452	72 460	72 469	72 477	72 485	72 493	72 501
531	509	518	526	534	542	550	558	567	575	583
532	591	599	607	616	624	632	640	648	656	665
533	673	681	689	697	705	713	722	730	738	746
534	754	762	770	779	787	795	803	811	819	827
535	72 835	72 843	72 852	72 860	72 868	72 876	72 884	72 892	72 900	72 908
536	916	925	933	941	949	957	965	973	981	989
537	997	73 006	73 014	73 022	73 030	73 038	73 046	73 054	73 062	73 070
538	73 078	086	094	102	111	119	127	135	143	151
539	159	167	175	183	191	199	207	215	223	231
540	73 239	73 247	73 255	73 263	73 272	73 280	73 288	73 296	73 304	73 312
541	320	328	336	344	352	360	368	376	384	392
542	400	408	416	424	432	440	448	456	464	472
543	480	488	496	504	512	520	528	536	544	552
544	560	568	576	584	592	600	608	616	624	632
545	73 640	73 648	73 656	73 664	73 672	73 679	73 687	73 695	73 703	73 711
546	719	727	735	743	751	759	767	775	783	791
547	799	807	815	823	830	838	846	854	862	870
548	878	886	894	902	910	918	926	933	941	949
549	957	965	973	981	989	997	74 005	74 013	74 020	74 028
550	74 036	74 044	74 052	74 060	74 068	74 076	74 084	74 092	74 099	74 107

Prop. Pts.

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
550	74 036	74 044	74 052	74 060	74 068	74 076	74 084	74 092	74 099	74 107
551	115	123	131	139	147	155	162	170	178	186
552	194	202	210	218	225	233	241	249	257	265
553	273	280	288	296	304	312	320	327	335	343
554	351	359	367	374	382	390	398	406	414	421
555	74 429	74 437	74 445	74 453	74 461	74 468	74 476	74 484	74 492	74 500
556	507	515	523	531	539	547	554	562	570	578
557	586	593	601	609	617	624	632	640	648	656
558	663	671	679	687	695	702	710	718	726	733
559	741	749	757	764	772	780	788	796	803	811
560	74 819	74 827	74 834	74 842	74 850	74 858	74 865	74 873	74 881	74 889
561	896	904	912	920	927	935	943	950	958	966
562	974	981	989	997	75 005	75 012	75 020	75 028	75 035	75 043
563	75 051	75 059	75 066	75 074	082	089	097	105	113	120
564	128	136	143	151	159	166	174	182	189	197
565	75 205	75 213	75 220	75 228	75 236	75 243	75 251	75 259	75 266	75 274
566	282	289	297	305	312	320	328	335	343	351
567	358	366	374	381	389	397	404	412	420	427
568	435	442	450	458	465	473	481	488	496	504
569	511	519	526	534	542	549	557	565	572	580
570	75 587	75 595	75 603	75 610	75 618	75 626	75 633	75 641	75 648	75 656
571	664	671	679	686	694	702	709	717	724	732
572	740	747	755	762	770	778	785	793	800	808
573	815	823	831	838	846	853	861	868	876	884
574	891	899	906	914	921	929	937	944	952	959
575	75 967	75 974	75 982	75 989	75 997	76 005	76 012	76 020	76 027	76 035
576	76 042	76 050	76 057	76 065	76 072	080	087	095	103	110
577	118	125	133	140	148	155	163	170	178	185
578	193	200	208	215	223	230	238	245	253	260
579	268	275	283	290	298	305	313	320	328	335
580	76 343	76 350	76 358	76 365	76 373	76 380	76 388	76 395	76 403	76 410
581	418	425	433	440	448	455	462	470	477	485
582	492	500	507	515	522	530	537	545	552	559
583	567	574	582	589	597	604	612	619	626	634
584	641	649	656	664	671	678	686	693	701	708
585	76 716	76 723	76 730	76 738	76 745	76 753	76 760	76 768	76 775	76 782
586	790	797	805	812	819	827	834	842	849	856
587	864	871	879	886	893	901	908	916	923	930
588	938	945	953	960	967	975	982	989	997	77 004
589	77 012	77 019	77 026	77 034	77 041	77 048	77 056	77 063	77 070	078
590	77 085	77 093	77 100	77 107	77 115	77 122	77 129	77 137	77 144	77 151
591	159	166	173	181	188	195	203	210	217	225
592	232	240	247	254	262	269	276	283	291	298
593	305	313	320	327	335	342	349	357	364	371
594	379	386	393	401	408	415	422	430	437	444
595	77 452	77 459	77 466	77 474	77 481	77 488	77 495	77 503	77 510	77 517
596	525	532	539	546	554	561	568	576	583	590
597	597	605	612	619	627	634	641	648	656	663
598	670	677	685	692	699	706	714	721	728	735
599	743	750	757	764	772	779	786	793	801	808
600	77 815	77 822	77 830	77 837	77 844	77 851	77 859	77 866	77 873	77 880

Prop. Pts.

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
600	77 815	77 822	77 830	77 837	77 844	77 851	77 859	77 866	77 873	77 880
601	887	895	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	78 003	78 010	78 017	78 025
603	78 032	78 039	78 046	78 053	78 061	78 068	075	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	78 176	78 183	78 190	78 197	78 204	78 211	78 219	78 226	78 233	78 240
606	247	254	262	269	276	283	290	297	305	312
607	319	326	333	340	347	355	362	369	376	383
608	390	398	405	412	419	426	433	440	447	455
609	462	469	476	483	490	497	504	512	519	526
610	78 533	78 540	78 547	78 554	78 561	78 569	78 576	78 583	78 590	78 597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	78 888	78 895	78 902	78 909	78 916	78 923	78 930	78 937	78 944	78 951
616	958	965	972	979	986	993	79 000	79 007	79 014	79 021
617	79 029	79 036	79 043	79 050	79 057	79 064	071	078	085	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	79 239	79 246	79 253	79 260	79 267	79 274	79 281	79 288	79 295	79 302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435	442
623	449	456	463	470	477	484	491	498	505	511
624	518	525	532	539	546	553	560	567	574	581
625	79 588	79 595	79 602	79 609	79 616	79 623	79 630	79 637	79 644	79 650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
630	79 934	79 941	79 948	79 955	79 962	79 969	79 975	79 982	79 989	79 996
631	80 003	80 010	80 017	80 024	80 030	80 037	80 044	80 051	80 058	80 065
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	80 277	80 284	80 291	80 298	80 305	80 312	80 318	80 325	80 332	80 339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611
640	80 618	80 625	80 632	80 638	80 645	80 652	80 659	80 665	80 672	80 679
641	686	693	699	706	713	720	726	733	740	747
642	754	760	767	774	781	787	794	801	808	814
643	821	828	835	841	848	855	862	868	875	882
644	889	895	902	909	916	922	929	936	943	949
645	80 956	80 963	80 969	80 976	80 983	80 990	80 996	81 003	81 010	81 017
646	81 023	81 030	81 037	81 043	81 050	81 057	81 064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245	251	258	265	271	278	285
650	81 291	81 298	81 305	81 311	81 318	81 325	81 331	81 338	81 345	81 351

Prop. Pts.

	8	7	6
1	0.8	0.7	0.6
2	1.6	1.4	1.2
3	2.4	2.1	1.8
4	3.2	2.8	2.4
5	4.0	3.5	3.0
6	4.8	4.2	3.6
7	5.6	4.9	4.2
8	6.4	5.6	4.8
9	7.2	6.3	5.4

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
650	81 291	81 298	81 305	81 311	81 318	81 325	81 331	81 338	81 345	81 351
651	358	365	371	378	385	391	398	405	411	418
652	425	431	438	445	451	458	465	471	478	485
653	491	498	505	511	518	525	531	538	544	551
654	558	564	571	578	584	591	598	604	611	617
655	81 624	81 631	81 637	81 644	81 651	81 657	81 664	81 671	81 677	81 684
656	690	697	704	710	717	723	730	737	743	750
657	757	763	770	776	783	790	796	803	809	816
658	823	829	836	842	849	856	862	869	875	882
659	889	895	902	908	915	921	928	935	941	948
660	81 954	81 961	81 968	81 974	81 981	81 987	81 994	82 000	82 007	82 014
661	82 020	82 027	82 033	82 040	82 046	82 053	82 060	066	073	079
662	086	092	099	105	112	119	125	132	138	145
663	151	158	164	171	178	184	191	197	204	210
664	217	223	230	236	243	249	256	263	269	276
665	82 282	82 289	82 295	82 302	82 308	82 315	82 321	82 328	82 334	82 341
666	347	354	360	367	373	380	387	393	400	406
667	413	419	426	432	439	445	452	458	465	471
668	478	484	491	497	504	510	517	523	530	536
669	543	549	556	562	569	575	582	588	595	601
670	82 607	82 614	82 620	82 627	82 633	82 640	82 646	82 653	82 659	82 666
671	672	679	685	692	698	705	711	718	724	730
672	737	743	750	756	763	769	776	782	789	795
673	802	808	814	821	827	834	840	847	853	860
674	866	872	879	885	892	898	905	911	918	924
675	82 930	82 937	82 943	82 950	82 956	82 963	82 969	82 975	82 982	82 988
676	995	83 001	83 008	83 014	83 020	83 027	83 033	83 040	83 046	83 052
677	83 059	065	072	078	085	091	097	104	110	117
678	123	129	136	142	149	155	161	168	174	181
679	187	193	200	206	213	219	225	232	238	245
680	83 251	83 257	83 264	83 270	83 276	83 283	83 289	83 296	83 302	83 308
681	315	321	327	334	340	347	353	359	366	372
682	378	385	391	398	404	410	417	423	429	436
683	442	448	455	461	467	474	480	487	493	499
684	506	512	518	525	531	537	544	550	556	563
685	83 569	83 575	83 582	83 588	83 594	83 601	83 607	83 613	83 620	83 626
686	632	639	645	651	658	664	670	677	683	689
687	696	702	708	715	721	727	734	740	746	753
688	759	765	771	778	784	790	797	803	809	816
689	822	828	835	841	847	853	860	866	872	879
690	83 885	83 891	83 897	83 904	83 910	83 916	83 923	83 929	83 935	83 942
691	948	954	960	967	973	979	985	992	998	84 004
692	84 011	84 017	84 023	84 029	84 036	84 042	84 048	84 055	84 061	067
693	073	080	086	092	098	105	111	117	123	130
694	136	142	148	155	161	167	173	180	186	192
695	84 198	84 205	84 211	84 217	84 223	84 230	84 236	84 242	84 248	84 255
696	261	267	273	280	286	292	298	305	311	317
697	323	330	336	342	348	354	361	367	373	379
698	386	392	398	404	410	417	423	429	435	442
699	448	454	460	466	473	479	485	491	497	504
700	84 510	84 516	84 522	84 528	84 535	84 541	84 547	84 553	84 559	84 566

Prop. Pts.

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
700	84 510	84 516	84 522	84 528	84 535	84 541	84 547	84 553	84 559	84 566
701	572	578	584	590	597	603	609	615	621	628
702	634	640	646	652	658	665	671	677	683	689
703	696	702	708	714	720	726	733	739	745	751
704	757	763	770	776	782	788	794	800	807	813
705	84 819	84 825	84 831	84 837	84 844	84 850	84 856	84 862	84 868	84 874
706	880	887	893	899	905	911	917	924	930	936
707	942	948	954	960	967	973	979	985	991	997
708	85 003	85 009	85 016	85 022	85 028	85 034	85 040	85 046	85 052	85 058
709	065	071	077	083	089	095	101	107	114	120
710	85 126	85 132	85 138	85 144	85 150	85 156	85 163	85 169	85 175	85 181
711	187	193	199	205	211	217	224	230	236	242
712	248	254	260	266	272	278	285	291	297	303
713	309	315	321	327	333	339	345	352	358	364
714	370	376	382	388	394	400	406	412	418	425
715	85 431	85 437	85 443	85 449	85 455	85 461	85 467	85 473	85 479	85 485
716	491	497	503	509	516	522	528	534	540	546
717	552	558	564	570	576	582	588	594	600	606
718	612	618	625	631	637	643	649	655	661	667
719	673	679	685	691	697	703	709	715	721	727
720	85 733	85 739	85 745	85 751	85 757	85 763	85 769	85 775	85 781	85 788
721	794	800	806	812	818	824	830	836	842	848
722	854	860	866	872	878	884	890	896	902	908
723	914	920	926	932	938	944	950	956	962	968
724	974	980	986	992	998	86 004	86 010	86 016	86 022	86 028
725	86 034	86 040	86 046	86 052	86 058	86 064	86 070	86 076	86 082	86 088
726	094	100	106	112	118	124	130	136	141	147
727	153	159	165	171	177	183	189	195	201	207
728	213	219	225	231	237	243	249	255	261	267
729	273	279	285	291	297	303	308	314	320	326
730	86 332	86 338	86 344	86 350	86 356	86 362	86 368	86 374	86 380	86 386
731	392	398	404	410	415	421	427	433	439	445
732	451	457	463	469	475	481	487	493	499	504
733	510	516	522	528	534	540	546	552	558	564
734	570	576	581	587	593	599	605	611	617	623
735	86 629	86 635	86 641	86 646	86 652	86 658	86 664	86 670	86 676	86 682
736	688	694	700	705	711	717	723	729	735	741
737	747	753	759	764	770	776	782	788	794	800
738	806	812	817	823	829	835	841	847	853	859
739	864	870	876	882	888	894	900	906	911	917
740	86 923	86 929	86 935	86 941	86 947	86 953	86 958	86 964	86 970	86 976
741	982	988	994	999	87 005	87 011	87 017	87 023	87 029	87 035
742	87 040	87 046	87 052	87 058	064	070	075	081	087	093
743	099	105	111	116	122	128	134	140	146	151
744	157	163	169	175	181	186	192	198	204	210
745	87 216	87 221	87 227	87 233	87 239	87 245	87 251	87 256	87 262	87 268
746	274	280	286	291	297	303	309	315	320	326
747	332	338	344	349	355	361	367	373	379	384
748	390	396	402	408	413	419	425	431	437	442
749	448	454	460	466	471	477	483	489	495	500
750	87 506	87 512	87 518	87 523	87 529	87 535	87 541	87 547	87 552	87 558

Prop. Pts.

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
750	87 506	87 512	87 518	87 523	87 529	87 535	87 541	87 547	87 552	87 558
751	564	570	576	581	587	593	599	604	610	616
752	622	628	633	639	645	651	656	662	668	674
753	679	685	691	697	703	708	714	720	726	731
754	737	743	749	754	760	766	772	777	783	789
755	87 795	87 800	87 806	87 812	87 818	87 823	87 829	87 835	87 841	87 846
756	852	858	864	869	875	881	887	892	898	904
757	910	915	921	927	933	938	944	950	955	961
758	967	973	978	984	990	996	88 001	88 007	88 013	88 018
759	88 024	88 030	88 036	88 041	88 047	88 053	058	064	070	076
760	88 081	88 087	88 093	88 098	88 104	88 110	88 116	88 121	88 127	88 133
761	138	144	150	156	161	167	173	178	184	190
762	195	201	207	213	218	224	230	235	241	247
763	252	258	264	270	275	281	287	292	298	304
764	309	315	321	326	332	338	343	349	355	360
765	88 366	88 372	88 377	88 383	88 389	88 395	88 400	88 406	88 412	88 417
766	423	429	434	440	446	451	457	463	468	474
767	480	485	491	497	502	508	513	519	525	530
768	536	542	547	553	559	564	570	576	581	587
769	593	598	604	610	615	621	627	632	638	643
770	88 649	88 655	88 660	88 666	88 672	88 677	88 683	88 689	88 694	88 700
771	705	711	717	722	728	734	739	745	750	756
772	762	767	773	779	784	790	795	801	807	812
773	818	824	829	835	840	846	852	857	863	868
774	874	880	885	891	897	902	908	913	919	925
775	88 930	88 936	88 941	88 947	88 953	88 958	88 964	88 969	88 975	88 981
776	986	992	997	89 003	89 009	89 014	89 020	89 025	89 031	89 037
777	89 042	89 048	89 053	059	064	070	076	081	087	092
778	098	104	109	115	120	126	131	137	143	148
779	154	159	165	170	176	182	187	193	198	204
780	89 209	89 215	89 221	89 226	89 232	89 237	89 243	89 248	89 254	89 260
781	265	271	276	282	287	293	298	304	310	315
782	321	326	332	337	343	348	354	360	365	371
783	376	382	387	393	398	404	409	415	421	426
784	432	437	443	448	454	459	465	470	476	481
785	89 487	89 492	89 498	89 504	89 509	89 515	89 520	89 526	89 531	89 537
786	542	548	553	559	564	570	575	581	586	592
787	597	603	609	614	620	625	631	636	642	647
788	653	658	664	669	675	680	686	691	697	702
789	708	713	719	724	730	735	741	746	752	757
790	89 763	89 768	89 774	89 779	89 785	89 790	89 796	89 801	89 807	89 812
791	818	823	829	834	840	845	851	856	862	867
792	873	878	883	889	894	900	905	911	916	922
793	927	933	938	944	949	955	960	966	971	977
794	982	988	993	998	90 004	90 009	90 015	90 020	90 026	90 031
795	90 037	90 042	90 048	90 053	90 059	90 064	90 069	90 075	90 080	90 086
796	091	097	102	108	113	119	124	129	135	140
797	146	151	157	162	168	173	179	184	189	195
798	200	206	211	217	222	227	233	238	244	249
799	255	260	266	271	276	282	287	293	298	304
800	90 309	90 314	90 320	90 325	90 331	90 336	90 342	90 347	90 352	90 358

Prop. Pts.

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
800	90 309	90 314	90 320	90 325	90 331	90 336	90 342	90 347	90 352	90 358
801	363	369	374	380	385	390	396	401	407	412
802	417	423	428	434	439	445	450	455	461	466
803	472	477	482	488	493	499	504	509	515	520
804	526	531	536	542	547	553	558	563	569	574
805	90 580	90 585	90 590	90 596	90 601	90 607	90 612	90 617	90 623	90 628
806	634	639	644	650	655	660	666	671	677	682
807	687	693	698	703	709	714	720	725	730	736
808	741	747	752	757	763	768	773	779	784	789
809	795	800	806	811	816	822	827	832	838	843
810	90 849	90 854	90 859	90 865	90 870	90 875	90 881	90 886	90 891	90 897
811	902	907	913	918	924	929	934	940	945	950
812	956	961	966	972	977	982	988	993	998	91 004
813	91 009	91 014	91 020	91 025	91 030	91 036	91 041	91 046	91 052	057
814	062	068	073	078	084	089	094	100	105	110
815	91 116	91 121	91 126	91 132	91 137	91 142	91 148	91 153	91 158	91 164
816	169	174	180	185	190	196	201	206	212	217
817	222	228	233	238	243	249	254	259	265	270
818	275	281	286	291	297	302	307	312	318	323
819	328	334	339	344	350	355	360	365	371	376
820	91 381	91 387	91 392	91 397	91 403	91 408	91 413	91 418	91 424	91 429
821	434	440	445	450	455	461	466	471	477	482
822	487	492	498	503	508	514	519	524	529	535
823	540	545	551	556	561	566	572	577	582	587
824	593	598	603	609	614	619	624	630	635	640
825	91 645	91 651	91 656	91 661	91 666	91 672	91 677	91 682	91 687	91 693
826	698	703	709	714	719	724	730	735	740	745
827	751	756	761	766	772	777	782	787	793	798
828	803	808	814	819	824	829	834	840	845	850
829	855	861	866	871	876	882	887	892	897	903
830	91 908	91 913	91 918	91 924	91 929	91 934	91 939	91 944	91 950	91 955
831	960	965	971	976	981	986	991	997	92 002	92 007
832	92 012	92 018	92 023	92 028	92 033	92 038	92 044	92 049	054	059
833	065	070	075	080	085	091	096	101	106	111
834	117	122	127	132	137	143	148	153	158	163
835	92 169	92 174	92 179	92 184	92 189	92 195	92 200	92 205	92 210	92 215
836	221	226	231	236	241	247	252	257	262	267
837	273	278	283	288	293	298	304	309	314	319
838	324	330	335	340	345	350	355	361	366	371
839	376	381	387	392	397	402	407	412	418	423
840	92 428	92 433	92 438	92 443	92 449	92 454	92 459	92 464	92 469	92 474
841	480	485	490	495	500	505	511	516	521	526
842	531	536	542	547	552	557	562	567	572	578
843	583	588	593	598	603	609	614	619	624	629
844	634	639	645	650	655	660	665	670	675	681
845	92 686	92 691	92 696	92 701	92 706	92 711	92 716	92 722	92 727	92 732
846	737	742	747	752	758	763	768	773	778	783
847	788	793	799	804	809	814	819	824	829	834
848	840	845	850	855	860	865	870	875	881	886
849	891	896	901	906	911	916	921	927	932	937
850	92 942	92 947	92 952	92 957	92 962	92 967	92 973	92 978	92 983	92 988

Prop. Pts.

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
850	92 942	92 947	92 952	92 957	92 962	92 967	92 973	92 978	92 983	92 988
851	993	998	93 003	93 008	93 013	93 018	93 024	93 029	93 034	93 039
852	93 044	93 049	054	059	064	069	075	080	085	090
853	095	100	105	110	115	120	125	131	136	141
854	146	151	156	161	166	171	176	181	186	192
855	93 197	93 202	93 207	93 212	93 217	93 222	93 227	93 232	93 237	93 242
856	247	252	258	263	268	273	278	283	288	293
857	298	303	308	313	318	323	328	334	339	344
858	349	354	359	364	369	374	379	384	389	394
859	399	404	409	414	420	425	430	435	440	445
860	93 450	93 455	93 460	93 465	93 470	93 475	93 480	93 485	93 490	93 495
861	500	505	510	515	520	526	531	536	541	546
862	551	556	561	566	571	576	581	586	591	596
863	601	606	611	616	621	626	631	636	641	646
864	651	656	661	666	671	676	682	687	692	697
865	93 702	93 707	93 712	93 717	93 722	93 727	93 732	93 737	93 742	93 747
866	752	757	762	767	772	777	782	787	792	797
867	802	807	812	817	822	827	832	837	842	847
868	852	857	862	867	872	877	882	887	892	897
869	902	907	912	917	922	927	932	937	942	947
870	93 952	93 957	93 962	93 967	93 972	93 977	93 982	93 987	93 992	93 997
871	94 002	94 007	94 012	94 017	94 022	94 027	94 032	94 037	94 042	94 047
872	052	057	062	067	072	077	082	086	091	096
873	101	106	111	116	121	126	131	136	141	146
874	151	156	161	166	171	176	181	186	191	196
875	94 201	94 206	94 211	94 216	94 221	94 226	94 231	94 236	94 240	94 245
876	250	255	260	265	270	275	280	285	290	295
877	300	305	310	315	320	325	330	335	340	345
878	349	354	359	364	369	374	379	384	389	394
879	399	404	409	414	419	424	429	433	438	443
880	94 448	94 453	94 458	94 463	94 468	94 473	94 478	94 483	94 488	94 493
881	498	503	507	512	517	522	527	532	537	542
882	547	552	557	562	567	571	576	581	586	591
883	596	601	606	611	616	621	626	630	635	640
884	645	650	655	660	665	670	675	680	685	689
885	94 694	94 699	94 704	94 709	94 714	94 719	94 724	94 729	94 734	94 738
886	743	748	753	758	763	768	773	778	783	787
887	792	797	802	807	812	817	822	827	832	836
888	841	846	851	856	861	866	871	876	880	885
889	890	895	900	905	910	915	919	924	929	934
890	94 939	94 944	94 949	94 954	94 959	94 963	94 968	94 973	94 978	94 983
891	988	993	998	95 002	95 007	95 012	95 017	95 022	95 027	95 032
892	95 036	95 041	95 046	051	056	061	066	071	075	080
893	085	090	095	100	105	109	114	119	124	129
894	134	139	143	148	153	158	163	168	173	177
895	95 182	95 187	95 192	95 197	95 202	95 207	95 211	95 216	95 221	95 226
896	231	236	240	245	250	255	260	265	270	274
897	279	284	289	294	299	303	308	313	318	323
898	328	332	337	342	347	352	357	361	366	371
899	376	381	386	390	395	400	405	410	415	419
900	95 424	95 429	95 434	95 439	95 444	95 448	95 453	95 458	95 463	95 468

Prop. Pts.

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
900	95 424	95 429	95 434	95 439	95 444	95 448	95 453	95 458	95 463	95 468
901	472	477	482	487	492	497	501	506	511	516
902	521	525	530	535	540	545	550	554	559	564
903	569	574	578	583	588	593	598	602	607	612
904	617	622	626	631	636	641	646	650	655	660
905	95 665	95 670	95 674	95 679	95 684	95 689	95 694	95 698	95 703	95 708
906	713	718	722	727	732	737	742	746	751	756
907	761	766	770	775	780	785	789	794	799	804
908	809	813	818	823	828	832	837	842	847	852
909	856	861	866	871	875	880	885	890	895	899
910	95 904	95 909	95 914	95 918	95 923	95 928	95 933	95 938	95 942	95 947
911	952	957	961	966	971	976	980	985	990	995
912	999	96 004	96 009	96 014	96 019	96 023	96 028	96 033	96 038	96 042
913	96 047	052	057	061	066	071	076	080	085	090
914	095	099	104	109	114	118	123	128	133	137
915	96 142	96 147	96 152	96 156	96 161	96 166	96 171	96 175	96 180	96 185
916	190	194	199	204	209	213	218	223	227	232
917	237	242	246	251	256	261	265	270	275	280
918	284	289	294	298	303	308	313	317	322	327
919	332	336	341	346	350	355	360	365	369	374
920	96 379	96 384	96 388	96 393	96 398	96 402	96 407	96 412	96 417	96 421
921	426	431	435	440	445	450	454	459	464	468
922	473	478	483	487	492	497	501	506	511	515
923	520	525	530	534	539	544	548	553	558	562
924	567	572	577	581	586	591	595	600	605	609
925	96 614	96 619	96 624	96 628	96 633	96 638	96 642	96 647	96 652	96 656
926	661	666	670	675	680	685	689	694	699	703
927	708	713	717	722	727	731	736	741	745	750
928	755	759	764	769	774	778	783	788	792	797
929	802	806	811	816	820	825	830	834	839	844
930	96 848	96 853	96 858	96 862	96 867	96 872	96 876	96 881	96 886	96 890
931	895	900	904	909	914	918	923	928	932	937
932	942	946	951	956	960	965	970	974	979	984
933	988	993	997	97 002	97 007	97 011	97 016	97 021	97 025	97 030
934	97 035	97 039	97 044	049	053	058	063	067	072	077
935	97 081	97 086	97 090	97 095	97 100	97 104	97 109	97 114	97 118	97 123
936	128	132	137	142	146	151	155	160	165	169
937	174	179	183	188	192	197	202	206	211	216
938	220	225	230	234	239	243	248	253	257	262
939	267	271	276	280	285	290	294	299	304	308
940	97 313	97 317	97 322	97 327	97 331	97 336	97 340	97 345	97 350	97 354
941	359	364	368	373	377	382	387	391	396	400
942	405	410	414	419	424	428	433	437	442	447
943	451	456	460	465	470	474	479	483	488	493
944	497	502	506	511	516	520	525	529	534	539
945	97 543	97 548	97 552	97 557	97 562	97 566	97 571	97 575	97 580	97 585
946	589	594	598	603	607	612	617	621	626	630
947	635	640	644	649	653	658	663	667	672	676
948	681	685	690	695	699	704	708	713	717	722
949	727	731	736	740	745	749	754	759	763	768
950	97 772	97 777	97 782	97 786	97 791	97 795	97 800	97 804	97 809	97 813

Prop. Pts.

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

TABLE I

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
950	97 772	97 777	97 782	97 786	97 791	97 795	97 800	97 804	97 809	97 813
951	818	823	827	832	836	841	845	850	855	859
952	864	868	873	877	882	886	891	896	900	905
953	909	914	918	923	928	932	937	941	946	950
954	955	959	964	968	973	978	982	987	991	996
955	98 000	98 005	98 009	98 014	98 019	98 023	98 028	98 032	98 037	98 041
956	046	050	055	059	064	068	073	078	082	087
957	091	096	100	105	109	114	118	123	127	132
958	137	141	146	150	155	159	164	168	173	177
959	182	186	191	195	200	204	209	214	218	223
960	98 227	98 232	98 236	98 241	98 245	98 250	98 254	98 259	98 263	98 268
961	272	277	281	286	290	295	299	304	308	313
962	318	322	327	331	336	340	345	349	354	358
963	363	367	372	376	381	385	390	394	399	403
964	408	412	417	421	426	430	435	439	444	448
965	98 453	98 457	98 462	98 466	98 471	98 475	98 480	98 484	98 489	98 493
966	498	502	507	511	516	520	525	529	534	538
967	543	547	552	556	561	565	570	574	579	583
968	588	592	597	601	605	610	614	619	623	628
969	632	637	641	646	650	655	659	664	668	673
970	98 677	98 682	98 686	98 691	98 695	98 700	98 704	98 709	98 713	98 717
971	722	726	731	735	740	744	749	753	758	762
972	767	771	776	780	784	789	793	798	802	807
973	811	816	820	825	829	834	838	843	847	851
974	856	860	865	869	874	878	883	887	892	896
975	98 900	98 905	98 909	98 914	98 918	98 923	98 927	98 932	98 936	98 941
976	945	949	954	958	963	967	972	976	981	985
977	989	994	998	99 003	99 007	99 012	99 016	99 021	99 025	99 029
978	99 034	99 038	99 043	047	052	056	061	065	069	074
979	078	083	087	092	096	100	105	109	114	118
980	99 123	99 127	99 131	99 136	99 140	99 145	99 149	99 154	99 158	99 162
981	167	171	176	180	185	189	193	198	202	207
982	211	216	220	224	229	233	238	242	247	251
983	255	260	264	269	273	277	282	286	291	295
984	300	304	308	313	317	322	326	330	335	339
985	99 344	99 348	99 352	99 357	99 361	99 366	99 370	99 374	99 379	99 383
986	388	392	396	401	405	410	414	419	423	427
987	432	436	441	445	449	454	458	463	467	471
988	476	480	484	489	493	498	502	506	511	515
989	520	524	528	533	537	542	546	550	555	559
990	99 564	99 568	99 572	99 577	99 581	99 585	99 590	99 594	99 599	99 603
991	607	612	616	621	625	629	634	638	642	647
992	651	656	660	664	669	673	677	682	686	691
993	695	699	704	708	712	717	721	726	730	734
994	739	743	747	752	756	760	765	769	774	778
995	99 782	99 787	99 791	99 795	99 800	99 804	99 808	99 813	99 817	99 822
996	826	830	835	839	843	848	852	856	861	865
997	870	874	878	883	887	891	896	900	904	909
998	913	917	922	926	930	935	939	944	948	952
999	957	961	965	970	974	978	983	987	991	996
1000	00 000	00 004	00 009	00 013	00 017	00 022	00 026	00 030	00 035	00 039

Prop. Pts.

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

TABLE II

TRIGONOMETRIC FUNCTIONS AND THEIR LOGARITHMS

ANGLE		Sine		Tangent		Cotangent		Cosine			
Radians	Degrees	Value	Log	Value	Log	Value	Log	Value	Log		
.0000	0°00′	.0000	—	.0000	—	—	—	1.0000	.0000	90°00′	1.5708
.0029	10	.0029	.4637	.0029	.4637	343.77	.5363	1.0000	.0000	50	1.5679
.0058	20	.0058	.7648	.0058	.7648	171.89	.2352	1.0000	.0000	40	1.5650
.0087	30	.0087	.9408	.0087	.9409	114.59	.0591	1.0000	.0000	30	1.5621
.0116	40	.0116	.0658	.0116	.0658	85.940	.9342	.9999	.0000	20	1.5592
.0145	50	.0145	.1627	.0145	.1627	68.750	.8373	.9999	.0000	10	1.5563
.0175	1°00′	.0175	.2419	.0175	.2419	57.290	.7581	.9998	.9999	89°00′	1.5533
.0204	10	.0204	.3088	.0204	.3089	49.104	.6911	.9998	.9999	50	1.5504
.0233	20	.0233	.3668	.0233	.3669	42.964	.6331	.9997	.9999	40	1.5475
.0262	30	.0262	.4179	.0262	.4181	38.188	.5819	.9997	.9999	30	1.5446
.0291	40	.0291	.4637	.0291	.4638	34.368	.5362	.9996	.9998	20	1.5417
.0320	50	.0320	.5050	.0320	.5053	31.242	.4947	.9995	.9998	10	1.5388
.0349	2°00′	.0349	.5428	.0349	.5431	28.636	.4569	.9994	.9997	88°00′	1.5359
.0378	10	.0378	.5776	.0378	.5779	26.432	.4221	.9993	.9997	50	1.5330
.0407	20	.0407	.6097	.0407	.6101	24.542	.3899	.9992	.9996	40	1.5301
.0436	30	.0436	.6397	.0437	.6401	22.904	.3599	.9990	.9996	30	1.5272
.0465	40	.0465	.6677	.0466	.6682	21.470	.3318	.9989	.9995	20	1.5243
.0495	50	.0494	.6940	.0495	.6945	20.206	.3055	.9988	.9995	10	1.5213
.0524	3°00′	.0523	.7188	.0524	.7194	19.081	.2806	.9986	.9994	87°00′	1.5184
.0553	10	.0552	.7423	.0553	.7429	18.075	.2571	.9985	.9993	50	1.5155
.0582	20	.0581	.7645	.0582	.7652	17.169	.2348	.9983	.9993	40	1.5126
.0611	30	.0610	.7857	.0612	.7865	16.350	.2135	.9981	.9992	30	1.5097
.0640	40	.0640	.8059	.0641	.8067	15.605	.1933	.9980	.9991	20	1.5068
.0669	50	.0669	.8251	.0670	.8261	14.924	.1739	.9978	.9990	10	1.5039
.0698	4°00′	.0698	.8436	.0699	.8446	14.301	.1554	.9976	.9989	86°00′	1.5010
.0727	10	.0727	.8613	.0729	.8624	13.727	.1376	.9974	.9989	50	1.4981
.0756	20	.0756	.8783	.0758	.8795	13.197	.1205	.9971	.9988	40	1.4952
.0785	30	.0785	.8946	.0787	.8960	12.706	.1040	.9969	.9987	30	1.4923
.0814	40	.0814	.9104	.0816	.9118	12.251	.0882	.9967	.9986	20	1.4893
.0844	50	.0843	.9256	.0846	.9272	11.826	.0728	.9964	.9985	10	1.4864
.0873	5°00′	.0872	.9403	.0875	.9420	11.430	.0580	.9962	.9983	85°00′	1.4835
.0902	10	.0901	.9545	.0904	.9563	11.059	.0437	.9959	.9982	50	1.4806
.0931	20	.0929	.9682	.0934	.9701	10.712	.0299	.9957	.9981	40	1.4777
.0960	30	.0958	.9816	.0963	.9836	10.385	.0164	.9954	.9980	30	1.4748
.0989	40	.0987	.9945	.0992	.9966	10.078	.0034	.9951	.9979	20	1.4719
.1018	50	.1016	.0070	.1022	.0093	9.7882	.9907	.9948	.9977	10	1.4690
.1047	6°00′	.1045	.0192	.1051	.0216	9.5144	.9784	.9945	.9976	84°00′	1.4661
.1076	10	.1074	.0311	.1080	.0336	9.2553	.9664	.9942	.9975	50	1.4632
.1105	20	.1103	.0426	.1110	.0453	9.0098	.9547	.9939	.9973	40	1.4603
.1134	30	.1132	.0539	.1139	.0567	8.7769	.9433	.9936	.9972	30	1.4573
.1164	40	.1161	.0648	.1169	.0678	8.5555	.9322	.9932	.9971	20	1.4544
.1193	50	.1190	.0755	.1198	.0786	8.3450	.9214	.9929	.9969	10	1.4515
.1222	7°00′	.1219	.0859	.1228	.0891	8.1443	.9109	.9925	.9968	83°00′	1.4486
.1251	10	.1248	.0961	.1257	.0995	7.9530	.9005	.9922	.9966	50	1.4457
.1280	20	.1276	.1060	.1287	.1096	7.7704	.8904	.9918	.9964	40	1.4428
.1309	30	.1305	.1157	.1317	.1194	7.5958	.8806	.9914	.9963	30	1.4399
.1338	40	.1334	.1252	.1346	.1291	7.4287	.8709	.9911	.9961	20	1.4370
.1367	50	.1363	.1345	.1376	.1385	7.2687	.8615	.9907	.9959	10	1.4341
.1396	8°00′	.1392	.1436	.1405	.1478	7.1154	.8522	.9903	.9958	82°00′	1.4312
.1425	10	.1421	.1525	.1435	.1569	6.9682	.8431	.9899	.9956	50	1.4283
.1454	20	.1449	.1612	.1465	.1658	6.8269	.8342	.9894	.9954	40	1.4254
.1484	30	.1478	.1697	.1495	.1745	6.6912	.8255	.9890	.9952	30	1.4224
.1513	40	.1507	.1781	.1524	.1831	6.5606	.8169	.9886	.9950	20	1.4195
.1542	50	.1536	.1863	.1554	.1915	6.4348	.8085	.9881	.9948	10	1.4166
.1571	9°00′	.1564	.1943	.1584	.1997	6.3138	.8003	.9877	.9946	81°00′	1.4137
		Value	Log	Value	Log	Value	Log	Value	Log	Degrees	Radians
		Cosine		Cotangent		Tangent		Sine			

TABLE II
TRIGONOMETRIC FUNCTIONS AND THEIR LOGARITHMS

ANGLE Radians	ANGLE Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log		
.1571	9°00′	.1564	.1943	.1584	.1997	6.3138	.8003	.9877	.9946	81°00′	1.4137
.1600	10	.1593	.2022	.1614	.2078	6.1970	.7922	.9872	.9944	50	1.4108
.1629	20	.1622	.2100	.1644	.2158	6.0844	.7842	.9868	.9942	40	1.4079
.1658	30	.1650	.2176	.1673	.2236	5.9758	.7764	.9863	.9940	30	1.4050
.1687	40	.1679	.2251	.1703	.2313	5.8708	.7687	.9858	.9938	20	1.4021
.1716	50	.1708	.2324	.1733	.2389	5.7694	.7611	.9853	.9936	10	1.3992
.1745	10°00′	.1736	.2397	.1763	.2463	5.6713	.7537	.9848	.9934	80°00′	1.3963
.1774	10	.1765	.2468	.1793	.2536	5.5764	.7464	.9843	.9931	50	1.3934
.1804	20	.1794	.2538	.1823	.2609	5.4845	.7391	.9838	.9929	40	1.3904
.1833	30	.1822	.2606	.1853	.2680	5.3955	.7320	.9833	.9927	30	1.3875
.1862	40	.1851	.2674	.1883	.2750	5.3093	.7250	.9827	.9924	20	1.3846
.1891	50	.1880	.2740	.1914	.2819	5.2257	.7181	.9822	.9922	10	1.3817
.1920	11°00′	.1908	.2806	.1944	.2887	5.1446	.7113	.9816	.9919	79°00′	1.3788
.1949	10	.1937	.2870	.1974	.2953	5.0658	.7047	.9811	.9917	50	1.3759
.1978	20	.1965	.2934	.2004	.3020	4.9894	.6980	.9805	.9914	40	1.3730
.2007	30	.1994	.2997	.2035	.3085	4.9152	.6915	.9799	.9912	30	1.3701
.2036	40	.2022	.3058	.2065	.3149	4.8430	.6851	.9793	.9909	20	1.3672
.2065	50	.2051	.3119	.2095	.3212	4.7729	.6788	.9787	.9907	10	1.3643
.2094	12°00′	.2079	.3179	.2126	.3275	4.7046	.6725	.9781	.9904	78°00′	1.3614
.2123	10	.2108	.3238	.2156	.3336	4.6382	.6664	.9775	.9901	50	1.3584
.2153	20	.2136	.3296	.2186	.3397	4.5736	.6603	.9769	.9899	40	1.3555
.2182	30	.2164	.3353	.2217	.3458	4.5107	.6542	.9763	.9896	30	1.3526
.2211	40	.2193	.3410	.2247	.3517	4.4494	.6483	.9757	.9893	20	1.3497
.2240	50	.2221	.3466	.2278	.3576	4.3897	.6424	.9750	.9890	10	1.3468
.2269	13°00′	.2250	.3521	.2309	.3634	4.3315	.6366	.9744	.9887	77°00′	1.3439
.2298	10	.2278	.3575	.2339	.3691	4.2747	.6309	.9737	.9884	50	1.3410
.2327	20	.2306	.3629	.2370	.3748	4.2193	.6252	.9730	.9881	40	1.3381
.2356	30	.2334	.3682	.2401	.3804	4.1653	.6196	.9724	.9878	30	1.3352
.2385	40	.2363	.3734	.2432	.3859	4.1126	.6141	.9717	.9875	20	1.3323
.2414	50	.2391	.3786	.2462	.3914	4.0611	.6086	.9710	.9872	10	1.3294
.2443	14°00′	.2419	.3837	.2493	.3968	4.0108	.6032	.9703	.9869	76°00′	1.3265
.2473	10	.2447	.3887	.2524	.4021	3.9617	.5979	.9696	.9866	50	1.3235
.2502	20	.2476	.3937	.2555	.4074	3.9136	.5926	.9689	.9863	40	1.3206
.2531	30	.2504	.3986	.2586	.4127	3.8667	.5873	.9681	.9859	30	1.3177
.2560	40	.2532	.4035	.2617	.4178	3.8208	.5822	.9674	.9856	20	1.3148
.2589	50	.2560	.4083	.2648	.4230	3.7760	.5770	.9667	.9853	10	1.3119
.2618	15°00′	.2588	.4130	.2679	.4281	3.7321	.5719	.9659	.9849	75°00′	1.3090
.2647	10	.2616	.4177	.2711	.4331	3.6891	.5669	.9652	.9846	50	1.3061
.2676	20	.2644	.4223	.2742	.4381	3.6470	.5619	.9644	.9843	40	1.3032
.2705	30	.2672	.4269	.2773	.4430	3.6059	.5570	.9636	.9839	30	1.3003
.2734	40	.2700	.4314	.2805	.4479	3.5656	.5521	.9628	.9836	20	1.2974
.2763	50	.2728	.4359	.2836	.4527	3.5261	.5473	.9621	.9832	10	1.2945
.2793	16°00′	.2756	.4403	.2867	.4575	3.4874	.5425	.9613	.9828	74°00′	1.2915
.2822	10	.2784	.4447	.2899	.4622	3.4495	.5378	.9605	.9825	50	1.2886
.2851	20	.2812	.4491	.2931	.4669	3.4124	.5331	.9596	.9821	40	1.2857
.2880	30	.2840	.4533	.2962	.4716	3.3759	.5284	.9588	.9817	30	1.2828
.2909	40	.2868	.4576	.2994	.4762	3.3402	.5238	.9580	.9814	20	1.2799
.2938	50	.2896	.4618	.3026	.4808	3.3052	.5192	.9572	.9810	10	1.2770
.2967	17°00′	.2924	.4659	.3057	.4853	3.2709	.5147	.9563	.9806	73°00′	1.2741
.2996	10	.2952	.4700	.3089	.4898	3.2371	.5102	.9555	.9802	50	1.2712
.3025	20	.2979	.4741	.3121	.4943	3.2041	.5057	.9546	.9798	40	1.2683
.3054	30	.3007	.4781	.3153	.4987	3.1716	.5013	.9537	.9794	30	1.2654
.3083	40	.3035	.4821	.3185	.5031	3.1397	.4969	.9528	.9790	20	1.2625
.3113	50	.3062	.4861	.3217	.5075	3.1084	.4925	.9520	.9786	10	1.2595
.3142	18°00′	.3090	.4900	.3249	.5118	3.0777	.4882	.9511	.9782	72°00′	1.2566
		Value Cosine	Log Cosine	Value Cotangent	Log Cotangent	Value Tangent	Log Tangent	Value Sine	Log Sine	Degrees	Radians

TABLE II

TRIGONOMETRIC FUNCTIONS AND THEIR LOGARITHMS

ANGLE Radians	ANGLE Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log		
.3142	18°00′	.3090	.4900	.3249	.5118	3.0777	.4882	.9511	.9782	72°00′	1.2566
.3171	10	.3118	.4939	.3281	.5161	3.0475	.4839	.9502	.9778	50	1.2537
.3200	20	.3145	.4977	.3314	.5203	3.0178	.4797	.9492	.9774	40	1.2508
.3229	30	.3173	.5015	.3346	.5245	2.9887	.4755	.9483	.9770	30	1.2479
.3258	40	.3201	.5052	.3378	.5287	2.9600	.4713	.9474	.9765	20	1.2450
.3287	50	.3228	.5090	.3411	.5329	2.9319	.4671	.9465	.9761	10	1.2421
.3316	19°00′	.3256	.5126	.3443	.5370	2.9042	.4630	.9455	.9757	71°00′	1.2392
.3345	10	.3283	.5163	.3476	.5411	2.8770	.4589	.9446	.9752	50	1.2363
.3374	20	.3311	.5199	.3508	.5451	2.8502	.4549	.9436	.9748	40	1.2334
.3403	30	.3338	.5235	.3541	.5491	2.8239	.4509	.9426	.9743	30	1.2305
.3432	40	.3365	.5270	.3574	.5531	2.7980	.4469	.9417	.9739	20	1.2275
.3462	50	.3393	.5306	.3607	.5571	2.7725	.4429	.9407	.9734	10	1.2246
.3491	20°00′	.3420	.5341	.3640	.5611	2.7475	.4389	.9397	.9730	70°00′	1.2217
.3520	10	.3448	.5375	.3673	.5650	2.7228	.4350	.9387	.9725	50	1.2188
.3549	20	.3475	.5409	.3706	.5689	2.6985	.4311	.9377	.9721	40	1.2159
.3578	30	.3502	.5443	.3739	.5727	2.6746	.4273	.9367	.9716	30	1.2130
.3607	40	.3529	.5477	.3772	.5766	2.6511	.4234	.9356	.9711	20	1.2101
.3636	50	.3557	.5510	.3805	.5804	2.6279	.4196	.9346	.9706	10	1.2072
.3665	21°00′	.3584	.5543	.3839	.5842	2.6051	.4158	.9336	.9702	69°00′	1.2043
.3694	10	.3611	.5576	.3872	.5879	2.5826	.4121	.9325	.9697	50	1.2014
.3723	20	.3638	.5609	.3906	.5917	2.5605	.4083	.9315	.9692	40	1.1985
.3752	30	.3665	.5641	.3939	.5954	2.5386	.4046	.9304	.9687	30	1.1956
.3782	40	.3692	.5673	.3973	.5991	2.5172	.4009	.9293	.9682	20	1.1926
.3811	50	.3719	.5704	.4006	.6028	2.4960	.3972	.9283	.9677	10	1.1897
.3840	22°00′	.3746	.5736	.4040	.6064	2.4751	.3936	.9272	.9672	68°00′	1.1868
.3869	10	.3773	.5767	.4074	.6100	2.4545	.3900	.9261	.9667	50	1.1839
.3898	20	.3800	.5798	.4108	.6136	2.4342	.3864	.9250	.9661	40	1.1810
.3927	30	.3827	.5828	.4142	.6172	2.4142	.3828	.9239	.9656	30	1.1781
.3956	40	.3854	.5859	.4176	.6208	2.3945	.3792	.9228	.9651	20	1.1752
.3985	50	.3881	.5889	.4210	.6243	2.3750	.3757	.9216	.9646	10	1.1723
.4014	23°00′	.3907	.5919	.4245	.6279	2.3559	.3721	.9205	.9640	67°00′	1.1694
.4043	10	.3934	.5948	.4279	.6314	2.3369	.3686	.9194	.9635	50	1.1665
.4072	20	.3961	.5978	.4314	.6348	2.3183	.3652	.9182	.9629	40	1.1636
.4102	30	.3987	.6007	.4348	.6383	2.2998	.3617	.9171	.9624	30	1.1606
.4131	40	.4014	.6036	.4383	.6417	2.2817	.3583	.9159	.9618	20	1.1577
.4160	50	.4041	.6065	.4417	.6452	2.2637	.3548	.9147	.9613	10	1.1548
.4189	24°00′	.4067	.6093	.4452	.6486	2.2460	.3514	.9135	.9607	66°00′	1.1519
.4218	10	.4094	.6121	.4487	.6520	2.2286	.3480	.9124	.9602	50	1.1490
.4247	20	.4120	.6149	.4522	.6553	2.2113	.3447	.9112	.9596	40	1.1461
.4276	30	.4147	.6177	.4557	.6587	2.1943	.3413	.9100	.9590	30	1.1432
.4305	40	.4173	.6205	.4592	.6620	2.1775	.3380	.9088	.9584	20	1.1403
.4334	50	.4200	.6232	.4628	.6654	2.1609	.3346	.9075	.9579	10	1.1374
.4363	25°00′	.4226	.6259	.4663	.6687	2.1445	.3313	.9063	.9573	65°00′	1.1345
.4392	10	.4253	.6286	.4699	.6720	2.1283	.3280	.9051	.9567	50	1.1316
.4422	20	.4279	.6313	.4734	.6752	2.1123	.3248	.9038	.9561	40	1.1286
.4451	30	.4305	.6340	.4770	.6785	2.0965	.3215	.9026	.9555	30	1.1257
.4480	40	.4331	.6366	.4806	.6817	2.0809	.3183	.9013	.9549	20	1.1228
.4509	50	.4358	.6392	.4841	.6850	2.0655	.3150	.9001	.9543	10	1.1199
.4538	26°00′	.4384	.6418	.4877	.6882	2.0503	.3118	.8988	.9537	64°00′	1.1170
.4567	10	.4410	.6444	.4913	.6914	2.0353	.3086	.8975	.9530	50	1.1141
.4596	20	.4436	.6470	.4950	.6946	2.0204	.3054	.8962	.9524	40	1.1112
.4625	30	.4462	.6495	.4986	.6977	2.0057	.3023	.8949	.9518	30	1.1083
.4654	40	.4488	.6521	.5022	.7009	1.9912	.2991	.8936	.9512	20	1.1054
.4683	50	.4514	.6546	.5059	.7040	1.9768	.2960	.8923	.9505	10	1.1025
.4712	27°00′	.4540	.6570	.5095	.7072	1.9626	.2928	.8910	.9499	63°00′	1.0996
		Cosine Value	Cosine Log	Cotangent Value	Cotangent Log	Tangent Value	Tangent Log	Sine Value	Sine Log	Degrees	Radians

TABLE II

TRIGONOMETRIC FUNCTIONS AND THEIR LOGARITHMS

ANGLE Radians	ANGLE Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log		
.4712	27°00′	.4540	.6570	.5095	.7072	1.9626	.2928	.8910	.9499	63°00′	1.0996
.4741	10	.4566	.6595	.5132	.7103	1.9486	.2897	.8897	.9492	50	1.0966
.4771	20	.4592	.6620	.5169	.7134	1.9347	.2866	.8884	.9486	40	1.0937
.4800	30	.4617	.6644	.5206	.7165	1.9210	.2835	.8870	.9479	30	1.0908
.4829	40	.4643	.6668	.5243	.7196	1.9074	.2804	.8857	.9473	20	1.0879
.4858	50	.4669	.6692	.5280	.7226	1.8940	.2774	.8843	.9466	10	1.0850
.4887	28°00′	.4695	.6716	.5317	.7257	1.8807	.2743	.8829	.9459	62°00′	1.0821
.4916	10	.4720	.6740	.5354	.7287	1.8676	.2713	.8816	.9453	50	1.0792
.4945	20	.4746	.6763	.5392	.7317	1.8546	.2683	.8802	.9446	40	1.0763
.4974	30	.4772	.6787	.5430	.7348	1.8418	.2652	.8788	.9439	30	1.0734
.5003	40	.4797	.6810	.5467	.7378	1.8291	.2622	.8774	.9432	20	1.0705
.5032	50	.4823	.6833	.5505	.7408	1.8165	.2592	.8760	.9425	10	1.0676
.5061	29°00′	.4848	.6856	.5543	.7438	1.8040	.2562	.8746	.9418	61°00′	1.0647
.5091	10	.4874	.6878	.5581	.7467	1.7917	.2533	.8732	.9411	50	1.0617
.5120	20	.4899	.6901	.5619	.7497	1.7796	.2503	.8718	.9404	40	1.0588
.5149	30	.4924	.6923	.5658	.7526	1.7675	.2474	.8704	.9397	30	1.0559
.5178	40	.4950	.6946	.5696	.7556	1.7556	.2444	.8689	.9390	20	1.0530
.5207	50	.4975	.6968	.5735	.7585	1.7437	.2415	.8675	.9383	10	1.0501
.5236	30°00′	.5000	.6990	.5774	.7614	1.7321	.2386	.8660	.9375	60°00′	1.0472
.5265	10	.5025	.7012	.5812	.7644	1.7205	.2356	.8646	.9368	50	1.0443
.5294	20	.5050	7033	.5851	.7673	1.7090	.2327	.8631	.9361	40	1.0414
.5323	30	.5075	.7055	.5890	.7701	1.6977	.2299	.8616	.9353	30	1.0385
.5352	40	.5100	.7076	.5930	.7730	1.6864	.2270	.8601	.9346	20	1.0356
.5381	50	.5125	.7097	.5969	.7759	1.6753	.2241	.8587	.9338	10	1.0327
.5411	31°00′	.5150	.7118	.6009	.7788	1.6643	.2212	.8572	.9331	59°00′	1.0297
.5440	10	.5175	.7139	.6048	.7816	1.6534	.2184	.8557	.9323	50	1.0268
.5469	20	.5200	.7160	.6088	.7845	1.6426	.2155	.8542	.9315	40	1.0239
.5498	30	.5225	.7181	.6128	.7873	1.6319	.2127	.8526	.9308	30	1.0210
.5527	40	.5250	.7201	.6168	.7902	1.6212	.2098	.8511	.9300	20	1.0181
.5556	50	.5275	.7222	.6208	.7930	1.6107	.2070	.8496	.9292	10	1.0152
.5585	32°00′	.5299	.7242	.6249	.7958	1.6003	.2042	.8480	.9284	58°00′	1.0123
.5614	10	.5324	.7262	.6289	.7986	1.5900	.2014	.8465	.9276	50	1.0094
.5643	20	.5348	.7282	.6330	.8014	1.5798	.1986	.8450	.9268	40	1.0065
.5672	30	.5373	.7302	.6371	.8042	1.5697	.1958	.8434	.9260	30	1.0036
.5701	40	.5398	.7322	.6412	.8070	1.5597	.1930	.8418	.9252	20	1.0007
.5730	50	.5422	.7342	.6453	.8097	1.5497	.1903	.8403	.9244	10	.9977
.5760	33°00′	.5446	.7361	.6494	.8125	1.5399	.1875	.8387	.9236	57°00′	.9948
.5789	10	.5471	.7380	.6536	.8153	1.5301	.1847	.8371	.9228	50	.9919
.5818	20	.5495	.7400	.6577	.8180	1.5204	.1820	.8355	.9219	40	.9890
.5847	30	.5519	.7419	.6619	.8208	1.5108	.1792	.8339	.9211	30	.9861
.5876	40	.5544	.7438	.6661	.8235	1.5013	.1765	.8323	.9203	20	.9832
.5905	50	.5568	.7457	.6703	.8263	1.4919	.1737	.8307	.9194	10	.9803
.5934	34°00′	.5592	.7476	.6745	.8290	1.4826	.1710	.8290	.9186	56°00′	.9774
.5963	10	.5616	.7494	.6787	.8317	1.4733	.1683	.8274	.9177	50	.9745
.5992	20	.5640	.7513	.6830	.8344	1.4641	.1656	.8258	.9169	40	.9716
.6021	30	.5664	.7531	.6873	.8371	1.4550	.1629	.8241	.9160	30	.9687
.6050	40	.5688	.7550	.6916	.8398	1.4460	.1602	.8225	.9151	20	.9657
.6080	50	.5712	.7568	.6959	.8425	1.4370	.1575	.8208	.9142	10	.9628
.6109	35°00′	.5736	.7586	.7002	.8452	1.4281	.1548	.8192	.9134	55°00′	.9599
.6138	10	.5760	.7604	.7046	.8479	1.4193	.1521	.8175	.9125	50	.9570
.6167	20	.5783	.7622	.7089	.8506	1.4106	.1494	.8158	.9116	40	.9541
.6196	30	.5807	.7640	.7133	.8533	1.4019	.1467	.8141	.9107	30	.9512
.6225	40	.5831	.7657	.7177	.8559	1.3934	.1441	.8124	.9098	20	.9483
.6254	50	.5854	.7675	.7221	.8586	1.3848	.1414	.8107	.9089	10	.9454
.6283	36°00′	.5878	.7692	.7265	.8613	1.3764	.1387	.8090	.9080	54°00′	.9425
		Value Cosine	Log Cosine	Value Cotangent	Log Cotangent	Value Tangent	Log Tangent	Value Sine	Log Sine	Degrees	Radians

TABLE II

TRIGONOMETRIC FUNCTIONS AND THEIR LOGARITHMS

ANGLE Radians	Degrees	Sine Value	Log	Tangent Value	Log	Cotangent Value	Log	Cosine Value	Log		
.6283	36°00′	.5878	.7692	.7265	.8613	1.3764	.1387	.8090	.9080	54°00′	.9425
.6312	10	.5901	.7710	.7310	.8639	1.3680	.1361	.8073	.9070	50	.9396
.6341	20	.5925	.7727	.7355	.8666	1.3597	.1334	.8056	.9061	40	.9367
.6370	30	.5948	.7744	.7400	.8692	1.3514	.1308	.8039	.9052	30	.9338
.6400	40	.5972	.7761	.7445	.8718	1.3432	.1282	.8021	.9042	20	.9308
.6429	50	.5995	.7778	.7490	.8745	1.3351	.1255	.8004	.9033	10	.9279
.6458	37°00′	.6018	.7795	.7536	.8771	1.3270	.1229	.7986	.9023	53°00′	.9250
.6487	10	.6041	.7811	.7581	.8797	1.3190	.1203	.7969	.9014	50	.9221
.6516	20	.6065	.7828	.7627	.8824	1.3111	.1176	.7951	.9004	40	.9192
.6545	30	.6088	.7844	.7673	.8850	1.3032	.1150	.8934	.8995	30	.9163
.6574	40	.6111	.7861	.7720	.8876	1.2954	.1124	.7916	.8985	20	.9134
.6603	50	.6134	.7877	.7766	.8902	1.2876	.1098	.7898	.8975	10	.9105
.6632	38°00′	.6157	.7893	.7813	.8928	1.2799	.1072	.7880	.8965	52°00′	.9076
.6661	10	.6180	.7910	.7860	.8954	1.2723	.1046	.7862	.8955	50	.9047
.6690	20	.6202	.7926	.7907	.8980	1.2647	.1020	.7844	.8945	40	.9018
.6720	30	.6225	.7941	.7954	.9006	1.2572	.0994	.7826	.8935	30	.8988
.6749	40	.6248	.7957	.8002	.9032	1.2497	.0968	.7808	.8925	20	.8959
.6778	50	.6271	.7973	.8050	.9058	1.2423	.0942	.7790	.8915	10	.8930
.6807	39°00′	.6293	.7989	.8098	.9084	1.2349	.0916	.7771	.8905	51°00′	.8901
.6836	10	.6316	.8004	.8146	.9110	1.2276	.0890	.7753	.8895	50	.8872
.6865	20	.6338	.8020	.8195	.9135	1.2203	.0865	.7735	.8884	40	.8843
.6894	30	.6361	.8035	.8243	.9161	1.2131	.0839	.7716	.8874	30	.8814
.6923	40	.6383	.8050	.8292	.9187	1.2059	.0813	.7698	.8864	20	.8785
.6952	50	.6406	.8066	.8342	.9212	1.1988	.0788	.7679	.8853	10	.8756
.6981	40°00′	.6428	.8081	.8391	.9238	1.1918	.0762	.7660	.8843	50°00′	.8727
.7010	10	.6450	.8096	.8441	.9264	1.1847	.0736	.7642	.8832	50	.8698
.7039	20	.6472	.8111	.8491	.9289	1.1778	.0711	.7623	.8821	40	.8668
.7069	30	.6494	.8125	.8541	.9315	1.1708	.0685	.7604	.8810	30	.8639
.7098	40	.6517	.8140	.8591	.9341	1.1640	.0659	.7585	.8800	20	.8610
.7127	50	.6539	.8155	.8642	.9366	1.1571	.0634	.7566	.8789	10	.8581
.7156	41°00′	.6561	.8169	.8693	.9392	1.1504	.0608	.7547	.8778	49°00′	.8552
.7185	10	.6583	.8184	.8744	.9417	1.1436	.0583	.7528	.8767	50	.8523
.7214	20	.6604	.8198	.8796	.9443	1.1369	.0557	.7509	.8756	40	.8494
.7243	30	.6626	.8213	.8847	.9468	1.1303	.0532	.7490	.8745	30	.8465
.7272	40	.6648	.8227	.8899	.9494	1.1237	.0506	.7470	.8733	20	.8436
.7301	50	.6670	.8241	.8952	.9519	1.1171	.0481	.7451	.8722	10	.8407
.7330	42°00′	.6691	.8255	.9004	.9544	1.1106	.0456	.7431	.8711	48°00′	.8378
.7359	10	.6713	.8269	.9057	.9570	1.1041	.0430	.7412	.8699	50	.8348
.7389	20	.6734	.8283	.9110	.9595	1.0977	.0405	.7392	.8688	40	.8319
.7418	30	.6756	.8297	.9163	.9621	1.0913	.0379	.7373	.8676	30	.8290
.7447	40	.6777	.8311	.9217	.9646	1.0850	.0354	.7353	.8665	20	.8261
.7476	50	.6799	.8324	.9271	.9671	1.0786	.0329	.7333	.8653	10	.8232
.7505	43°00′	.6820	.8338	.9325	.9697	1.0724	.0303	.7314	.8641	47°00′	.8203
.7534	10	.6841	.8351	.9380	.9722	1.0661	.0278	.7294	.8629	50	.8174
.7563	20	.6862	.8365	.9435	.9747	1.0599	.0253	.7274	.8618	40	.8145
.7592	30	.6884	.8378	.9490	.9772	1.0538	.0228	.7254	.8606	30	.8116
.7621	40	.6905	.8391	.9545	.9798	1.0477	.0202	.7234	.8594	20	.8087
.7650	50	.6926	.8405	.9601	.9823	1.0416	.0177	.7214	.8582	10	.8058
.7679	44°00′	.6947	.8418	.9657	.9848	1.0355	.0152	.7193	.8569	46°00′	.8029
.7709	10	.6967	.8431	.9713	.9874	1.0295	.0126	.7173	.8557	50	.7999
.7738	20	.6988	.8444	.9770	.9899	1.0235	.0101	.7153	.8545	40	.7970
.7767	30	.7009	.8457	.9827	.9924	1.0176	.0076	.7133	.8532	30	.7941
.7796	40	.7030	.8469	.9884	.9949	1.0117	.0051	.7112	.8520	20	.7912
.7825	50	.7050	.8482	.9942	.9975	1.0058	.0025	.7092	.8507	10	.7883
.7854	45°00′	.7071	.8495	1.0000	.0000	1.0000	.0000	.7071	.8495	45°00′	.7854
		Value Cosine	Log	Value Cotangent	Log	Value Tangent	Log	Value Sine	Log	Degrees	Radians

TABLE III

Amount of One Dollar at Compound Interest for n Years

$s=(1+r)^n$

n	2%	2½%	3%	3½%	4%	4½%	5%	6%	7%
1	1.0200	1.0250	1.0300	1.0350	1.0400	1.0450	1.0500	1.0600	1.0700
2	1.0404	1.0506	1.0609	1.0712	1.0816	1.0920	1.1025	1.1236	1.1449
3	1.0612	1.0769	1.0927	1.1087	1.1249	1.1412	1.1576	1.1910	1.2250
4	1.0824	1.1038	1.1255	1.1475	1.1699	1.1925	1.2155	1.2625	1.3108
5	1.1041	1.1314	1.1593	1.1877	1.2167	1.2462	1.2763	1.3382	1.4026
6	1.12 2	1.1597	1.1941	1.2293	1.2653	1.3023	1.3401	1.4185	1.5007
7	1.1487	1.1887	1.2299	1.2723	1.3159	1.3609	1.4071	1.5036	1.6058
8	1.1717	1.2184	1.2668	1.3168	1.3686	1.4221	1.4775	1.5938	1.7182
9	1.1951	1.2489	1.3048	1.3629	1.4233	1.4861	1.5513	1.6895	1.8385
10	1.2190	1.2801	1.3439	1.4106	1.4802	1.5530	1.6289	1.7908	1.9672
11	1.2434	1.3121	1.3842	1.4600	1.5395	1.6229	1.7103	1.8983	2.1049
12	1.2682	1.3449	1.4258	1.5111	1.6010	1.6959	1.7959	2.0122	2.2522
13	1.2936	1.3785	1.4685	1.5640	1.6651	1.7722	1.8856	2.1329	2.4098
14	1.3195	1.4130	1.5126	1.6187	1.7317	1.8519	1.9799	2.2609	2.5785
15	1.3459	1.4483	1.5580	1.6753	1.8009	1.9353	2.0789	2.3966	2.7590
16	1.3728	1.4845	1.6047	1.7340	1.8730	2.0224	2.1829	2.5404	2.9522
17	1.4002	1.5216	1.6528	1.7947	1.9479	2.1134	2.2920	2.6928	3.1588
18	1.4282	1.5597	1.7024	1.8575	2.0258	2.2085	2.4066	2.8543	3.3799
19	1.4568	1.5987	1.7535	1.9225	2.1068	2.3079	2.5270	3.0256	3.6165
20	1.4859	1.6386	1.8061	1.9898	2.1911	2.4117	2.6533	3.2071	3.8697
21	1.5157	1.6796	1.8603	2.0594	2.2788	2.5202	2.7860	3.3996	4.1406
22	1.5460	1.7216	1.9161	2.1315	2.3699	2.6337	2.9253	3.6035	4.4304
23	1.5769	1.7646	1.9736	2.2061	2.4647	2.7522	3.0715	3.8197	4.7405
24	1.6084	1.8087	2.0328	2.2833	2.5633	2.8760	3.2251	4.0489	5.0724
25	1.6406	1.8539	2.0938	2.3632	2.6658	3.0054	3.3864	4.2919	5.4274
26	1.6734	1.9003	2.1566	2.4460	2.7725	3.1407	3.5557	4.5494	5.8074
27	1.7069	1.9478	2.2213	2.5316	2.8834	3.2820	3.7335	4.8223	6.2139
28	1.7410	1.9965	2.2879	2.6202	2.9987	3.4297	3.9201	5.1117	6.6488
29	1.7758	2.0464	2.3566	2.7119	3.1187	3.5840	4.1161	5.4184	7.1143
30	1.8114	2.0976	2.4273	2.8068	3.2434	3.7453	4.3219	5.7435	7.6123
31	1.8476	2.1500	2.5001	2.9050	3.3731	3.9139	4.5380	6.0881	8.1451
32	1.8845	2.2038	2.5751	3.0067	3.5081	4.0900	4.7649	6.4534	8.7153
33	1.9222	2.2589	2.6523	3.1119	3.6484	4.2740	5.0032	6.8406	9.3253
34	1.9607	2.3153	2.7319	3.2209	3.7943	4.4664	5.2533	7.2510	9.9781
35	1.9999	2.3732	2.8139	3.3336	3.9461	4.6673	5.5160	7.6861	10.6766
36	2.0399	2.4325	2.8983	3.4503	4.1039	4.8774	5.7918	8.1473	11.4239
37	2.0807	2.4933	2.9852	3.5710	4.2681	5.0969	6.0814	8.6361	12.2236
38	2.1223	2.5557	3.0748	3.6960	4.4388	5.3262	6.3855	9.1543	13.0793
39	2.1647	2.6196	3.1670	3.8254	4.6164	5.5659	6.7048	9.7035	13.9948
40	2.2080	2.6851	3.2620	3.9593	4.8010	5.8164	7.0400	10.2857	14.9745
41	2.2522	2.7522	3.3599	4.0978	4.9931	6.0781	7.3920	10.9029	16.0227
42	2.2972	2.8210	3.4607	4.2413	5.1928	6.3516	7.7616	11.5570	17.1443
43	2.3432	2.8915	3.5645	4.3897	5.4005	6.6374	8.1497	12.2505	18.3444
44	2.3901	2.9638	3.6715	4.5433	5.6165	6.9361	8.5572	12.9855	19.6285
45	2.4379	3.0379	3.7816	4.7024	5.8412	7.2482	8.9850	13.7646	21.0025
46	2.4866	3.1139	3.8950	4.8669	6.0748	7.5744	9.4343	14.5905	22.4726
47	2.5363	3.1917	4.0119	5.0373	6.3178	7.9153	9.9060	15.4659	24.0457
48	2.5871	3.2715	4.1323	5.2136	6.5705	8.2715	10.4013	16.3939	25.7289
49	2.6388	3.3533	4.2562	5.3961	6.8333	8.6437	10.9213	17.3775	27.5299
50	2.6916	3.4371	4.3839	5.5849	7.1067	9.0326	11.4674	18.4202	29.4570

TABLE IV

Present Value of One Dollar Due at End of n Years

$$p = 1/(1+r)^n$$

n	2%	2½%	3%	3½%	4%	4½%	5%	6%	7%
1	.98039	.97561	.97087	.96618	.96154	.95694	.95238	.94340	.93458
2	.96117	.95181	.94260	.93351	.92456	.91573	.90703	.89000	.87344
3	.94232	.92860	.91514	.90194	.88900	.87630	.86384	.83962	.81630
4	.92385	.90595	.88849	.87144	.85480	.83856	.82270	.79209	.76290
5	.90573	.88385	.86261	.84197	.82193	.80245	.78353	.74726	.71299
6	.88797	.86230	.83748	.81350	.79031	.76790	.74622	.70496	.66634
7	.87056	.84127	.81309	.78599	.75992	.73483	.71068	.66506	.62275
8	.85349	.82075	.78941	.75941	.73069	.70319	.67684	.62741	.58201
9	.83676	.80073	.76642	.73373	.70259	.67290	.64461	.59190	.54393
10	.82035	.78120	.74409	.70892	.67556	.64393	.61391	.55839	.50835
11	.80426	.76214	.72242	.68495	.64958	.61620	.58468	.52679	.47509
12	.78849	.74356	.70138	.66178	.62460	.58966	.55684	.49697	.44401
13	.77303	.72542	.68095	.63940	.60057	.56427	.53032	.46884	.41496
14	.75788	.70773	.66112	.61778	.57748	.53997	.50507	.44230	.38782
15	.74301	.69047	.64186	.59689	.55526	.51672	.48102	.41727	.36245
16	.72845	.67362	.62317	.57671	.53391	.49447	.45811	.39365	.33873
17	.71416	.65720	.60502	.55720	.51337	.47318	.43630	.37136	.31657
18	.70016	.64117	.58739	.53836	.49363	.45280	.41552	.35034	.29586
19	.68643	.62553	.57029	.52016	.47464	.43330	.39573	.33051	.27651
20	.67297	.61027	.55368	.50257	.45639	.41464	.37689	.31180	.25842
21	.65978	.59539	.53755	.48557	.43883	.39679	.35894	.29416	.24151
22	.64684	.58086	.52189	.46915	.42196	.37970	.34185	.27751	.22571
23	.63416	.56670	.50669	.45329	.40573	.36335	.32557	.26180	.21095
24	.62172	.55288	.49193	.43796	.39012	.34770	.31007	.24698	.19715
25	.60953	.53939	.47761	.42315	.37512	.33273	.29530	.23300	.18425
26	.59758	.52623	.46369	.40884	.36069	.31840	.28124	.21981	.17220
27	.58586	.51340	.45019	.39501	.34682	.30469	.26785	.20737	.16093
28	.57437	.50088	.43708	.38165	.33348	.29157	.25509	.19563	.15040
29	.56311	.48866	.42435	.36875	.32065	.27902	.24295	.18456	.14056
30	.55207	.47674	.41199	.35628	.30832	.26700	.23138	.17411	.13137
31	.54125	.46511	.39999	.34423	.29646	.25550	.22036	.16425	.12277
32	.53063	.45377	.38834	.33259	.28506	.24450	.20987	.15496	.11474
33	.52023	.44270	.37703	.32134	.27409	.23397	.19987	.14619	.10723
34	.51003	.43191	.36604	.31048	.26355	.22390	.19035	.13791	.10022
35	.50003	.42137	.35538	.29998	.25342	.21425	.18129	.13011	.09366
36	.49022	.41109	.34503	.28983	.24367	.20503	.17266	.12274	.08754
37	.48061	.40107	.33498	.28003	.23430	.19620	.16444	.11580	.08181
38	.47119	.39128	.32523	.27056	.22529	.18775	.15661	.10924	.07646
39	.46195	.38174	.31575	.26141	.21662	.17967	.14915	.10306	.07146
40	.45289	.37243	.30656	.25257	.20829	.17193	.14205	.09722	.06678
41	.44401	.36335	.29763	.24403	.20028	.16453	.13528	.09172	.06241
42	.43530	.35448	.28896	.23578	.19257	.15744	.12884	.08653	.05833
43	.42677	.34584	.28054	.22781	.18517	.15066	.12270	.08163	.05451
44	.41840	.33740	.27237	.22010	.17805	.14417	.11686	.07701	.05095
45	.41020	.32917	.26444	.21266	.17120	.13796	.11130	.07265	.04761
46	.40215	.32115	.25674	.20547	.16461	.13202	.10600	.06854	.04450
47	.39427	.31331	.24926	.19852	.15828	.12634	.10095	.06466	.04159
48	.38654	.30567	.24200	.19181	.15219	.12090	.09614	.06100	.03887
49	.37896	.29822	.23495	.18532	.14634	.11569	.09156	.05755	.03632
50	.37153	.29094	.22811	.17905	.14071	.11071	.08720	.05429	.03395

TABLE V

Amount of an Annuity of One Dollar Per Year for n Years

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

n	2%	2½%	3%	3½%	4%	4½%	5%	6%	7%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0200	2.0250	2.0300	2.0350	2.0400	2.0450	2.0500	2.0600	2.0700
3	3.0604	3.0756	3.0909	3.1062	3.1216	3.1370	3.1525	3.1836	3.2149
4	4.1216	4.1525	4.1836	4.2149	4.2465	4.2782	4.3101	4.3746	4.4399
5	5.2040	5.2563	5.3091	5.3625	5.4163	5.4707	5.5256	5.6371	5.7507
6	6.3081	6.3877	6.4684	6.5502	6.6330	6.7169	6.8019	6.9753	7.1533
7	7.4343	7.5474	7.6625	7.7794	7.8983	8.0192	8.1420	8.3938	8.6540
8	8.5830	8.7361	8.8923	9.0517	9.2142	9.3800	9.5491	9.8975	10.2598
9	9.7546	9.9545	10.1591	10.3685	10.5828	10.8021	11.0266	11.4913	11.9780
10	10.9497	11.2034	11.4639	11.7314	12.0061	12.2882	12.5779	13.1808	13.8164
11	12.1687	12.4835	12.8708	13.1420	13.4864	13.8412	14.2068	14.9716	15.7836
12	13.4121	13.7956	14.1920	14.6020	15.0258	15.4640	15.9171	16.8699	17.8885
13	14.6803	15.1404	15.6178	16.1130	16.6268	17.1599	17.7130	18.8821	20.1406
14	15.9739	16.5190	17.0863	17.6770	18.2919	18.9321	19.5986	21.0151	22.5505
15	17.2934	17.9319	18.5989	19.2957	20.0236	20.7841	21.5786	23.2760	25.1290
16	18.6393	19.3802	20.1569	20.9710	21.8245	22.7193	23.6575	25.6725	27.8881
17	20.0121	20.8647	21.7616	22.7050	23.6975	24.7417	25.8404	28.2129	30.8402
18	21.4123	22.3863	23.4144	24.4997	25.6454	26.8551	28.1324	30.9057	33.9990
19	22.8406	23.9460	25.1169	26.3572	27.6712	29.0636	30.5390	33.7600	37.3790
20	24.2974	25.5447	26.8704	28.2797	29.7781	31.3714	33.0660	36.7856	40.9955
21	25.7833	27.1833	28.6765	30.2695	31.9692	33.7831	35.7193	39.9927	44.8652
22	27.2990	28.8629	30.5368	32.3289	34.2480	36.3034	38.5052	43.3923	49.0057
23	28.8450	30.5844	32.4529	34.4604	36.6179	38.9370	41.4305	46.9958	53.4361
24	30.4219	32.3490	34.4265	36.6665	39.0826	41.6892	44.5020	50.8156	58.1767
25	32.0303	34.1578	36.4593	38.9499	41.6459	44.5652	47.7271	54.8645	63.2490
26	33.6709	36.0117	38.5530	41.3131	44.3117	47.5706	51.1135	59.1564	68.6765
27	35.3443	37.9120	40.7096	43.7591	47.0842	50.7113	54.6691	63.7058	74.4838
28	37.0512	39.8598	42.9309	46.2906	49.9676	53.9933	58.4026	68.5281	80.6977
29	38.7922	41.8563	45.2189	48.9108	52.9663	57.4230	62.3227	73.6398	87.3465
30	40.5681	43.9027	47.5754	51.6227	56.0849	61.0071	66.4388	79.0582	94.4608
31	42.3794	46.0003	50.0027	54.4295	59.3283	64.7524	70.7608	84.8017	102.0730
32	44.2270	48.1503	52.5028	57.3345	62.7015	68.6662	75.2988	90.8898	110.2182
33	46.1116	50.3540	55.0778	60.3412	66.2095	72.7562	80.0638	97.3432	118.9334
34	48.0338	52.6129	57.7302	63.4532	69.8579	77.0303	85.0670	104.1838	128.2588
35	49.9945	54.9282	60.4621	66.6740	73.6522	81.4966	90.3203	111.4348	138.2369
36	51.9944	57.3014	63.2759	70.0076	77.5983	86.1640	95.8363	119.1209	148.9135
37	54.0343	59.7339	66.1742	73.4579	81.7022	91.0413	101.6281	127.2681	160.3374
38	56.1149	62.2273	69.1594	77.0289	85.9703	96.1382	107.7095	135.9042	172.5610
39	58.2372	64.7830	72.2342	80.7249	90.4091	101.4644	114.0950	145.0585	185.6403
40	60.4020	67.4026	75.4013	84.5503	95.0255	107.0303	120.7998	154.7620	199.6351
41	62.6100	70.0876	78.6633	88.5095	99.8265	112.8467	127.8398	165.0477	214.6096
42	64.8622	72.8398	82.0232	92.6074	104.8196	118.9248	135.2318	175.9505	230.6322
43	67.1595	75.6608	85.4839	96.8486	110.0124	125.2764	142.9933	187.5076	247.7765
44	69.5027	78.5523	89.0484	101.2383	115.4129	131.9138	151.1430	199.7580	266.1209
45	71.8927	81.5161	92.7199	105.7817	121.0294	138.8500	159.7002	212.7435	285.7493
46	74.3306	84.5540	96.5015	110.4840	126.8706	146.0982	168.6852	226.5081	306.7518
47	76.8172	87.6679	100.3965	115.3510	132.9454	153.6726	178.1194	241.0986	329.2244
48	79.3535	90.8596	104.4084	120.3883	139.2632	161.5879	188.0254	256.5645	353.2701
49	81.9406	94.1311	108.5406	125.6018	145.8337	169.8594	198.4267	272.9584	378.9990
50	84.5794	97.4843	112.7969	130.9979	152.6671	178.5030	209.3480	290.3359	406.5289

TABLE VI

Present Value of an Annuity of One Dollar Per Year for n Years

$$a_{\overline{n}|i}=\frac{1-(1+i)^{-n}}{i}$$

n	2%	2½%	3%	3½%	4%	4½%	5%	6%	7%
1	.9804	.9756	.9709	.9662	.9615	.9569	.9524	.9434	.9346
2	1.9416	1.9274	1.9135	1.8997	1.8861	1.8727	1.8594	1.8334	1.8080
3	2.8839	2.8560	2.8286	2.8016	2.7751	2.7490	2.7232	2.6730	2.6243
4	3.8077	3.7620	3.7171	3.6731	3.6299	3.5875	3.5460	3.4651	3.3872
5	4.7135	4.6458	4.5797	4.5151	4.4518	4.3900	4.3295	4.2124	4.1002
6	5.6014	5.5081	5.4172	5.3286	5.2421	5.1579	5.0757	4.9173	4.7665
7	6.4720	6.3494	6.2303	6.1145	6.0021	5.8927	5.7864	5.5824	5.3893
8	7.3255	7.1701	7.0197	6.8740	6.7327	6.5959	6.4632	6.2098	5.9713
9	8.1622	7.9709	7.7861	7.6077	7.4353	7.2688	7.1078	6.8017	6.5152
10	8.9826	8.7521	8.5302	8.3166	8.1109	7.9127	7.7217	7.3601	7.0236
11	9.7868	9.5142	9.2526	9.0016	8.7605	8.5289	8.3064	7.8869	7.4987
12	10.5753	10.2578	9.9540	9.6633	9.3851	9.1186	8.8633	8.3838	7.9427
13	11.3484	10.9832	10.6350	10.3027	9.9856	9.6829	9.3936	8.8527	8.3577
14	12.1062	11.6909	11.2961	10.9205	10.5631	10.2228	9.8986	9.2950	8.7455
15	12.8493	12.3814	11.9379	11.5174	11.1184	10.7395	10.3797	9.7122	9.1079
16	13.5777	13.0550	12.5611	12.0941	11.6523	11.2340	10.8378	10.1059	9.4466
17	14.2919	13.7122	13.1661	12.6513	12.1657	11.7072	11.2741	10.4773	9.7632
18	14.9920	14.3534	13.7535	13.1897	12.6593	12.1600	11.6896	10.8276	10.0591
19	15.6785	14.9789	14.3238	13.7098	13.1339	12.5933	12.0853	11.1581	10.3356
20	16.3514	15.5892	14.8775	14.2124	13.5903	13.0079	12.4622	11.4699	10.5940
21	17.0112	16.1845	15.4150	14.6980	14.0292	13.4047	12.8212	11.7641	10.8355
22	17.6580	16.7654	15.9369	15.1671	14.4511	13.7844	13.1630	12.0416	11.0612
23	18.2922	17.3321	16.4436	15.6204	14.8568	14.1478	13.4886	12.3034	11.2722
24	18.9139	17.8850	16.9355	16.0584	15.2470	14.4955	13.7986	12.5504	11.4693
25	19.5235	18.4244	17.4131	16.4815	15.6221	14.8282	14.0939	12.7834	11.6536
26	20.1210	18.9506	17.8768	16.8904	15.9828	15.1466	14.3752	13.0032	11.8258
27	20.7069	19.4640	18.3270	17.2854	16.3296	15.4513	14.6430	13.2105	11.9867
28	21.2813	19.9649	18.7641	17.6670	16.6631	15.7429	14.8981	13.4062	12.1371
29	21.8444	20.4535	19.1885	18.0358	16.9837	16.0219	15.1411	13.5907	12.2777
30	22.3965	20.9303	19.6004	18.3920	17.2920	16.2889	15.3725	13.7648	12.4090
31	22.9377	21.3954	20.0004	18.7363	17.5885	16.5444	15.5928	13.9291	12.5318
32	23.4683	21.8392	20.3888	19.0689	17.8736	16.7889	15.8027	14.0840	12.6466
33	23.9886	22.2919	20.7658	19.3902	18.1476	17.0229	16.0025	14.2302	12.7538
34	24.4986	22.7238	21.1318	19.7007	18.4112	17.2468	16.1929	14.3681	12.8540
35	24.9986	23.1452	21.4872	20.0007	18.6646	17.4610	16.3742	14.4982	12.9477
36	25.4888	23.5563	21.8323	20.2905	18.9083	17.6660	16.5469	14.6210	13.0352
37	25.9695	23.9673	22.1672	20.5705	19.1426	17.8622	16.7113	14.7368	13.1170
38	26.4406	24.3486	22.4925	20.8411	19.3679	18.0500	16.8679	14.8460	13.1935
39	26.9026	24.7303	22.8082	21.1025	19.5845	18.2297	17.0170	14.9491	13.2649
40	27.3555	25.1028	23.1148	21.3551	19.7928	18.4016	17.1591	15.0463	13.3317
41	27.7995	25.4661	23.4124	21.5991	19.9931	18.5661	17.2944	15.1380	13.3941
42	28.2348	25.8206	23.7014	21.8349	20.1856	18.7236	17.4232	15.2245	13.4524
43	28.6616	26.1664	23.9819	22.0627	20.3708	18.8742	17.5459	15.3062	13.5070
44	29.0800	26.5038	24.2543	22.2828	20.5488	19.0184	17.6628	15.3832	13.5579
45	29.4902	26.8330	24.5187	22.4955	20.7200	19.1563	17.7741	15.4558	13.6055
46	29.8923	27.1542	24.7754	22.7009	20.8847	19.2884	17.8801	15.5244	13.6500
47	30.2866	27.4675	25.0247	22.8994	21.0429	19.4147	17.9810	15.5890	13.6910
48	30.6731	27.7732	25.2667	23.0912	21.1951	19.5356	18.0772	15.6500	13.7305
49	31.0521	28.0714	25.5017	23.2766	21.3415	19.6513	18.1687	15.7076	13.7668
50	31.4236	28.3623	25.7298	23.4556	21.4822	19.7620	18.2559	15.7619	13.8007

TABLE VII

AMERICAN EXPERIENCE TABLE OF MORTALITY

Age x	Number living l_x	Number of deaths d_x	Yearly probability of dying q_x	Yearly probability of living p_x	Age x	Number living l_x	Number of deaths d_x	Yearly probability of dying q_x	Yearly probability of living p_x
10	100,000	749	0.007 490	0.992 510	53	66,797	1,091	0.016 333	0.983 667
11	99,251	746	0.007 516	0.992 484	54	65,706	1,143	0.017 396	0.982 604
12	98,505	743	0.007 543	0.992 457	55	64,563	1,199	0.018 571	0.981 429
13	97,762	740	0.007 569	0.992 431	56	63,364	1,260	0.019 885	0.980 115
14	97,022	737	0.007 596	0.992 404	57	62,104	1,325	0.021 335	0.978 665
15	96,285	735	0.007 634	0.992 366	58	60.779	1,394	0.022 936	0.977 064
16	95,550	732	0.007 661	0.992 339	59	59,385	1,468	0.024 720	0.975 280
17	94,818	729	0.007 688	0.992 312	60	57,917	1,546	0.026 693	0.973 307
18	94,089	727	0,007 727	0.992 273	61	56,371	1,628	0.028 880	0.971 120
19	93,362	725	0.007 765	0.992 235	62	54,743	1,713	0.031 292	0.968 708
20	92,637	723	0.007 805	0.992 195	63	53,030	1,800	0.033 943	0.966 057
21	91,914	722	0.007 855	0.992 145	64	51,230	1,889	0.036 873	0.963 127
22	91,192	721	0.007 906	0.992 094	65	49,341	1,980	0.040 129	0.959 871
23	90,471	720	0.007 958	0.992 042	66	47,361	2,070	0.043 707	0.956 293
24	89,751	719	0.008 011	0.991 989	67	45,291	2,158	0.047 647	0.952 353
25	89,032	718	0.008 065	0.991 935	68	43,133	2,243	0.052 002	0.947 998
26	88,314	718	0.008 130	0.991 870	69	40,890	2,321	0.056 762	0.943 238
27	87,596	718	0.008 197	0.991 803	70	38,569	2,391	0.061 993	0.938 007
28	86,878	718	0.008 264	0.991 736	71	36,178	2,448	0.067 665	0.932 335
29	86,160	719	0.008 345	0.991 655	72	33,730	2,487	0.073 733	0.926 267
30	85,441	720	0.008 427	0.991 573	73	31,243	2,505	0.080 178	0.919 822
31	84,721	721	0.008 510	0.991 490	74	28,738	2,501	0.087 028	0.912 972
32	84,000	723	0.008 607	0.991 393	75	26,237	2,476	0.094 371	0.905 629
33	83,277	726	0.008 718	0.991 282	76	23,761	2,431	0.102 311	0.897 689
34	82,551	729	0.008 831	0.991 169	77	21,330	2,369	0.111 064	0.888 936
35	81,822	732	0.008 946	0.991 054	78	18,961	2,291	0.120 827	0.879 173
36	81,090	737	0.009 089	0.990 911	79	16,670	2,196	0.131 734	0.868 266
37	80,353	742	0.009 234	0.990 776	80	14,474	2,091	0.144 466	0.855 534
38	79,611	749	0.009 408	0.990 592	81	12,383	1,964	0.158 605	0.841 395
39	78,862	756	0.009 586	0.990 414	82	10,419	1,816	0.174 297	0.825 703
40	78,106	765	0.009 794	0.990 206	83	8,603	1,648	0.191 561	0.808 439
41	77,341	774	0.010 008	0.989 992	84	6,955	1,470	0.211 359	0.788 641
42	76,567	785	0.010 252	0.989 748	85	5,485	1,292	0.235 552	0.764 448
43	75,782	797	0.010 517	0.989 483	86	4,193	1,114	0.265 681	0.734 319
44	74,985	812	0.010 829	0.989 171	87	3,079	933	0.303 020	0.696 980
45	74,173	828	0.011 163	0.988 837	88	2,146	744	0.346 692	0.653 308
46	73,345	848	0.011 562	0.988 438	89	1,402	555	0.395 863	0.604 137
47	72,497	870	0.012 000	0.988 000	90	847	385	0.454 545	0.545 455
48	71,627	896	0.012 509	0.987 491	91	462	246	0.532 468	0.467 532
49	70,731	927	0.013 106	0.986 894	92	216	137	0.634 259	0.365 741
50	69,804	962	0.013 781	0.986 219	93	79	58	0.734 177	0.265 823
51	68,842	1,001	0.014 541	0.985 459	94	21	18	0.857 143	0.142 857
52	67,841	1,044	0.015 389	0.984 611	95	3	3	1.000 000	0.000 000

TABLE VIII

Squares, Cubes, Square Roots, and Cube Roots of Integers

No.	Square	Cube	Square root	Cube root
1	1	1	1.0000	1.00000
2	4	8	1.4142	1.25992
3	9	27	1.7321	1.44225
4	16	64	2.0000	1.58740
5	25	125	2.2361	1.70998
6	36	216	2.4495	1.81712
7	49	343	2.6458	1.91293
8	64	512	2.8284	2.00000
9	81	729	3.0000	2.08008
10	100	1 000	3.1623	2.15443
11	121	1 331	3.3166	2.22398
12	144	1 728	3.4641	2.28943
13	169	2 197	3.6056	2.35133
14	196	2 744	3.7417	2.41014
15	225	3 375	3.8730	2.46621
16	256	4 096	4.0000	2.51984
17	289	4 913	4.1231	2.57128
18	324	5 832	4.2426	2.62074
19	361	6 859	4.3589	2.66840
20	400	8 000	4.4721	2.71442
21	441	9 261	4.5826	2.75892
22	484	10 648	4.6904	2.80204
23	529	12 167	4.7958	2.84387
24	576	13 824	4.8990	2.88450
25	625	15 625	5.0000	2.92402
26	676	17 576	5.0990	2.96250
27	729	19 683	5.1962	3.00000
28	784	21 952	5.2915	3.03659
29	841	24 389	5.3852	3.07232
30	900	27 000	5.4772	3.10723
31	961	29 791	5.5678	3.14138
32	1 024	32 768	5.6569	3.17480
33	1 089	35 937	5.7446	3.20753
34	1 156	39 304	5.8310	3.23961
35	1 225	42 875	5.9161	3.27107
36	1 296	46 656	6.0000	3.30193
37	1 369	50 653	6.0828	3.33222
38	1 444	54 872	6.1644	3.36198
39	1 521	59 319	6.2450	3.39121
40	1 600	64 000	6.3246	3.41995
41	1 681	68 921	6.4031	3.44822
42	1 764	74 088	6.4807	3.47603
43	1 849	79 507	6.5574	3.50340
44	1 936	85 184	6.6332	3.53035
45	2 025	91 125	6.7082	3.55689
46	2 116	97 336	6.7823	3.58305
47	2 209	103 823	6.8557	3.60883
48	2 304	110 592	6.9282	3.63424
49	2 401	117 649	7.0000	3.65931
50	2 500	125 000	7.0711	3.68403
51	2 601	132 651	7.1414	3.70843
52	2 704	140 608	7.2111	3.73251
53	2 809	148 877	7.2801	3.75629
54	2 916	157 464	7.3485	3.77976
55	3 025	166 375	7.4162	3.80295
56	3 136	175 616	7.4833	3.82586
57	3 249	185 193	7.5498	3.84850
58	3 364	195 112	7.6158	3.87088
59	3 481	205 379	7.6811	3.89300
60	3 600	216 000	7.7460	3.91487
61	3 721	226 981	7.8102	3.93650
62	3 844	238 328	7.8740	3.95789
63	3 969	250 047	7.9373	3.97906
64	4 096	262 144	8.0000	4.00000
65	4 225	274 625	8.0623	4.02073
66	4 356	287 496	8.1240	4.04124
67	4 489	300 763	8.1854	4.06155
68	4 624	314 432	8.2462	4.08166
69	4 761	328 509	8.3066	4.10157
70	4 900	343 000	8.3666	4.12129
71	5 041	357 911	8.4261	4.14082
72	5 184	373 248	8.4853	4.16017
73	5 329	389 017	8.5440	4.17934
74	5 476	405 224	8.6023	4.19834
75	5 625	421 875	8.6603	4.21716
76	5 776	438 976	8.7178	4.23582
77	5 929	456 533	8.7750	4.25432
78	6 084	474 552	8.8318	4.27266
79	6 241	493 039	8.8882	4.29084
80	6 400	512 000	8.9443	4.30887
81	6 561	531 441	9.0000	4.32675
82	6 724	551 368	9.0554	4.34448
83	6 889	571 787	9.1104	4.36207
84	7 056	592 704	9.1652	4.37952
85	7 225	614 125	9.2195	4.39683
86	7 396	636 056	9.2736	4.41400
87	7 569	658 503	9.3274	4.43105
88	7 744	681 472	9.3808	4.44796
89	7 921	704 969	9.4340	4.46475
90	8 100	729 000	9.4868	4.48140
91	8 281	753 571	9.5394	4.49794
92	8 464	778 688	9.5917	4.51436
93	8 649	804 357	9.6437	4.53065
94	8 836	830 584	9.6954	4.54684
95	9 025	857 375	9.7468	4.56290
96	9 216	884 736	9.7980	4.57886
97	9 409	912 673	9.8489	4.59470
98	9 604	941 192	9.8995	4.61044
99	9 801	970 299	9.9499	4.62607

DENOMINATE NUMBERS

LENGTH

1 hand = 4 inches
1 palm = 3 inches (sometimes 4 in.)
1 span = 9 inches
12 inches (in.) = 1 foot (ft.)
1 (military) pace = $2\frac{1}{2}$ feet
3 feet = 1 yard (yd.)
$16\frac{1}{2}$ feet = $5\frac{1}{2}$ yards = 1 rod (rd.)
40 rods = 1 furlong
5280 feet = 320 rods = 1 mile (mi.)

METRIC LINEAR MEASURE

myr'i·a-me'ter = 10,000 m. = 6.214 miles
kil'o-meter = 1,000 m. = 0.6214 mile
hec'to-me'ter = 100 m. = 328 feet, 1 inch
dec'a-me'ter = 10 m. = 393.7 inches
me'ter = 1 m. = 39.37 inches
dec'i-me'ter = 1/10 m. = 3.937 inches
cen'ti-me'ter = 1/100 m. = 0.3937 inch
mil'li-me'ter = 1/1000 m. = 0.03937 inch
mi'cron = 1/1,000,000 m. = .00003937 inch

SURVEYORS' MEASURE

7.92 inches = 1 link (li.)
25 links = 1 rod (rd.)
4 rods = 1 (Gunter's) chain
100 links = 1 (Gunter's) chain
66 feet = 1 (Gunter's) chain
80 chains = 1 mile
(An engineers' chain, or measuring tape, is usually 100 feet long).
625 square links = 1 square rod
16 square rods = 1 square chain
10 square chains = 1 acre
36 square miles = 1 township

MARINER'S MEASURE

6 feet = 1 fathom
120 fathoms = 1 cable lgth. (or cable)
$7\frac{1}{2}$ cable lgths. = 1 mile
5,280 feet = 1 statute mile
6,080.27 feet = 1 nautical mile (U. S. Coast Survey)
3 nautical miles = 1 marine league.

AREA

144 square inches = 1 square foot
9 square feet = 1 square yard
$30\frac{1}{4}$ square yards = 1 square rod
40 sq. rods = 1 rood
43,560 square ft. = 160 square rds. = 1 acre
640 acres = 1 square mile

CUBIC MEASURE

1728 cubic inches = 1 cubic foot
27 cubic feet = 1 cubic yard
$24\frac{3}{4}$ cubic feet = 1 perch

METRIC VOLUME MEASURE

kil'o-li'ter (stere) = 1,000 liters
hec'to-li'ter = 100 liters
dec'a-li'ter = 10 liters
li'ter = 1 cu. dm.
dec'i-li'ter = 1/10 liter
cen'ti-li'ter = 1/100 liter
mil'li-li'ter = 1/1000 liter
(One *liter* equals .908 qt. *dry measure* and 1.0567 qt. *liquid measure.*)

DRY MEASURE

2 pints = 1 quart
8 quarts = 1 peck
4 pecks = 1 bushel
In the United States one *bushel* contains 2150.42 cubic inches; in Great Britain, 2218.2.
One bushel of:
corn, shelled = 56 lbs.
corn on cob = 70 lbs.
wheat, beans, peas, Irish potatoes = 60 lbs.
barley = 48 lbs.
oats = 32 lbs.
sweet potatoes = 55 lbs.

LIQUID MEASURE

4 gills = 1 pint
2 pints = 1 quart
4 quarts = 1 gallon
1 gallon = 231 cubic inches
$31\frac{1}{2}$ gallons = 1 barrel
63 gallons = 1 hogshead

APOTHECARIES' FLUID MEASURE

60 minims = 1 fluid dram
8 fluid drams = 1 fluid ounce
16 fluid ounces = 1 pint
8 pints = 1 gallon

WOOD MEASURE

16 cubic feet = 1 cord foot
8 cord feet or 128 cubic feet = 1 cord
A cord of wood is 8 ft. long, 4 ft. wide, and 4 ft. high.

BOARD MEASURE

1 ft. B. M. = 1 piece 1 ft. square and 1 inch, or less, thick. For pieces more than 1 inch thick, the B. M. is the product of the area in feet by the thickness in inches.

AVOIRDUPOIS WEIGHT

(used in weighing all articles except drugs, gold, silver, and precious stones).
$27\frac{11}{32}$ grains (gr.) = 1 dram (dr.)

16 drams = 1 ounce (oz.) = 28.3495 grams
16 ounces = 1 pound (lb)
14 pounds = 1 stone (English)
25 pounds = 1 quarter
4 quarters = 1 hundredweight (cwt.)
20 hundredweight = 1 ton (T.)
2000 pounds = 1 ton
2240 pounds = 1 long ton, or gross ton.
2352 pounds = 1 Cornish mining ton.
2204.6 pounds = 1000 kilograms = 1 metric ton

TROY WEIGHT

24 grains = 1 pennyweight
20 pennyweight = 1 ounce
12 ounces = 1 pound
1 pound = 5760 grains
1 carat = 3.168 grains

APOTHECARIES' WEIGHT

20 grains = 1 scruple
3 scruples = 1 dram
8 drams = 1 ounce
12 ounces = 1 pound
The pound, ounce and grain have the same weight as those of Troy Weight.

METRIC WEIGHT

mil'lier' = 1,000,000 gr. = 2,204.6 lbs.
quin'tal = 100,000 gr. = 220.46 lbs.
myr'i-a-gram = 10,000 gr. = 22.046 lbs.
kil'o-gram = 1,000 gr. = 2.205 lbs.
hec'to-gram = 100 gr. = 3.527 ozs.
dec'a-gram = 10 gr. = 0.353 oz.
gram = 1 gr. = 15.432 grains = 0.035 oz.
dec'i-gram = 1/10 gr. = 1.543 grs.
cen'ti-gram = 1/100 gr. = 0.154 gr.
mil'li-gram = 1/1000 gr. = 0.015 gr.

PAPER MEASURE

24 sheets = 1 quire
20 sheets = 1 quire of outsides
25 sheets = 1 printers' quire
20 quires = 1 ream
21½ quires = 1 printers' ream
2 reams = 1 bundle
4 reams = 1 printers' bundle
10 reams = 1 bale
60 skins = 1 roll of parchment
480 sheets = 1 short ream
500 sheets = 1 long ream

TIME

60 seconds = 1 minute
60 minutes = 1 hour
24 hours = 1 day
7 days = 1 week
365 days = 1 common year
366 days = 1 leap year
12 months = 1 year
360 days = 1 commercial year
1 sidereal year = 365 days, 6 hrs., 9 min., 9.5 sec.
1 tropical (equinoctial) year = 365 days, 5 hrs., 48 min., 45.7 sec.
100 years = 1 century
10 years = 1 decade

UNITED STATES MONEY

10 mills (m.) = 1 cent (ct. or ¢)
10 cents = 1 dime (di.)
10 dimes = 1 dollar ($)
10 dollars = 1 eagle
5 cents = 1 nickel
5 nickels = 1 quarter
2 quarters = 1 half-dollar

ENGLISH MONEY

(the left column consists of the coins in actual use).
1 farthing (copper) = ¼ penny
1 halfpenny (copper) = ½ penny
1 penny (copper) = $\frac{1}{12}$ shilling
6 pence (silver) = ½ shilling
1 shilling (silver) = 12 pence
1 florin (silver) = 2 shillings
½ crown (silver) = 2½ shillings
1 crown (silver) = 5 shillings
1 pound sterling (gold) = 20 shillings
The *pound* (£) is normally about $5, the shilling 25¢.

CIRCULAR MEASURE

60 seconds (″) = 1 minute (′)
60 minutes = 1 degree (°)
360 degrees = 1 circumference
1 radian = 57° 17′ 44″.806
1 degree = .01745329 radians
1 mil = 1/6400 of a circumference
69 miles = 1 degree of latitude, approximately.

MISCELLANEOUS

12 units = 1 dozen
12 dozen = 1 gross
12 gross = 1 great gross
20 units = 1 score
1 teaspoonful = ¼ tablespoonful (medical)
= ⅓ tablespoonful (cooking)
1 tablespoonfull = ½ fluid ounce
1 bushel = 2150.42 cu. in. or ⅘ cu. ft.
1 U. S. gallon = 231 cu. in.
1 gallon = 231 cu. in.
1 cu. ft. = 7.48 gal.
1 cu. ft. water = 62.425 lbs. (max. density)
1 gallon water = 8.337 lbs.
1 cu. ft. air = 0.0807 lb. (at 32° F.)
1 liter = .264178 gallon (U. S.)

1 inch = 2.54 centimeters

g (acceleration of gravity) = 32.16 feet per sec. per sec.

g (legal) = 980.665 cm./sec.^{2}

1 horse power = 550 ft. lb per sec. (legal) = 745. 70 watts (legal)

e = Naperian Base = 2.71828 18284

π = 3.14159 26535 89793 23846

M = $\log_{10} e$ = 0.43429 44819 03251

1/M = 2.30258 50929 94045 68402

1 astronomical unit = 93,000,000 miles (approx.) = the mean distance between the sun and the earth.

1 light year = 59×10^{11} miles = the distance traveled by light in 1 year.

1 parsec = the distance of a star at which the angle subtended by the radius of the earth's orbit is 1″ (about 3.3 light years).

DIFFERENTIATION FORMULAS

(See DERIVATIVE)

In the following formulas, u, v and y are functions of x which possess derivatives with respect to x, the other letters are constants and $\log u = \log_e u$.

$$\frac{dc}{dx} = 0.$$

$$\frac{d}{dx} x = 1.$$

$$\frac{d}{dx}(cv) = c\frac{dv}{dx}.$$

$$\frac{d}{dx} x^n = nx^{n-1}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx},$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx},$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2},$$

$$\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}.$$

n any number (positive or negative, integral or fractional),

$$\frac{d}{dx}(\sin u) = \cos u \cdot \frac{du}{dx}.$$

$$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}.$$

$$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}.$$

$$\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}.$$

$$\frac{d}{ax}(\sec u) = \sec u \tan u \frac{du}{dx},$$

$$\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}.$$

$$\frac{d}{ax}(\log u) = \frac{\frac{du}{dx}}{u},$$

$$\frac{d}{dx}(\log_a u) = \log_a e \cdot \frac{\frac{du}{dx}}{u},$$

$$\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{ax}.$$

$$\frac{d}{dx}(a^u) = \log a \cdot a^u \cdot \frac{du}{dx},$$

$$\frac{d}{dx}(\text{arc sin } u) = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$$

$$\frac{d}{dx}(\text{arc cos } u) = -\frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$$

$$\frac{d}{dx}(\text{arc tan } u) = \frac{\frac{du}{ax}}{1+u^2}.$$

$$\frac{d}{dx}(\text{arc cot } u) = -\frac{\frac{du}{dx}}{1+u^2}.$$

$$\frac{d}{dx}(\text{arc sec } u) = \frac{\frac{du}{dx}}{u\sqrt{u^2-1}},$$

$$-\pi < \sec^{-1} u < -\frac{\pi}{2}, \quad 0 < \sec^{-1} u < \frac{\pi}{2}.$$

$$\frac{d}{dx}(\text{arc csc } u) = -\frac{\frac{du}{dx}}{u\sqrt{u^2-1}}.$$

$$-\pi < \csc^{-1} u < -\frac{\pi}{2}, \quad 0 < \csc^{-1} u < \frac{\pi}{2}.$$

INTEGRAL TABLES*

In the following tables, the constant of integration, **C**, is omitted but should be added to the result of every integration. The letter **x** represents any variable; **u** represents any function of **x**; the remaining letters represent arbitrary constants, unless otherwise indicated; all angles are in radians. **Unless otherwise mentioned** $\log_e u \equiv \log u$.

Short Table of Integrals.

1. $\int df(x) = f(x)$.

2. $d\int f(x)dx = f(x)dx$.

3. $\int 0 \cdot dx = C$.

4. $\int af(x)\,dx = a\int f(x)\,dx$.

5. $\int (u \pm v)dx = \int u dx \pm \int v dx$.

6. $\int u\,dv = uv - \int v\,du$.

7. $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$.

8. $\int f(y)dx = \int \dfrac{f(y)dy}{\frac{dy}{dx}}$.

9. $\int u^n\,du = \frac{u^{n+1}}{n+1},\ n \neq -1$.

10. $\int \frac{du}{u} = \log\ u$.

11. $\int e^u\,du = e^u$.

12. $\int b^u\,du = \frac{b^u}{\log\ b}$.

13. $\int \sin u\,du = -\cos u$.

14. $\int \cos u\,du = \sin u$.

15. $\int \tan u\,du = \log\ \sec u = -\log\ \cos u$.

16. $\int \operatorname{ctn} u\,du = \log\ \sin u = -\log\ \csc u$.

17. $\int \sec u\,du = \log\ (\sec u + \tan u) = \log\ \tan\left(\frac{u}{2} + \frac{\pi}{4}\right)$.

18. $\int \csc u\,du = \log\ (\csc u - \operatorname{ctn} u) = \log\ \tan\frac{u}{2}$.

19. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{2}\sin u \cos u$.

20. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{2}\sin u \cos u$.

21. $\int \sec^2 u\,du = \tan u$.

22. $\int \csc^2 u\,du = -\operatorname{ctn} u$.

23. $\int \tan^2 u\,du = \tan u - u$.

24. $\int \operatorname{ctn}^2 u\,du = -\operatorname{ctn} u - u$.

25. $\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$.

26. $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\log\left(\frac{u-a}{u+a}\right) = -\frac{1}{a}\operatorname{ctnh}^{-1}\left(\frac{u}{a}\right)$, if $u^2 > a^2$,
$= \frac{1}{2a}\log\left(\frac{a-u}{a+u}\right) = -\frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right)$, if $u^2 < a^2$.

27. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right)$.

28. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log\ \left(u + \sqrt{u^2 \pm a^2}\right)$.

29. $\int \frac{du}{\sqrt{2\,au - u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right)$.

30. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right)$.

31. $\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 \pm u^2}}{u}\right)$. •

32. $\int \sqrt{a^2 - u^2}\cdot du = \frac{1}{2}\left(u\sqrt{a^2 - u^2} + a^2\sin^{-1}\frac{u}{a}\right)$.

33. $\int \sqrt{u^2 \pm a^2}\,du = \frac{1}{2}\left[u\sqrt{u^2 \pm a^2} \pm a^2\log\ \left(u + \sqrt{u^2 \pm a^2}\right)\right]$. •

34. $\int \sinh u\,du = \cosh u$.

35. $\int \cosh u\,du = \sinh u$.

36. $\int \tanh u\,du = \log\ (\cosh u)$.

37. $\int \operatorname{ctnh} u\,du = \log\ (\sinh u)$.

38. $\int \operatorname{sech} u\,du = \sin^{-1}(\tanh u)$.

39. $\int \operatorname{csch} u\,du = \log\ \left(\tanh\frac{u}{2}\right)$.

40. $\int \operatorname{sech} u \cdot \tanh u \cdot du = -\operatorname{sech} u$.

41. $\int \operatorname{csch} u \cdot \operatorname{ctnh} u \cdot du = -\operatorname{csch} u$.

Expressions Containing $(ax + b)$.

42. $\int (ax+b)^n dx = \frac{1}{a\,(n+1)}(ax+b)^{n+1},\ n \neq -1$.

43. $\int \frac{dx}{ax+b} = \frac{1}{a}\log_e\,(ax+b)$.

44. $\int \frac{dx}{(ax+b)^2} = -\frac{1}{a(ax+b)}$.

45. $\int \frac{dx}{(ax+b)^3} = -\frac{1}{2a(ax+b)^2}$.

46. $\int x(ax+b)^n\,dx = \frac{1}{a^2(n+2)}(ax+b)^{n+2} - \frac{b}{a^2(n+1)}(ax+b)^{n+1},\ n \neq -1, -2$

• $\log\left(\frac{u+\sqrt{u^2+a^2}}{a}\right) = \sinh^{-1}\left(\frac{u}{a}\right)$; $\log\left(\frac{a+\sqrt{a^2-u^2}}{u}\right) = \operatorname{sech}^{-1}\left(\frac{u}{a}\right)$;
$\log\left(\frac{u+\sqrt{u^2-a^2}}{a}\right) = \cosh^{-1}\left(\frac{u}{a}\right)$; $\log\left(\frac{a+\sqrt{a^2+u^2}}{u}\right) = \operatorname{csch}^{-1}\left(\frac{u}{a}\right)$.

* *These tables were taken, with permission, from* "HANDBOOK OF MATHEMATICAL TABLES AND FORMULAS," *Handbook Publishers, Inc., Sandusky, Ohio, edited by* RICHARD S. BURINGTON, *Associate Professor of Mathematics, Case School of Applied Science.*

47. $\displaystyle\int \frac{x\,dx}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\log(ax+b).$

48. $\displaystyle\int \frac{x\,dx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\log(ax+b).$

49. $\displaystyle\int \frac{x\,dx}{(ax+b)^3} = \frac{b}{2a^2(ax+b)^2} - \frac{1}{a^2(ax+b)}.$

50. $\displaystyle\int x^2(ax+b)^n dx = \frac{1}{a^3}\left[\frac{(ax+b)^{n+3}}{n+3} - 2b\frac{(ax+b)^{n+2}}{n+2} + b^2\frac{(ax+b)^{n+1}}{n+1}\right],\ n \neq -1, -2, -3.$

51. $\displaystyle\int \frac{x^2dx}{ax+b} = \frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2 - 2b(ax+b) + b^2\log(ax+b)\right].$

52. $\displaystyle\int \frac{x^2dx}{(ax+b)^2} = \frac{1}{a^3}\left[(ax+b) - 2b\log(ax+b) - \frac{b^2}{ax+b}\right].$

53. $\displaystyle\int \frac{x^2\,dx}{(ax+b)^3} = \frac{1}{a^3}\left[\log(ax+b) + \frac{2b}{ax+b} - \frac{b^2}{2(ax+b)^2}\right].$

54. $\displaystyle\int x^m(ax+b)^n\,dx$
$\displaystyle= \frac{1}{a(m+n+1)}\left[x^m(ax+b)^{n+1} - mb\int x^{m-1}(ax+b)^n dx\right],$
$\displaystyle= \frac{1}{m+n+1}\left[x^{m+1}(ax+b)^n + nb\int x^m(ax+b)^{n-1}dx\right],$
$m>0,\ m+n+1 \neq 0.$

55. $\displaystyle\int \frac{dx}{x(ax+b)} = \frac{1}{b}\log\frac{x}{ax+b}.$

56. $\displaystyle\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2}\log\frac{ax+b}{x}.$

57. $\displaystyle\int \frac{dx}{x^3(ax+b)} = \frac{2ax-b}{2b^2x^2} + \frac{a^2}{b^3}\log\frac{x}{ax+b}.$

58. $\displaystyle\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} - \frac{1}{b^2}\log\frac{ax+b}{x}.$

59. $\displaystyle\int \frac{dx}{x(ax+b)^3} = \frac{1}{b^3}\left[\frac{1}{2}\left(\frac{ax+2b}{ax+b}\right)^2 + \log\frac{x}{ax+b}\right].$

60. $\displaystyle\int \frac{dx}{x^2(ax+b)^2} = -\frac{b+2ax}{b^2x(ax+b)} + \frac{2a}{b^3}\log\frac{ax+b}{x}.$

61. $\displaystyle\int \sqrt{ax+b}\,dx = \frac{2}{3a}\sqrt{(ax+b)^3}$

62. $\displaystyle\int x\sqrt{ax+b}\,dx = \frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}$

63. $\displaystyle\int x^2\sqrt{ax+b}\,dx = \frac{2(15a^2x^2-12abx+8b^2)\sqrt{(ax+b)^3}}{105a^3}$

64. $\displaystyle\int x^3\sqrt{ax+b}\,dx$
$\displaystyle= \frac{2(35a^3x^3-30a^2bx^2+24ab^2x-16b^3)\sqrt{(ax+b)^3}}{315a^4}$

65. $\displaystyle\int x^n\sqrt{ax+b}\,dx = \frac{2}{a^{n+1}}\int u^2(u^2-b)^n du,\ u = \sqrt{ax+b}.$

66. $\displaystyle\int \frac{\sqrt{ax+b}}{x}dx = 2\sqrt{ax+b} + b\int\frac{dx}{x\sqrt{ax+b}}.$

67. $\displaystyle\int \frac{dx}{\sqrt{ax+b}} = \frac{2\sqrt{ax+b}}{a}$

68. $\displaystyle\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}.$

69. $\displaystyle\int \frac{x^2dx}{\sqrt{ax+b}} = \frac{2(3a^2x^2-4abx+8b^2)}{15a^3}\sqrt{ax+b}.$

70. $\displaystyle\int \frac{x^3dx}{\sqrt{ax+b}} = \frac{2(5a^3x^3-6a^2bx^2+8ab^2x-16b^3)}{35a^4}\sqrt{ax+b}.$

71. $\displaystyle\int \frac{x^n dx}{\sqrt{ax+b}} = \frac{2}{a^{n+1}}\int (u^2-b)^n du,\ u = \sqrt{ax+b}.$

72. $\displaystyle\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\log\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}},$ for $b>0.$

73. $\displaystyle\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\frac{ax+b}{-b}},$ or $\displaystyle\frac{-2}{\sqrt{b}}\tanh^{-1}\sqrt{\frac{ax+b}{b}},\ b<0.$

74. $\displaystyle\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int\frac{dx}{x\sqrt{ax+b}}.$

75. $\displaystyle\int \frac{dx}{x^3\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{2bx^2} + \frac{3a\sqrt{ax+b}}{4b^2x} + \frac{3a^2}{8b^2}\int\frac{dx}{x\sqrt{ax+b}}$

76. $\displaystyle\int \frac{dx}{x^n(ax+b)^m} = -\frac{1}{b^{m+n-1}}\int\frac{(u-a)^{m+n-2}du}{u^m},\ u = \frac{ax+b}{x}$

77. $\displaystyle\int (ax+b)^{\pm\frac{n}{2}}dx = \frac{2(ax+b)^{\frac{2\pm n}{2}}}{a(2\pm n)}$

78. $\displaystyle\int x(ax+b)^{\pm\frac{n}{2}}dx = \frac{2}{a^2}\left[\frac{(ax+b)^{\frac{4\pm n}{2}}}{4\pm n} - \frac{b(ax+b)^{\frac{2\pm n}{2}}}{2\pm n}\right].$

79. $\displaystyle\int \frac{dx}{x(ax+b)^{\frac{n}{2}}} = \frac{1}{b}\int\frac{dx}{x(ax+b)^{\frac{n-2}{2}}} - \frac{a}{b}\int\frac{dx}{(ax+b)^{\frac{n}{2}}}.$

80. $\displaystyle\int \frac{x^m dx}{\sqrt{ax+b}} = \frac{2x^m\sqrt{ax+b}}{(2m+1)a} - \frac{2mb}{(2m+1)a}\int\frac{x^{m-1}dx}{\sqrt{ax+b}}.$

81. $\displaystyle\int \frac{dx}{x^n\sqrt{ax+b}} = \frac{-\sqrt{ax+b}}{(n-1)bx^{n-1}} - \frac{(2n-3)a}{(2n-2)b}\int\frac{dx}{x^{n-1}\sqrt{ax+b}}.$

82. $\displaystyle\int \frac{(ax+b)^{\frac{n}{2}}}{x}dx = a\int(ax+b)^{\frac{n-2}{2}}dx + b\int\frac{(ax+b)^{\frac{n-2}{2}}}{x}dx.$

83. $\displaystyle\int \frac{dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad}\log\frac{cx+d}{ax+b},\ bc-ad \neq 0.$

84. $\displaystyle\int \frac{dx}{(ax+b)^2(cx+d)}$
$\displaystyle= \frac{1}{bc-ad}\left[\frac{1}{ax+b} + \frac{c}{bc-ad}\log\left(\frac{cx+d}{ax+b}\right)\right],\ bc-ad \neq 0.$

85. $\displaystyle\int (ax+b)^n(cx+d)^m\,dx = \frac{1}{(m+n+1)a}\left[(ax+b)^{n+1}(cx+d)^m\right.$
$\displaystyle\left. - m(bc-ad)\int(ax+b)^n(cx+d)^{m-1}\,dx\right].$

86. $\displaystyle\int \frac{dx}{(ax+b)^n(cx+d)^m} = \frac{1}{(m-1)(bc-ad)}\left[\frac{1}{(ax+b)^{n-1}(cx+d)^{m-1}}\right.$
$\displaystyle\left. - a(m+n-2)\int\frac{dx}{(ax+b)^n(cx+d)^{m-1}}\right],\ m>1,\ n>0,\ bc-ad\neq 0.$

87. $\displaystyle\int \frac{(ax+b)^n}{(cx+d)^m}dx$
$\displaystyle= -\frac{1}{(m-1)(bc-ad)}\left[\frac{(ax+b)^{n+1}}{(cx+d)^{m-1}} + (m-n-2)a\int\frac{(ax+b)^n dx}{(cx+d)^{m-1}}\right],$
$\displaystyle= \frac{-1}{(m-n-1)c}\left[\frac{(ax+b)^n}{(cx+d)^{m-1}} + n(bc-ad)\int\frac{(ax+b)^{n-1}}{(cx+d)^m}dx\right].$

88. $\displaystyle\int \frac{x\,dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad}\left[\frac{b}{a}\log(ax+b)\right.$
$\displaystyle\left. - \frac{d}{c}\log(cx+d)\right],\ bc-ad \neq 0.$

89. $\displaystyle\int \frac{x\,dx}{(ax+b)^2(cx+d)} = \frac{1}{bc-ad}\left[-\frac{b}{a(ax+b)}\right.$
$\displaystyle\left. - \frac{d}{bc-ad}\log\frac{cx+d}{ax+b}\right],\ bc-ad \neq 0.$

90. $\displaystyle\int \frac{cx+d}{\sqrt{ax+b}}dx = \frac{2}{3a^2}(3ad-2bc+acx)\sqrt{ax+b}$

91. $\int \frac{\sqrt{ax+b}}{cx+d}\,dx = \frac{2\sqrt{ax+b}}{c} - \frac{2}{c}\sqrt{\frac{ad-bc}{c}}\,\tan^{-1}\sqrt{\frac{c(ax+b)}{ad-bc}}, \; c>0,\; ad>bc.$

92. $\int \frac{\sqrt{ax+b}}{cx+d}\,dx = \frac{2\sqrt{ax+b}}{c} + \frac{1}{c}\sqrt{\frac{bc-ad}{c}}\,\log\frac{\sqrt{c(ax+b)}-\sqrt{bc-ad}}{\sqrt{c(ax+b)}+\sqrt{bc-ad}}, \; c>0,\; bc>ad.$

93. $\int \frac{dx}{(cx+d)\sqrt{ax+b}} = \frac{2}{\sqrt{c}\sqrt{ad-bc}}\tan^{-1}\sqrt{\frac{c(ax+b)}{ad-bc}}, \; c>0,\; ad>bc.$

94. $\int \frac{dx}{(cx+d)\sqrt{ax+b}} = \frac{1}{\sqrt{c}\sqrt{bc-ad}}\log\frac{\sqrt{c(ax+b)}-\sqrt{bc-ad}}{\sqrt{c(ax+b)}+\sqrt{bc-ad}}, \; c>0,\; bc>ad.$

pressions Containing ax^2+c, ax^n+c, $x^2\pm p^2$, and p^2-x^2.

95. $\int \frac{dx}{p^2+x^2} = \frac{1}{p}\tan^{-1}\frac{x}{p}$, or $-\frac{1}{p}\operatorname{ctn}^{-1}\left(\frac{x}{p}\right)$.

96. $\int \frac{dx}{p^2-x^2} = \frac{1}{2p}\log\frac{p+x}{p-x}$, or $\frac{1}{p}\tanh^{-1}\left(\frac{x}{p}\right)$.

97. $\int \frac{dx}{ax^2+c} = \frac{1}{\sqrt{ac}}\tan^{-1}\left(x\sqrt{\frac{a}{c}}\right)$, a and $c>0$.

98. $\int \frac{dx}{ax^2+c} = \frac{1}{2\sqrt{-ac}}\log\frac{x\sqrt{a}-\sqrt{-c}}{x\sqrt{a}+\sqrt{-c}}, \; a>0,\; c<0.$

$= \frac{1}{2\sqrt{-ac}}\log\frac{\sqrt{c}+x\sqrt{-a}}{\sqrt{c}-x\sqrt{-a}}, \; a<0,\; c>0.$

99. $\int \frac{dx}{(ax^2+c)^n} = \frac{1}{2(n-1)c}\cdot\frac{x}{(ax^2+c)^{n-1}} + \frac{2n-3}{2(n-1)c}\int\frac{dx}{(ax^2+c)^{n-1}}$, n a positive integer.

100. $\int x(ax^2+c)^n\,dx = \frac{1}{2a}\,\frac{(ax^2+c)^{n+1}}{n+1}, \; n\neq -1.$

101. $\int \frac{x}{ax^2+c}\,dx = \frac{1}{2a}\log(ax^2+c).$

102. $\int \frac{dx}{x(ax^2+c)} = \frac{1}{2c}\log\frac{ax^2}{ax^2+c}$

103. $\int \frac{dx}{x^2(ax^2+c)} = -\frac{1}{cx} - \frac{a}{c}\int\frac{dx}{ax^2+c}$

104. $\int \frac{x^2\,dx}{ax^2+c} = \frac{x}{a} - \frac{c}{a}\int\frac{dx}{ax^2+c}$

105. $\int \frac{x^n\,dx}{ax^2+c} = \frac{x^{n-1}}{a(n-1)} - \frac{c}{a}\int\frac{x^{n-2}\,dx}{ax^2+c}, \; n\neq 1.$

106. $\int \frac{x^2\,dx}{(ax^2+c)^n} = -\frac{1}{2(n-1)a}\cdot\frac{x}{(ax^2+c)^{n-1}} + \frac{1}{2(n-1)a}\int\frac{dx}{(ax^2+c)^{n-1}}$

107. $\int \frac{dx}{x^2(ax^2+c)^n} = \frac{1}{c}\int\frac{dx}{x^2(ax^2+c)^{n-1}} - \frac{a}{c}\int\frac{dx}{(ax^2+c)^n}$

108. $\int \sqrt{x^2\pm p^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2\pm p^2} \pm p^2\log\left(x+\sqrt{x^2\pm p^2}\right)\right].$

109. $\int \sqrt{p^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{p^2-x^2} + p^2\sin^{-1}\left(\frac{x}{p}\right)\right]$

110. $\int \frac{dx}{\sqrt{x^2\pm p^2}} = \log\left(x+\sqrt{x^2\pm p^2}\right).$

111. $\int \frac{dx}{\sqrt{p^2-x^2}} = \sin^{-1}\left(\frac{x}{p}\right)$ or $-\cos^{-1}\left(\frac{x}{p}\right)$.

112. $\int \sqrt{ax^2+c}\,dx = \frac{x}{2}\sqrt{ax^2+c} + \frac{c}{2\sqrt{a}}\log\left(x\sqrt{a}+\sqrt{ax^2+c}\right), \; a>0.$

113. $\int \sqrt{ax^2+c}\,dx = \frac{x}{2}\sqrt{ax^2+c} + \frac{c}{2\sqrt{-a}}\sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \; a<0.$

114. $\int \frac{dx}{\sqrt{ax^2+c}} = \frac{1}{\sqrt{a}}\log\left(x\sqrt{a}+\sqrt{ax^2+c}\right), \; a>0.$

115. $\int \frac{dx}{\sqrt{ax^2+c}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \; a<0.$

116. $\int x\sqrt{ax^2+c}\cdot dx = \frac{1}{3a}(ax^2+c)^{\frac{3}{2}}.$

117. $\int x^2\sqrt{ax^2+c}\,dx = \frac{x}{4a}\sqrt{(ax^2+c)^3} - \frac{cx}{8a}\sqrt{ax^2+c} - \frac{c^2}{8\sqrt{a^3}}\log\left(x\sqrt{a}+\sqrt{ax^2+c}\right), \; a>0.$

118. $\int x^2\sqrt{ax^2+c}\,dx = \frac{x}{4a}\sqrt{(ax^2+c)^3} - \frac{cx}{8a}\sqrt{ax^2+c} - \frac{c^2}{8a\sqrt{-a}}\sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right), \; a<0.$

119. $\int \frac{x\,dx}{\sqrt{ax^2+c}} = \frac{1}{a}\sqrt{ax^2+c}.$

120. $\int \frac{x^2\,dx}{\sqrt{ax^2+c}} = \frac{x}{a}\sqrt{ax^2+c} - \frac{1}{a}\int\sqrt{ax^2+c}\,dx.$

121. $\int \frac{\sqrt{ax^2+c}}{x}\,dx = \sqrt{ax^2+c} + \sqrt{c}\log\frac{\sqrt{ax^2+c}-\sqrt{c}}{x}, \; c>0.$

122. $\int \frac{\sqrt{ax^2+c}}{x}\,dx = \sqrt{ax^2+c} - \sqrt{-c}\tan^{-1}\frac{\sqrt{ax^2+c}}{\sqrt{-c}}, \; c<0.$

123. $\int \frac{dx}{x\sqrt{p^2\pm x^2}} = -\frac{1}{p}\log\left(\frac{p+\sqrt{p^2\pm x^2}}{x}\right).$

124. $\int \frac{dx}{x\sqrt{x^2-p^2}} = \frac{1}{p}\cos^{-1}\left(\frac{p}{x}\right)$, or $-\frac{1}{p}\sin^{-1}\left(\frac{p}{x}\right)$.

125. $\int \frac{dx}{x\sqrt{ax^2+c}} = \frac{1}{\sqrt{c}}\log\frac{\sqrt{ax^2+c}-\sqrt{c}}{x}, \; c>0.$

126. $\int \frac{dx}{x\sqrt{ax^2+c}} = \frac{1}{\sqrt{-c}}\sec^{-1}\left(x\sqrt{-\frac{a}{c}}\right), \; c<0.$

127. $\int \frac{dx}{x^2\sqrt{ax^2+c}} = -\frac{\sqrt{ax^2+c}}{cx}$

128. $\int \frac{x^n\,dx}{\sqrt{ax^2+c}} = \frac{x^{n-1}\sqrt{ax^2+c}}{na} - \frac{(n-1)c}{na}\int\frac{x^{n-2}\,dx}{\sqrt{ax^2+c}}, \; n>0.$

129. $\int x^n\sqrt{ax^2+c}\,dx = \frac{x^{n-1}(ax^2+c)^{\frac{3}{2}}}{(n+2)a} - \frac{(n-1)c}{(n+2)a}\int x^{n-2}\sqrt{ax^2+c}\cdot dx, \; n>0.$

130. $\int \frac{\sqrt{ax^2+c}}{x^n}\,dx = -\frac{(ax^2+c)^{\frac{3}{2}}}{c(n-1)x^{n-1}} - \frac{(n-4)a}{(n-1)c}\int\frac{\sqrt{ax^2+c}}{x^{n-2}}\,dx, \; n>1.$

131. $$\int \frac{dx}{x^n\sqrt{ax^2+c}} = -\frac{\sqrt{ax^2+c}}{c(n-1)x^{n-1}} - \frac{(n-2)}{(n-1)}\frac{a}{c}\int\frac{dx}{x^{n-2}\sqrt{ax^2+c}},\ n > 1.$$

132. $$\int (ax^2+c)^{\frac{3}{2}}\,dx = \frac{x}{8}(2ax^2+5c)\sqrt{ax^2+c} + \frac{3c^2}{8\sqrt{a}}\log\left(x\sqrt{a}+\sqrt{ax^2+c}\right),\ a > 0.$$

133. $$\int (ax^2+c)^{\frac{3}{2}}\,dx = \frac{x}{8}(2ax^2+5c)\sqrt{ax^2+c} + \frac{3c^2}{8\sqrt{-a}}\sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right),\ a < 0.$$

134. $$\int \frac{dx}{(ax^2+c)^{\frac{3}{2}}} = \frac{x}{c\sqrt{ax^2+c}}$$

135. $$\int x(ax^2+c)^{\frac{3}{2}}\,dx = \frac{1}{5a}(ax^2+c)^{\frac{5}{2}}.$$

136. $$\int x^2(ax^2+c)^{\frac{3}{2}}\,dx = \frac{x^3}{6}(ax^2+c)^{\frac{3}{2}} + \frac{c}{2}\int x^2\sqrt{ax^2+c}\,dx.$$

137. $$\int x^n(ax^2+c)^{\frac{3}{2}}\,dx = \frac{x^{n+1}(ax^2+c)^{\frac{3}{2}}}{n+4} + \frac{3c}{n+4}\int x^n\sqrt{ax^2+c}\,dx.$$

138. $$\int \frac{x\,dx}{(ax^2+c)^{\frac{3}{2}}} = -\frac{1}{a\sqrt{ax^2+c}}.$$

139. $$\int \frac{x^2dx}{(ax^2+c)^{\frac{3}{2}}} = -\frac{x}{a\sqrt{ax^2+c}} + \frac{1}{a\sqrt{a}}\log\left(x\sqrt{a}+\sqrt{ax^2+c}\right),\quad a > 0.$$

140. $$\int \frac{x^2dx}{(ax^2+c)^{\frac{3}{2}}} = -\frac{x}{a\sqrt{ax^2+c}} + \frac{1}{a\sqrt{-a}}\sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right),\quad a < 0.$$

141. $$\int \frac{x^3dx}{(ax^2+c)^{\frac{3}{2}}} = -\frac{x^2}{a\sqrt{ax^2+c}} + \frac{2}{a^2}\sqrt{ax^2+c}.$$

142. $$\int \frac{dx}{x(ax^n+c)} = \frac{1}{cn}\log\frac{x^n}{ax^n+c}$$

143. $$\int \frac{dx}{(ax^n+c)^m} = \frac{1}{c}\int\frac{dx}{(ax^n+c)^{m-1}} - \frac{a}{c}\int\frac{x^n dx}{(ax^n+c)^m}$$

144. $$\int \frac{dx}{x\sqrt{ax^n+c}} = \frac{1}{n\sqrt{c}}\log\frac{\sqrt{ax^n+c}-\sqrt{c}}{\sqrt{ax^n+c}+\sqrt{c}},\quad c > 0.$$

145. $$\int \frac{dx}{x\sqrt{ax^n+c}} = \frac{2}{n\sqrt{-c}}\sec^{-1}\sqrt{\frac{-ax^n}{c}},\quad c < 0.$$

146. $$\int x^{m-1}(ax^n+c)^p dx$$
$$= \frac{1}{m+np}\left[x^m(ax^n+c)^p + npc\int x^{m-1}(ax^n+c)^{p-1}dx\right]$$
$$= \frac{1}{cn(p+1)}\left[-x^m(ax^n+c)^{p+1} + (m+np+n)\int x^{m-1}(ax^n+c)^{p+1}dx\right]$$
$$= \frac{1}{a(m+np)}\left[x^{m-n}(ax^n+c)^{p+1} - (m-n)c\int x^{m-n-1}(ax^n+c)^p dx\right].$$
$$= \frac{1}{mc}\left[x^m(ax^n+c)^{p+1} - (m+np+n)\int x^{m+n-1}(ax^n+c)^p dx\right].$$

147. $$\int \frac{x^m dx}{(ax^n+c)^p} = \frac{1}{a}\int\frac{x^{m-n}dx}{(ax^n+c)^{p-1}} - \frac{c}{a}\int\frac{x^{m-n}dx}{(ax^n+c)^p}$$

148. $$\int \frac{dx}{x^m(ax^n+c)^p} = \frac{1}{c}\int\frac{dx}{x^m(ax^n+c)^{p-1}} - \frac{a}{c}\int\frac{dx}{x^{m-n}(ax^n+c)^p}.$$

Expressions Containing (ax^2+bx+c).

149. $$\int \frac{dx}{ax^2+bx+c} = \frac{1}{\sqrt{b^2-4ac}}\log\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}},\ b^2 > 4ac.$$

150. $$\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}},\quad b^2 < 4ac.$$

151. $$\int \frac{dx}{ax^2+bx+c} = -\frac{2}{2ax+b},\quad b^2 = 4ac.$$

152. $$\int \frac{dx}{(ax^2+bx+c)^{n+1}} = \frac{2ax+b}{n(4ac-b^2)(ax^2+bx+c)^n} + \frac{2(2n-1)a}{n(4ac-b^2)}\int\frac{dx}{(ax^2+bx+c)^n}$$

153. $$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a}\log(ax^2+bx+c) - \frac{b}{2a}\int\frac{dx}{ax^2+bx+c}$$

154. $$\int \frac{x^2dx}{ax^2+bx+c} = \frac{x}{a} - \frac{b}{2a^2}\log(ax^2+bx+c) + \frac{b^2-2ac}{2a^2}\int\frac{dx}{ax^2+bx+c}$$

155. $$\int \frac{x^n dx}{ax^2+bx+c} = \frac{x^{n-1}}{(n-1)a} - \frac{c}{a}\int\frac{x^{n-2}dx}{ax^2+bx+c} - \frac{b}{a}\int\frac{x^{n-1}dx}{ax^2+bx+c}$$

156. $$\int \frac{x\,dx}{(ax^2+bx+c)^{n+1}} = \frac{-(2c+bx)}{n(4ac-b^2)(ax^2+bx+c)^n} - \frac{b(2n-1)}{n(4ac-b^2)}\int\frac{dx}{(ax^2+bx+c)^n}$$

157. $$\int \frac{x^m dx}{(ax^2+bx+c)^{n+1}} = -\frac{x^{m-1}}{a(2n-m+1)(ax^2+bx+c)^n} - \frac{n-m+1}{2n-m+1}\cdot\frac{b}{a}\int\frac{x^{m-1}dx}{(ax^2+bx+c)^{n+1}} + \frac{m-1}{2n-m+1}\cdot\frac{c}{a}\int\frac{x^{m-2}dx}{(ax^2+bx+c)^{n+1}}$$

158. $$\int \frac{dx}{x(ax^2+bx+c)} = \frac{1}{2c}\log\frac{x^2}{ax^2+bx+c} - \frac{b}{2c}\int\frac{dx}{(ax^2+bx+c)}.$$

159. $$\int \frac{dx}{x^2(ax^2+bx+c)} = \frac{b}{2c^2}\log\left(\frac{ax^2+bx+c}{x^2}\right) - \frac{1}{cx} + \left(\frac{b^2}{2c^2}-\frac{a}{c}\right)\int\frac{dx}{(ax^2+bx+c)}$$

160. $$\int \frac{dx}{x^m(ax^2+bx+c)^{n+1}} = -\frac{1}{(m-1)cx^{m-1}(ax^2+bx+c)^n} - \frac{(n+m-1)}{m-1}\cdot\frac{b}{c}\int\frac{dx}{x^{m-1}(ax^2+bx+c)^{n+1}} - \frac{(2n+m-1)}{m-1}\cdot\frac{a}{c}\int\frac{dx}{x^{m-2}(ax^2+bx+c)^{n+1}}$$

161. $$\int \frac{dx}{x(ax^2+bx+c)^n} = \frac{1}{2c(n-1)(ax^2+bx+c)^{n-1}} - \frac{b}{2c}\int\frac{dx}{(ax^2+bx+c)^n} + \frac{1}{c}\int\frac{dx}{x(ax^2+bx+c)^{n-1}}$$

162. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\log\left(2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right),\ a>0.$$

163. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-a}}\sin^{-1}\frac{-2ax-b}{\sqrt{b^2-4ac}},\ a<0.$$

164. $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}}.$$

165. $$\int \frac{x^n dx}{\sqrt{ax^2+bx+c}} = \frac{x^{n-1}}{an}\sqrt{ax^2+bx+c} - \frac{b(2n-1)}{2an}\int\frac{x^{n-1}dx}{\sqrt{ax^2+bx+c}} - \frac{c(n-1)}{an}\int\frac{x^{n-2}dx}{\sqrt{ax^2+bx+c}}.$$

166. $$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}}.$$

167. $$\int x\sqrt{ax^2+bx+c}\,dx = \frac{(ax^2+bx+c)^{\frac{3}{2}}}{3a} - \frac{b}{2a}\int\sqrt{ax^2+bx+c}\,dx.$$

168. $$\int x^2\sqrt{ax^2+bx+c}\cdot dx = \left(x-\frac{5b}{6a}\right)\frac{(ax^2+bx+c)^{\frac{3}{2}}}{4a} + \frac{(5b^2-4ac)}{16a^2}\int\sqrt{ax^2+bx+c}\,dx.$$

169. $\int \frac{dx}{x\sqrt{ax^2+bx+c}} = -\frac{1}{\sqrt{c}}\log\left(\frac{\sqrt{ax^2+bx+c}+\sqrt{c}}{x}+\frac{b}{2\sqrt{c}}\right), c>0.$

170. $\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-c}}\sin^{-1}\frac{bx+2c}{x\sqrt{b^2-4ac}},\ c<0.$

171. $\int \frac{dx}{x\sqrt{ax^2+bx}} = -\frac{2}{bx}\sqrt{ax^2+bx},\ c=0.$

172. $\int \frac{dx}{x^n\sqrt{ax^2+bx+c}} = -\frac{\sqrt{ax^2+bx+c}}{c(n-1)x^{n-1}}$

$+\frac{b(3-2n)}{2c(n-1)}\int\frac{dx}{x^{n-1}\sqrt{ax^2+bx+c}}+\frac{a(2-n)}{c(n-1)}\int\frac{dx}{x^{n-2}\sqrt{ax^2+bx+c}}$

173. $\int \frac{dx}{(ax^2+bx+c)^{\frac{3}{2}}} = -\frac{2(2ax+b)}{(b^2-4ac)\sqrt{ax^2+bx+c}},\ b^2\neq 4ac.$

174. $\int \frac{dx}{(ax^2+bx+c)^{\frac{3}{2}}} = -\frac{1}{2\sqrt{a^3}(x+b/2a)^2},\ b^2=4ac.$

Miscellaneous Algebraic Expressions.

175. $\int\sqrt{2px-x^2}\,dx=\frac{1}{2}\left[(x-p)\sqrt{2px-x^2}+p^2\sin^{-1}[(x-p)/p]\right].$

176. $\int\frac{dx}{\sqrt{2px-x^2}}=\cos^{-1}\left(\frac{p-x}{p}\right).$

177. $\int\frac{dx}{\sqrt{ax+b}\cdot\sqrt{cx+d}}=\frac{2}{\sqrt{-ac}}\tan^{-1}\sqrt{\frac{-c(ax+b)}{a(cx+d)}}$

or $\frac{2}{\sqrt{ac}}\tanh^{-1}\sqrt{\frac{c(ax+b)}{a(cx+d)}}$

178. $\int\sqrt{ax+b}\cdot\sqrt{cx+d}\,dx=$
$\frac{(2acx+bc+ad)\sqrt{ax+b}\cdot\sqrt{cx+d}}{4ac}$

$+\frac{(ad-bc)^2}{8ac}\int\frac{dx}{\sqrt{ax+b}\cdot\sqrt{cx+d}}.$

179. $\int\sqrt{\frac{cx+d}{ax+b}}\,dx=\frac{\sqrt{ax+b}\cdot\sqrt{cx+d}}{a}$

$+\frac{(ad-bc)}{2a}\int\frac{dx}{\sqrt{ax+b}\cdot\sqrt{cx+d}}.$

180. $\int\sqrt{\frac{x+b}{x+d}}\,dx=\sqrt{x+d}\cdot\sqrt{x+b}$

$+(b-d)\log[\sqrt{x+d}+\sqrt{x+b}].$

181. $\int\sqrt{\frac{1+x}{1-x}}\,dx=\sin^{-1}x-\sqrt{1-x^2}.$

182. $\int\sqrt{\frac{p-x}{q+x}}dx=\sqrt{p-x}\cdot\sqrt{q+x}+(p+q)\sin^{-1}\sqrt{\frac{x+q}{p+q}}.$

183. $\int\sqrt{\frac{p+x}{q-x}}dx=-\sqrt{p+x}\cdot\sqrt{q-x}+(p+q)\sin^{-1}\sqrt{\frac{q-x}{p+q}}.$

184. $\int\frac{dx}{\sqrt{x-p}\cdot\sqrt{q-x}}=2\sin^{-1}\sqrt{\frac{x-p}{q-p}}.$

Expressions Containing sin ax.

185. $\int\sin ax\,dx=-\frac{1}{a}\cos ax.$

186. $\int\sin^2 ax\,dx=\frac{x}{2}-\frac{\sin 2ax}{4a}.$

187. $\int\sin^3 ax\,dx=-\frac{1}{a}\cos ax+\frac{1}{3a}\cos^3 ax.$

188. $\int\sin^4 ax\,dx=\frac{3x}{8}-\frac{3\sin 2ax}{16a}-\frac{\sin^3 ax\cos ax}{4a}.$

189. $\int\sin^n ax\,dx=-\frac{\sin^{n-1}ax\cos ax}{na}+\frac{n-1}{n}\int\sin^{n-2}ax\,dx,$ (n pos. integer)

190. $\int\frac{dx}{\sin ax}=\frac{1}{a}\log\tan\frac{ax}{2}=\frac{1}{a}\log(\csc ax-\operatorname{ctn} ax).$

191. $\int\frac{dx}{\sin^2 ax}=\int\csc^2 ax\,dx=-\frac{1}{a}\operatorname{ctn} ax.$

192. $\int\frac{dx}{\sin^n ax}=-\frac{1}{a(n-1)}\frac{\cos ax}{\sin^{n-1}ax}+\frac{n-2}{n-1}\int\frac{dx}{\sin^{n-2}ax},$ n integer > 1.

193. $\int\frac{dx}{1\pm\sin ax}=\mp\frac{1}{a}\tan\left(\frac{\pi}{4}\mp\frac{ax}{2}\right).$

194. $\int\frac{dx}{b+c\sin ax}=\frac{-2}{a\sqrt{b^2-c^2}}\tan^{-1}\left[\sqrt{\frac{b-c}{b+c}}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right)\right],\ b^2>c^2.$

195. $\int\frac{dx}{b+c\sin ax}=\frac{-1}{a\sqrt{c^2-b^2}}\log\frac{c+b\sin ax+\sqrt{c^2-b^2}\cdot\cos ax}{b+c\sin ax},\ c^2>b^2.$

196. $\int\sin ax\sin bx\,dx=\frac{\sin(a-b)x}{2(a-b)}-\frac{\sin(a+b)x}{2(a+b)},\ a^2\neq b^2.$

197. $\int\sqrt{1+\sin x}\,dx=\pm2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)$; use + sign when $(8k-1)\frac{\pi}{2}<x\leqq(8k+3)\frac{\pi}{2}$, otherwise −, k an integer.

198. $\int\sqrt{1-\sin x}\,dx=\pm2\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)$; use + sign when $(8k-3)\frac{\pi}{2}<x\leqq(8k+1)\frac{\pi}{2}$, otherwise −, k an integer.

Expressions Involving cos ax.

199. $\int\cos ax\,dx=\frac{1}{a}\sin ax.$

200. $\int\cos^2 ax\,dx=\frac{x}{2}+\frac{\sin 2ax}{4a}.$

201. $\int\cos^3 ax\,dx=\frac{1}{a}\sin ax-\frac{1}{3a}\sin^3 ax.$

202. $\int\cos^4 ax\,dx=\frac{3x}{8}+\frac{3\sin 2ax}{16a}+\frac{\cos^3 ax\sin ax}{4a}$

203. $\int\cos^n ax\,dx=\frac{\cos^{n-1}ax\sin ax}{na}+\frac{n-1}{n}\int\cos^{n-2}ax\,dx,$ (n pos. integer).

204. $\int\frac{dx}{\cos ax}=\frac{1}{a}\log\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)=\frac{1}{a}\log(\tan ax+\sec ax).$

205. $\int\frac{dx}{\cos^2 ax}=\frac{1}{a}\tan ax.$

206. $\int\frac{dx}{\cos^n ax}=\frac{1}{a(n-1)}\frac{\sin ax}{\cos^{n-1}ax}+\frac{n-2}{n-1}\int\frac{dx}{\cos^{n-2}ax},$ n integer > 1.

207. $\int\frac{dx}{1+\cos ax}=\frac{1}{a}\tan\frac{ax}{2}$ 210. $\int\frac{dx}{1-\cos ax}=-\frac{1}{a}\operatorname{ctn}\frac{ax}{2}$

208. $\int\sqrt{1+\cos x}\cdot dx=\pm\sqrt{2}\int\cos\frac{x}{2}\,dx=\pm2\sqrt{2}\sin\frac{x}{2}$
Use + when $(4k-1)\pi<x\leqq(4k+1)\pi$, otherwise −, k an integer.

209. $\int\sqrt{1-\cos x}\cdot dx=\pm\sqrt{2}\int\sin\frac{x}{2}\,dx=\mp2\sqrt{2}\cos\frac{x}{2}$
Use top signs when $4k\pi<x\leqq(4k+2)\pi$, otherwise bottom signs.

210. $\int\frac{dx}{b+c\cos ax}=\frac{1}{a\sqrt{b^2-c^2}}\tan^{-1}\left(\frac{\sqrt{b^2-c^2}\cdot\sin ax}{c+b\cos ax}\right),\ b^2>c^2.$

211. $\int\frac{dx}{b+c\cos ax}=\frac{1}{a\sqrt{c^2-b^2}}\tanh^{-1}\left[\frac{\sqrt{c^2-b^2}\cdot\sin ax}{c+b\cos ax}\right],c^2>b^2.$

212. $\int\cos ax\cdot\cos bx\,dx=\frac{\sin(a-b)x}{2(a-b)}+\frac{\sin(a+b)x}{2(a+b)},a^2\neq b^2.$

Expressions Containing sin ax and cos ax.

213. $\int\sin ax\cos bx\,dx=-\frac{1}{2}\left[\frac{\cos(a-b)x}{a-b}+\frac{\cos(a+b)x}{a+b}\right],a^2\neq b^2.$

214. $\int\sin^n ax\cos ax\,dx=\frac{1}{a(n+1)}\sin^{n+1}ax,\ n\neq-1.$

215. $\int\cos^n ax\sin ax\,dx=-\frac{1}{a(n+1)}\cos^{n+1}ax,\ n\neq-1.$

216. $\int \frac{\sin ax}{\cos ax}dx = -\frac{1}{a}\log \cos ax.$

217. $\int \frac{\cos ax}{\sin ax}dx = \frac{1}{a}\log \sin ax.$

218. $\int (b+c\sin ax)^n \cos ax\,dx = \frac{1}{ac(n+1)}(b+c\sin ax)^{n+1}, n \neq -1.$

219. $\int (b+c\cos ax)^n \sin ax\,dx = -\frac{1}{ac(n+1)}(b+c\cos ax)^{n+1}, n \neq -1.$

220. $\int \frac{\cos ax\,dx}{b+c\sin ax} = \frac{1}{ac}\log(b+c\sin ax).$

221. $\int \frac{\sin ax}{b+c\cos ax}dx = -\frac{1}{ac}\log(b+c\cos ax).$

222. $\int \frac{dx}{b\sin ax + c\cos ax} = \frac{1}{a\sqrt{b^2+c^2}}\left[\log \tan \frac{1}{2}\left(ax + \tan^{-1}\frac{c}{b}\right)\right].$

223. $\int \frac{dx}{b + c\cos ax + d\sin ax} = \frac{-1}{a\sqrt{b^2-c^2-d^2}}\sin^{-1} U,$

$U = \left[\frac{c^2+d^2+b(c\cos ax + a\sin ax)}{\sqrt{c^2+d^2}\,(b+c\cos ax + d\sin ax)}\right]$; or $= \frac{1}{a\sqrt{c^2+d^2-b^2}}\log V,$

$V = \left[\frac{c^2+d^2+b(c\cos ax + d\sin ax) + \sqrt{c^2+d^2-b^2}\,(c\sin ax - d\cos ax)}{\sqrt{c^2+d^2}\,(b + c\cos ax + d\sin ax)}\right],$

$b^2 \neq c^2+d^2, -\pi < ax < \pi.$

224. $\int \frac{dx}{b + c\cos ax + d\sin ax}$

$= \frac{1}{ab}\left[\frac{b-(c+d)\cos ax + (c-d)\sin ax}{b+(c-d)\cos ax + (c+d)\sin ax}\right], b^2 = c^2+d^2.$

225. $\int \frac{\sin^2 ax\,dx}{b + c\cos^2 ax} = \frac{1}{ac}\sqrt{\frac{b+c}{b}}\tan^{-1}\left(\sqrt{\frac{b}{b+c}}\cdot \tan ax\right) - \frac{x}{c}$

226. $\int \frac{\sin ax\cos ax\,dx}{b\cos^2 ax + c\sin^2 ax} = \frac{1}{2a(c-b)}\log(b\cos^2 ax + c\sin^2 ax).$

227. $\int \frac{dx}{b^2\cos^2 ax - c^2\sin^2 ax} = \frac{1}{2abc}\log\frac{b\cos ax + c\sin ax}{b\cos ax - c\sin ax}$

228. $\int \frac{dx}{b^2\cos^2 ax + c^2\sin^2 ax} = \frac{1}{abc}\tan^{-1}\left(\frac{c\tan ax}{b}\right).$

229. $\int \sin^2 ax\cos^2 ax\,dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$

230. $\int \frac{dx}{\sin ax\cos ax} = \frac{1}{a}\log \tan ax.$

231. $\int \frac{dx}{\sin^2 ax\cos^2 ax} = \frac{1}{a}(\tan ax - \operatorname{ctn} ax).$

232. $\int \frac{\sin^2 ax}{\cos ax}dx = \frac{1}{a}\left[-\sin ax + \log\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right].$

233. $\int \frac{\cos^2 ax}{\sin ax}dx = \frac{1}{a}\left[\cos ax + \log\tan\frac{ax}{2}\right].$

234. $\int \sin^m ax\cos^n ax\,dx = -\frac{\sin^{m-1} ax\cos^{n+1} ax}{a(m+n)}$

$+ \frac{m-1}{m+n}\int \sin^{m-2} ax\cos^n ax\,dx, \quad m, n > 0.$

235. $\int \sin^m ax\cos^n ax\,dx = \frac{\sin^{m+1} ax\cos^{n-1} ax}{a(m+n)}$

$+ \frac{n-1}{m+n}\int \sin^m ax\cos^{n-2} ax\,dx, \quad m, n > 0.$

236. $\int \frac{\sin^m ax}{\cos^n ax}dx = \frac{\sin^{m+1} ax}{a(n-1)\cos^{n-1} ax}$

$- \frac{m-n+2}{n-1}\int \frac{\sin^m ax}{\cos^{n-2} ax}dx, \quad m, n > 0, n \neq 1.$

237. $\int \frac{\cos^n ax}{\sin^m ax}dx = \frac{-\cos^{n+1} ax}{a(m-1)\sin^{m-1} ax}$

$+ \frac{m-n-2}{(m-1)}\int \frac{\cos^n ax}{\sin^{m-2} ax}dx, \quad m, n, > 0, m \neq 1.$

238. $\int \frac{dx}{\sin^m ax\cos^n ax} = \frac{1}{a(n-1)}\,\frac{1}{\sin^{m-1} ax\cos^{n-1} ax}$

$+ \frac{m+n-2}{(n-1)}\int \frac{dx}{\sin^m ax\cos^{n-2} ax}$

239. $\int \frac{dx}{\sin^m ax\cos^n ax} = -\frac{1}{a(m-1)}\,\frac{1}{\sin^{m-1} ax\cos^{n-1} ax}$

$+ \frac{m+n-2}{(m-1)}\int \frac{dx}{\sin^{m-2} ax\cos^n ax}$

240. $\int \frac{\sin^{2n} ax}{\cos ax}dx = \int \frac{(1-\cos^2 ax)^n}{\cos ax}dx.$ (Expand, divide, and use 203).

241. $\int \frac{\cos^{2n} ax}{\sin ax}dx = \int \frac{(1-\sin^2 ax)^n}{\sin ax}dx.$ (Expand, divide, and use 189).

242. $\int \frac{\sin^{2n+1} ax}{\cos ax}dx = \int \frac{(1-\cos^2 ax)^n}{\cos ax}\sin ax\,dx.$ (Expand, divide, and use 215).

243. $\int \frac{\cos^{2n+1} ax}{\sin ax}dx = \int \frac{(1-\sin^2 ax)^n}{\sin ax}\cos ax\,dx.$ (Expand, divide, and use 214).

Expressions Containing $\tan ax$ *or* $\operatorname{ctn} ax$ ($\tan ax = 1/\operatorname{ctn} ax$).

244. $\int \tan ax\,dx = -\frac{1}{a}\log\cos ax.$

245. $\int \tan^2 ax\,dx = \frac{1}{a}\tan ax - x.$

246. $\int \tan^3 ax\,dx = \frac{1}{2a}\tan^2 ax + \frac{1}{a}\log\cos ax.$

247. $\int \tan^n ax\,dx = \frac{1}{a(n-1)}\tan^{n-1} ax - \int \tan^{n-2} ax\,dx,$ n integer>1.

248. $\int \operatorname{ctn} u\,du = \log\sin u$, or $-\log\csc u$, where u is any function of x.

249. $\int \operatorname{ctn}^2 ax\,dx = \int \frac{dx}{\tan^2 ax} = -\frac{1}{a}\operatorname{ctn} ax - x.$

250. $\int \operatorname{ctn}^3 ax\,dx = -\frac{1}{2a}\operatorname{ctn}^2 ax - \frac{1}{a}\log\sin ax.$

251. $\int \operatorname{ctn}^n ax\,dx = \int \frac{dx}{\tan^n ax} = -\frac{1}{a(n-1)}\operatorname{ctn}^{n-1} ax$

$-\int \operatorname{ctn}^{n-2} ax\,dx,$ n integer>1.

252. $\int \frac{dx}{b + c\tan ax} = \int \frac{\operatorname{ctn} ax\,dx}{b\operatorname{ctn} ax + c}$

$= \frac{1}{b^2+c^2}\left[bx + \frac{c}{a}\log(b\cos ax + c\sin ax)\right].$

253. $\int \frac{dx}{b + c\operatorname{ctn} ax} = \int \frac{\tan ax\,dx}{b\tan ax + c}$

$= \frac{1}{b^2+c^2}\left[bx - \frac{c}{a}\log(c\cos ax + b\sin ax)\right].$

254. $\int \frac{dx}{\sqrt{b+c\tan^2 ax}} = \frac{1}{a\sqrt{b-c}}\sin^{-1}\left(\sqrt{\frac{b-c}{b}}\sin ax\right)$, b pos., $b^2>c^2$.

Expressions Containing sec $ax = 1/\cos ax$ or csc $ax = 1/\sin ax$.

255. $\int \sec ax\, dx = \frac{1}{a}\log \tan\left(\frac{ax}{2} + \frac{\pi}{4}\right).$

256. $\int \sec^2 ax\, dx = \frac{1}{a}\tan ax.$

257. $\int \sec^3 ax\, dx = \frac{1}{2a}\left[\tan ax \sec ax + \log \tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right].$

258. $\int \sec^n ax\, dx = \frac{1}{a(n-1)}\frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1}\int \sec^{n-2} ax\, dx$, n integer > 1.

259. $\int \csc ax\, dx = \frac{1}{a}\log \tan\frac{ax}{2}.$

260. $\int \csc^2 ax\, dx = -\frac{1}{a}\operatorname{ctn} ax.$

261. $\int \csc^3 ax\, dx = \frac{1}{2a}\left[-\operatorname{ctn} ax \csc ax + \log \tan\frac{ax}{2}\right].$

262. $\int \csc^n ax\, dx = -\frac{1}{a(n-1)}\frac{\cos ax}{\sin^{n-1} ax} + \frac{n-2}{n-1}\int \csc^{n-2} ax\, dx$, n integer > 1.

Expressions Containing tan ax and sec ax or ctn ax and csc ax.

263. $\int \tan ax \sec ax\, dx = \frac{1}{a}\sec ax.$

264. $\int \tan^n ax \sec^2 ax\, dx = \frac{1}{a(n+1)}\tan^{n+1} ax,\ n \neq -1.$

265. $\int \tan ax \sec^n ax\, dx = \frac{1}{an}\sec^n ax,\ n \neq 0.$

266. $\int \operatorname{ctn} ax \csc ax\, dx = -\frac{1}{a}\csc ax.$

267. $\int \operatorname{ctn}^n ax \csc^2 ax\, dx = -\frac{1}{a(n+1)}\operatorname{ctn}^{n+1} ax,\ n \neq -1.$

268. $\int \operatorname{ctn} ax \csc^n ax\, dx = -\frac{1}{an}\csc^n ax,\ n \neq 0.$

269. $\int \frac{\csc^2 ax\, dx}{\operatorname{ctn} ax} = -\frac{1}{a}\log \operatorname{ctn} ax.$

Expressions Containing Algebraic and Trigonometric Functions.

270. $\int x \sin ax\, dx = \frac{1}{a^2}\sin ax - \frac{1}{a}x\cos ax.$

271. $\int x^2 \sin ax\, dx = \frac{2x}{a^2}\sin ax + \frac{2}{a^3}\cos ax - \frac{x^2}{a}\cos ax.$

272. $\int x^3 \sin ax\, dx = \frac{3x^2}{a^2}\sin ax - \frac{6}{a^4}\sin ax - \frac{x^3}{a}\cos ax + \frac{6x}{a^3}\cos ax.$

273. $\int x \sin^2 ax\, dx = \frac{x^2}{4} - \frac{x\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}.$

274. $\int x^2 \sin^2 ax\, dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax - \frac{x\cos 2ax}{4a^2}.$

275. $\int x^3 \sin^2 ax\, dx = \frac{x^4}{8} - \left(\frac{x^3}{4a} - \frac{3x}{8a^3}\right)\sin 2ax - \left(\frac{3x^2}{8a^2} - \frac{3}{16a^4}\right)\cos 2ax.$

276. $\int x \sin^3 ax\, dx = \frac{x\cos 3ax}{12a} - \frac{\sin 3ax}{36a^2} - \frac{3x\cos ax}{4a} + \frac{3\sin ax}{4a^2}.$

277. $\int x^n \sin ax\, dx = -\frac{1}{a}x^n\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\, dx,\ n > 0.$

278. $\int \frac{\sin ax\, dx}{x} = ax - \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} - \cdots.$

279. $\int \frac{\sin ax\, dx}{x^m} = \frac{-1}{(m-1)}\frac{\sin ax}{x^{m-1}} + \frac{a}{(m-1)}\int \frac{\cos ax\, dx}{x^{m-1}}.$

280. $\int x \cos ax\, dx = \frac{1}{a^2}\cos ax + \frac{1}{a}x\sin ax.$

281. $\int x^2 \cos ax\, dx = \frac{2x}{a^2}\cos ax - \frac{2}{a^3}\sin ax + \frac{x^2}{a}\sin ax.$

282. $\int x^3 \cos ax\, dx = \frac{(3a^2x^2 - 6)\cos ax}{a^4} + \frac{(a^2x^3 - 6x)\sin ax}{a^3}.$

283. $\int x \cos^2 ax\, dx = \frac{x^2}{4} + \frac{x\sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}.$

284. $\int x^2 \cos^2 ax\, dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax + \frac{x\cos 2ax}{4a^2}.$

285. $\int x^3 \cos^2 ax\, dx = \frac{x^4}{8} + \left(\frac{x^3}{4a} - \frac{3x}{8a^3}\right)\sin 2ax + \left(\frac{3x^2}{8a^2} - \frac{3}{16a^4}\right)\cos 2ax.$

286. $\int x \cos^3 ax\, dx = \frac{x\sin 3ax}{12a} + \frac{\cos 3ax}{36a^2} + \frac{3x\sin ax}{4a} + \frac{3\cos ax}{4a^2}.$

287. $\int x^n \cos ax\, dx = \frac{1}{a}x^n\sin ax - \frac{n}{a}\int x^{n-1}\sin ax\, dx$, n pos.

288. $\int \frac{\cos ax\, dx}{x} = \log ax - \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} - \cdots.$

289. $\int \frac{\cos ax}{x^m}dx = -\frac{1}{(m-1)}\cdot\frac{\cos ax}{x^{m-1}} - \frac{a}{(m-1)}\int \frac{\sin ax\, dx}{x^{m-1}}.$

Expressions Containing Exponential and Logarithmic Functions.

290. $\int e^{ax}\, dx = \frac{1}{a}e^{ax}, \quad \int b^{ax}\, dx = \frac{b^{ax}}{a\log b}.$

291. $\int xe^{ax}\, dx = \frac{e^{ax}}{a^2}(ax - 1), \int xb^{ax}\, dx = \frac{xb^{ax}}{a\log b} - \frac{b^{ax}}{a^2(\log b)^2}.$

292. $\int x^2e^{ax}dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2).$

293. $\int x^ne^{ax}\, dx = \frac{1}{a}x^ne^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx$, n pos.

294. $\int x^ne^{ax}dx = \frac{e^{ax}}{a^{n+1}}\left[(ax)^n - n(ax)^{n-1} + n(n-1)(ax)^{n-2} - \cdots + (-1)^n n!\right]$, n pos. integ.

295. $\int x^ne^{-ax}dx = -\frac{e^{-ax}}{a^{n+1}}\left[(ax)^n + n(ax)^{n-1} + n(n-1)(ax)^{n-2} + \cdots + n!\right]$, n pos. integ.

296. $\int x^nb^{ax}\, dx = \frac{x^nb^{ax}}{a\log b} - \frac{n}{a\log b}\int x^{n-1}b^{ax}\, dx$, n pos.

297. $\int \frac{e^{ax}}{x}dx = \log x + ax + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^3}{3\cdot 3!} + \cdots.$

298. $\int \frac{e^{ax}}{x^n}dx = \frac{1}{n-1}\left[-\frac{e^{ax}}{x^{n-1}} + a\int \frac{e^{ax}}{x^{n-1}}dx\right]$, n integ > 1.

299. $\int \frac{dx}{b + ce^{ax}} = \frac{1}{ab}[ax - \log(b + ce^{ax})].$

300. $\int \frac{e^{ax}\, dx}{b + ce^{ax}} = \frac{1}{ac}\log(b + ce^{ax}).$

301. $\int \frac{dx}{be^{ax} + ce^{-ax}} = \frac{1}{a\sqrt{bc}}\tan^{-1}\left(e^{ax}\sqrt{\frac{b}{c}}\right)$, b and c pos.

302. $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx).$

303. $\int e^{ax}\sin bx \sin cx\,dx = \frac{e^{ax}[(b-c)\sin(b-c)x + a\cos(b-c)x]}{2[a^2+(b-c)^2]}$
$\qquad - \frac{e^{ax}[(b+c)\sin(b+c)x + a\cos(b+c)x]}{2[a^2+(b+c)^2]}$

304. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx).$

305. $\int e^{ax}\cos bx \cos cx\,dx = \frac{e^{ax}[(b-c)\sin(b-c)x + a\cos(b-c)x]}{2[a^2+(b-c)^2]}$
$\qquad + \frac{e^{ax}[(b+c)\sin(b+c)x + a\cos(b+c)x]}{2[a^2+(b+c)^2]}$

306. $\int e^{ax}\sin bx \cos cx\,dx = \frac{e^{ax}[a\sin(b-c)x - (b-c)\cos(b-c)x]}{2[a^2+(b-c)^2]}$
$\qquad + \frac{e^{ax}[a\sin(b+c)x - (b+c)\cos(b+c)x]}{2[a^2+(b+c)^2]}$

307. $\int e^{ax}\sin bx \sin(bx+c)\,dx =$
$\qquad \frac{e^{ax}\cos c}{2a} - \frac{e^{ax}[a\cos(2bx+c) + 2b\sin(2bx+c)]}{2(a^2+4b^2)}$

308. $\int e^{ax}\cos bx \cos(bx+c)\,dx =$
$\qquad \frac{e^{ax}\cos c}{2a} + \frac{e^{ax}[a\cos(2bx+c) + 2b\sin(2bx+c)]}{2(a^2+4b^2)}$

309. $\int e^{ax}\cos bx \sin(bx+c)\,dx =$
$\qquad \frac{e^{ax}\sin c}{2a} + \frac{e^{ax}[a\sin(2bx+c) - 2b\cos(2bx+c)]}{2(a^2+4b^2)}$

310. $\int e^{ax}\sin bx \cos(bx+c)\,dx =$
$\qquad -\frac{e^{ax}\sin c}{2a} + \frac{e^{ax}[a\sin(2bx+c) - 2b\cos(2bx+c)]}{2(a^2+4b^2)}$

311. $\int xe^{ax}\sin bx\,dx = \frac{xe^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$
$\qquad - \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\sin bx - 2ab\cos bx].$

312. $\int xe^{ax}\cos bx\,dx = \frac{xe^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$
$\qquad - \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\cos bx + 2ab\sin bx].$

313. $\int e^{ax}\cos^n bx\,dx = \frac{e^{ax}\cos^{n-1}bx\,(a\cos bx + nb\sin bx)}{a^2+n^2b^2}$
$\qquad + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\cos^{n-2}bx\,dx.$

314. $\int e^{ax}\sin^n bx\,dx = \frac{e^{ax}\sin^{n-1}bx\,(a\sin bx - nb\cos bx)}{a^2+n^2b^2}$
$\qquad + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\sin^{n-2}bx\,dx.$

315. $\int \log ax\,dx = x\log ax - x.$

316. $\int x\log ax\,dx = \frac{x^2}{2}\log ax - \frac{x^2}{4}.$

317. $\int x^2\log ax\,dx = \frac{x^3}{3}\log ax - \frac{x^3}{9}.$

318. $\int (\log ax)^2\,dx = x(\log ax)^2 - 2x\log ax + 2x.$

319. $\int (\log ax)^n\,dx = x(\log ax)^n - n\int(\log ax)^{n-1}\,dx,\quad n$ pos.

320. $\int x^n\log ax\,dx = x^{n+1}\left[\frac{\log ax}{n+1} - \frac{1}{(n+1)^2}\right],\quad n \neq -1.$

321. $\int x^n(\log ax)^m\,dx = \frac{x^{n+1}}{n+1}(\log ax)^m - \frac{m}{n+1}\int x^n(\log ax)^{m-1}\,dx.$

322. $\int \frac{(\log ax)^n}{x}\,dx = \frac{(\log ax)^{n+1}}{n+1},\quad n \neq -1.$

323. $\int \frac{dx}{x\log ax} = \log(\log ax).$

324. $\int \frac{dx}{x(\log ax)^n} = -\frac{1}{(n-1)(\log ax)^{n-1}}$

325. $\int \frac{x^n\,dx}{(\log ax)^m} = \frac{-x^{n+1}}{(m-1)(\log ax)^{m-1}} + \frac{n+1}{m-1}\int\frac{x^n\,dx}{(\log ax)^{m-1}},\ m \neq 1.$

326. $\int \frac{x^n\,dx}{\log ax} = \frac{1}{a^{n+1}}\int\frac{e^y\,dy}{y},\ y = (n+1)\log ax.$

327. $\int \frac{x^n\,dx}{\log ax} = \frac{1}{a^{n+1}}\Big[\log|\log ax| + (n+1)\log ax$
$\qquad + \frac{(n+1)^2(\log ax)^2}{2\cdot 2!} + \frac{(n+1)^3(\log ax)^3}{3\cdot 3!} + \cdots\Big].$

328. $\int \frac{dx}{\log ax} = \frac{1}{a}\Big[\log(\log ax) + \log ax + \frac{(\log ax)^2}{2\cdot 2!}$
$\qquad + \frac{(\log ax)^3}{3\cdot 3!} + \cdots\Big].$

329. $\int \sin(\log ax)\,dx = \frac{x}{2}[\sin(\log ax) - \cos(\log ax)].$

330. $\int \cos(\log ax)\,dx = \frac{x}{2}[\sin(\log ax) + \cos(\log ax)].$

331. $\int e^{ax}\log bx\,dx = \frac{1}{a}e^{ax}\log bx - \frac{1}{a}\int\frac{e^{ax}}{x}\,dx.$

Expressions Containing Inverse Trigonometric Functions

332. $\int \sin^{-1}ax\,dx = x\sin^{-1}ax + \frac{1}{a}\sqrt{1-a^2x^2}.$

333. $\int (\sin^{-1}ax)^2\,dx = x(\sin^{-1}ax)^2 - 2x + \frac{2}{a}\sqrt{1-a^2x^2}\sin^{-1}ax.$

334. $\int x\sin^{-1}ax\,dx = \frac{x^2}{2}\sin^{-1}ax - \frac{1}{4a^2}\sin^{-1}ax + \frac{x}{4a}\sqrt{1-a^2x^2}.$

335. $\int x^n\sin^{-1}ax\,dx = \frac{x^{n+1}}{n+1}\sin^{-1}ax$
$\qquad - \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}},\ n \neq -1.$

336. $\int \frac{\sin^{-1}ax\,dx}{x} = ax + \frac{1}{2\cdot 3\cdot 3}(ax)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}(ax)^5$
$\qquad + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}(ax)^7 + \cdots,\ a^2x^2 < 1.$

337. $\int \frac{\sin^{-1}ax\,dx}{x^2} = -\frac{1}{x}\sin^{-1}ax - a\log\left|\frac{1+\sqrt{1-a^2x^2}}{ax}\right|.$

338. $\int \cos^{-1}ax\,dx = x\cos^{-1}ax - \frac{1}{a}\sqrt{1-a^2x^2}.$

339. $\int (\cos^{-1}ax)^2\,dx = x(\cos^{-1}ax)^2 - 2x - \frac{2}{a}\sqrt{1-a^2x^2}\cos^{-1}ax.$

340. $\int x\cos^{-1}ax\,dx = \frac{x^2}{2}\cos^{-1}ax - \frac{1}{4a^2}\cos^{-1}ax - \frac{x}{4a}\sqrt{1-a^2x^2}.$

341. $\int x^n\cos^{-1}ax\,dx = \frac{x^{n+1}}{n+1}\cos^{-1}ax$
$\qquad + \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}},\ n \neq -1.$

342. $\displaystyle\int \frac{\cos^{-1} ax\,dx}{x} = \frac{\pi}{2}\log|ax| - ax - \frac{1}{2\cdot3\cdot3}(ax)^3 - \frac{1\cdot3}{2\cdot4\cdot5\cdot5}(ax)^5 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot7\cdot7}(ax)^7 - \cdots,\ a^2x^2<1.$

343. $\displaystyle\int \frac{\cos^{-1} ax\,dx}{x^2} = -\frac{1}{x}\cos^{-1} ax + a\log\left|\frac{1+\sqrt{1-a^2x^2}}{ax}\right|.$

344. $\displaystyle\int \tan^{-1} ax\,dx = x\tan^{-1} ax - \frac{1}{2a}\log(1+a^2x^2).$

345. $\displaystyle\int x^n\tan^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\tan^{-1} ax - \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{1+a^2x^2},\ n\neq-1.$

346. $\displaystyle\int \frac{\tan^{-1} ax\,dx}{x^2} = -\frac{1}{x}\tan^{-1} ax - \frac{a}{2}\log\left(\frac{1+a^2x^2}{a^2x^2}\right).$

347. $\displaystyle\int \mathrm{ctn}^{-1} ax\,dx = x\,\mathrm{ctn}^{-1} ax + \frac{1}{2a}\log(1+a^2x^2).$

348. $\displaystyle\int x^n\,\mathrm{ctn}^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\mathrm{ctn}^{-1} ax + \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{1+a^2x^2},\ n\neq-1.$

349. $\displaystyle\int \frac{\mathrm{ctn}^{-1} ax\,dx}{x^2} = -\frac{1}{x}\mathrm{ctn}^{-1} ax + \frac{a}{2}\log\left(\frac{1+a^2x^2}{a^2x^2}\right).$

350. $\displaystyle\int \sec^{-1} ax\,dx = x\sec^{-1} ax - \frac{1}{a}\log\left(ax+\sqrt{a^2x^2-1}\right).$

351. $\displaystyle\int x^n\sec^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\sec^{-1} ax \pm \frac{1}{n+1}\int\frac{x^n\,dx}{\sqrt{a^2x^2-1}},\ n\neq-1.$

Use + sign when $\frac{\pi}{2} < \sec^{-1} ax < \pi$; − sign when $0 < \sec^{-1} ax < \frac{\pi}{2}$.

352. $\displaystyle\int \csc^{-1} ax\,dx = x\csc^{-1} ax + \frac{1}{a}\log\left(ax+\sqrt{a^2x^2-1}\right).$

353. $\displaystyle\int x^n\csc^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\csc^{-1} ax \pm \frac{1}{n+1}\int\frac{x^n\,dx}{\sqrt{a^2x^2-1}},\ n\neq-1.$

Use + sign when $0 < \csc^{-1} ax < \frac{\pi}{2}$; − sign when $-\frac{\pi}{2} < \csc^{-1} ax < 0$.

Definite Integrals

354. $\displaystyle\int_0^\infty \frac{a\,dx}{a^2+x^2} = \frac{\pi}{2},$ if $a>0$; 0, if $a=0$; $\frac{-\pi}{2}$, if $a<0$.

355. $\displaystyle\int_0^\infty x^{n-1}e^{-x}\,dx = \int_0^1\left[\log\frac{1}{x}\right]^{n-1}dx = \Gamma(n).$

$\Gamma(n+1) = n\cdot\Gamma(n)$, if $n>0$. $\quad\Gamma(2) = \Gamma(1) = 1.$

$\Gamma(n+1) = n!$, if n is an integer. $\quad\Gamma(\frac{1}{2}) = \sqrt{\pi}.$

356. $\displaystyle\int_0^\infty e^{-zx}\cdot z^n\cdot x^{n-1}\,dx = \Gamma(n),\ z>0.$

357. $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^\infty\frac{x^{m-1}\,dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

358. $\displaystyle\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi},\ 0<n<1.$

359. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}}\cos^n x\,dx$

$\displaystyle= \frac{1}{2}\sqrt{\pi}\cdot\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)},$ if $n>-1$;

$\displaystyle= \frac{1\cdot3\cdot5\cdots(n-1)}{2\cdot4\cdot6\cdots(n)}\cdot\frac{\pi}{2},$ if n is an even integer;

$\displaystyle= \frac{2\cdot4\cdot6\cdots(n-1)}{1\cdot3\cdot5\cdot7\cdots n},$ if n is an odd integer.

360. $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$ $\qquad$ 361. $\displaystyle\int_0^\infty\frac{\sin ax}{x}\,dx = \frac{\pi}{2},$ if $a>0$.

362. $\displaystyle\int_0^\infty \frac{\sin x\cos ax}{x}\,dx = 0,$ if $a<-1$, or $a>1$;
$\displaystyle= \frac{\pi}{4},$ if $a=-1$, or $a=1$;
$\displaystyle= \frac{\pi}{2},$ if $-1<a<1$.

363. $\displaystyle\int_0^\pi \sin^2 ax\,dx = \int_0^\pi\cos^2 ax\,dx = \frac{\pi}{2}$

364. $\displaystyle\int_0^{\pi/a} \sin ax\cdot\cos ax\,dx = \int_0^\pi\sin ax\cdot\cos ax\,dx = 0.$

365. $\displaystyle\int_0^\pi \sin ax\sin bx\,dx = \int_0^\pi\cos ax\cos bx\,dx = 0,\quad a\neq b.$

366. $\displaystyle\int_0^\pi \sin ax\cos bx\,dx = \frac{2a}{a^2-b^2},$ if $a-b$ is odd;
$= 0,$ if $a-b$ is even.

367. $\displaystyle\int_0^\infty \frac{\sin ax\sin bx}{x^2}\,dx = \frac{1}{2}\pi a.$ if $a<b$.

368. $\displaystyle\int_0^\infty \cos(x^2)\,dx = \int_0^\infty\sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

369. $\displaystyle\int_0^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a} = \frac{1}{2a}\Gamma\left(\frac{1}{2}\right),$ if $a>0$.

370. $\displaystyle\int_0^\infty x^n\cdot e^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}},$
$\displaystyle= \frac{n!}{a^{n+1}},$ if n is a positive integer, $a>0$.

371. $\displaystyle\int_0^\infty x^{2n}e^{-ax^2}\,dx = \frac{1\cdot3\cdot5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$

372. $\displaystyle\int_0^\infty \sqrt{x}\,e^{-ax}\,dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}.$ $\qquad$ 373. $\displaystyle\int_0^\infty\frac{e^{-ax}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{a}}$

374. $\displaystyle\int_0^\infty e^{(-x^2-a^2/x^2)}\,dx = \frac{1}{2}e^{-2a}\sqrt{\pi},$ if $a>0$.

375. $\displaystyle\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2},$ if $a>0$.

376. $\displaystyle\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2},$ if $a>0$.

377. $\displaystyle\int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx = \mathrm{ctn}^{-1} a,\ a>0.$

378. $\displaystyle\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}\cdot e^{-\frac{b^2}{4a^2}}}{2a},$ if $a>0$.

379. $\displaystyle\int_0^1 (\log x)^n\,dx = (-1)^n\cdot n!,\ n$ pos. integ.

380. $\displaystyle\int_0^1 \frac{\log x}{1-x}\,dx = -\frac{\pi^2}{6}$ $\qquad$ 381. $\displaystyle\int_0^1\frac{\log x}{1+x}\,dx = -\frac{\pi^2}{12}$

382. $\displaystyle\int_0^1 \frac{\log x}{1-x^2}\,dx = -\frac{\pi^2}{8}$ $\qquad$ 383. $\displaystyle\int_0^1\frac{\log x}{\sqrt{1-x^2}}\,dx = -\frac{\pi}{2}\log 2$

384. $\displaystyle\int_0^1 \log\left(\frac{1+x}{1-x}\right)\cdot\frac{dx}{x} = \frac{\pi^2}{4}$ $\qquad$ 385. $\displaystyle\int_0^\infty\log\left(\frac{e^x+1}{e^x-1}\right)dx = \frac{\pi^2}{4}$

386. $\displaystyle\int_0^1 \frac{dx}{\sqrt{\log(1/x)}} = \sqrt{\pi}.$

387. $\displaystyle\int_0^{\frac{\pi}{2}} \log\sin x\,dx = \int_0^{\frac{\pi}{2}}\log\cos x\,dx = -\frac{\pi}{2}\log_e 2.$

388. $\displaystyle\int_0^\pi x\log\sin x\,dx = -\frac{\pi^2}{2}\cdot\log_e 2.$

389. $\int_0^1 \log(\log x)\,dx = \int_0^\infty e^{-x}\log x\,dx = -\gamma = -0.5772157\cdots$.

390. $\int_0^1\left(\log\frac{1}{x}\right)^{\frac{1}{2}}dx = \frac{\sqrt{\pi}}{2}$.

391. $\int_0^1\left(\log\frac{1}{x}\right)^{-\frac{1}{2}}dx = \sqrt{\pi}$.

392. $\int_0^1 x^m\log\left(\frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$, if $m+1>0$, $n+1>0$.

393. $\int_0^\pi \log(a\pm b\cos x)\,dx = \pi\log\left(\frac{a+\sqrt{a^2-b^2}}{2}\right)$, $a\geqq b$.

394. $\int_0^\pi \frac{\log(1+\sin a\cos x)}{\cos x}dx = \pi a$.

395. $\int_0^1 \frac{x^b-x^a}{\log x}dx = \log\frac{1+b}{1+a}$.

396. $\int_0^\pi \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}$, if $a>b>0$.

397. $\int_0^{\frac{\pi}{2}} \frac{dx}{a+b\cos x} = \frac{\cos^{-1}\left(\frac{b}{a}\right)}{\sqrt{a^2-b^2}}$, $a>b$.

398. $\int_0^\infty \frac{\cos ax\,dx}{1+x^2} = \frac{\pi}{2}\cdot e^{-a}$, if $a>0$; $=\frac{\pi}{2}e^{a}$, if $a<0$.

399. $\int_0^\infty \frac{\cos x\,dx}{\sqrt{x}} = \int_0^\infty \frac{\sin x\,dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}$.

400. $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx = \log\frac{b}{a}$.

401. $\int_0^\infty \frac{\tan^{-1}ax-\tan^{-1}bx}{x}dx = \frac{\pi}{2}\log\frac{a}{b}$.

402. $\int_0^\infty \frac{\cos ax-\cos bx}{x}dx = \log\frac{b}{a}$.

403. $\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2x+b^2\sin^2x} = \frac{\pi}{2ab}$.

404. $\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2x+b^2\sin^2x)^2} = \frac{\pi(a^2+b^2)}{4a^3b^3}$.

405. $\int_0^\pi \frac{(a-b\cos x)dx}{a^2-2ab\cos x+b^2} = 0$, if $a^2<b^2$;

$= \frac{\pi}{a}$, if $a^2>b^2$;

$= \frac{\pi}{2a}$, if $a=b$.

406. $\int_0^1 \frac{1+x^2}{1+x^4}dx = \frac{\pi}{4}\sqrt{2}$.

407. $\int_0^1 \frac{\log(1+x)}{x}dx = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots = \frac{\pi^2}{12}$.

408. $\int_{-\infty}^1 \frac{e^{-xu}}{u}du = \gamma+\log x-x+\frac{x^2}{2\cdot 2!}-\frac{x^3}{3\cdot 3!}+\frac{x^4}{4\cdot 4!}-\cdots$,

where $\gamma = \lim_{t\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{t}-\log t\right) = 0.5772157\cdots$.

409. $\int_{-\infty}^1 \frac{\cos xu}{u}du = \gamma+\log x-\frac{x^2}{2\cdot 2!}+\frac{x^4}{4\cdot 4!}-\frac{x^6}{6\cdot 6!}+\cdots$,

where $\gamma = 0.5772157\cdots$.

410. $\int_0^1 \frac{e^{xu}-e^{-xu}}{u}du = 2\left(x+\frac{x^3}{3\cdot 3!}+\frac{x^5}{5\cdot 5!}+\cdots\right)$.

411. $\int_0^1 \frac{1-e^{-xu}}{u}du = x-\frac{x^2}{2\cdot 2!}+\frac{x^3}{3\cdot 3!}-\frac{x^4}{4\cdot 4!}+\cdots$.

412. $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-K^2\sin^2x}} = \frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2K^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2K^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2K^6+\cdots\right]$, if $K^2<1$.

413. $\int_0^{\frac{\pi}{2}} \sqrt{1-K^2\sin^2x}\,dx = \frac{\pi}{2}\left[1-\left(\frac{1}{2}\right)^2K^2 -\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{K^4}{3}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{K^6}{5}-\cdots\right]$, if $K^2<1$.

414. $\int_0^\infty e^{-ax}\cosh bx\,dx = \frac{a}{a^2-b^2}$, $a>0$, $a^2\neq b^2$.

415. $\int_0^\infty e^{-ax}\sinh bx\,dx = \frac{b}{a^2-b^2}$, $a>0$, $a^2\neq b^2$.

416. $\int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}$, $a>0$.

417. $\int_0^\infty xe^{-ax}\cos bx\,dx = \frac{a^2-b^2}{(a^2+b^2)^2}$, $a>0$.

418. $\int_0^\infty x^2e^{-ax}\sin bx\,dx = \frac{2b(3a^2-b^2)}{(a^2+b^2)^3}$, $a>0$.

419. $\int_0^\infty x^2e^{-ax}\cos bx\,dx = \frac{2a(a^2-3b^2)}{(a^2+b^2)^3}$, $a>0$.

420. $\int_0^\infty x^3e^{-ax}\sin bx\,dx = \frac{24ab(a^2-b^2)}{(a^2+b^2)^4}$, $a>0$.

421. $\int_0^\infty x^3e^{-ax}\cos bx\,dx = \frac{6(a^4-6a^2b^2+b^4)}{(a^2+b^2)^4}$, $a>0$.

422. $\int_0^\infty x^ne^{-ax}\sin bx\,dx = \frac{i\cdot n!\,[(a-ib)^{n+1}-(a+ib)^{n+1}]}{2(a^2+b^2)^{n+1}}$, $a>0$.

423. $\int_0^\infty x^ne^{-ax}\cos bx\,dx = \frac{n!\,[(a-ib)^{n+1}+(a+ib)^{n+1}]}{2(a^2+b^2)^{n+1}}$, $a>0$.

GREEK ALPHABET

Letters	Names	Letters	Names	Letters	Names
Α α	Alpha	Ι ι	Iota	Ρ ρ	Rho
Β β	Beta	Κ κ	Kappa	Σ σ ς	Sigma
Γ γ	Gamma	Λ λ	Lambda	Τ τ	Tau
Δ δ	Delta	Μ μ	Mu	Υ υ	Upsilon
Ε ε	Epsilon	Ν ν	Nu	Φ φ	Phi
Ζ ζ	Zeta	Ξ ξ	Xi	Χ χ	Chi
Η η	Eta	Ο ο	Omicron	Ψ ψ	Psi
Θ θ	Theta	Π π	Pi	Ω ω	Omega

MATHEMATICAL SYMBOLS

Arithmetic, Algebra, Number Theory

$+$ Plus; positive.

$-$ Minus; negative.

$\pm$ Plus or minus; positive or negative.

$\mp$ Minus or plus; negative or positive.

ab, $a \cdot b$, $a \times b$ a times b; a multiplied by b.

a/b, $a \div b$, $a:b$ a divided by b; the ratio of a to b.

$=$, $::$ Equals (the symbol $::$ is practically obsolete).

$a/b = c/d$ or $a:b::c:d$ A proportion: a is to b as c is to d (the second form is seldom used).

$\equiv$ Identical; identically equal to.

$\neq$ Does not equal.

$\cong$ or $\risingdotseq$ Congruent; approximately equal (not common).

$\sim$ or $\backsim$ Equivalent; similar.

$>$ Greater than.

$<$ Less than.

$\geqq$ or $\geq$ Greater than or equal to.

$\leqq$ or $\leq$ Less than or equal to.

a^n $a\,a\,a \cdots$ to n factors.

$\sqrt{a}$, $a^{1/2}$ The positive square root of a, for positive a.

$\sqrt[n]{a}$, $a^{1/n}$ nth root of a, usually means the *principal* nth root.

a^{-n} The reciprocal of a^n; $1/a^n$.

( ) Parentheses.

[ ] Brackets.

{ } Braces.

$\overline{}$ Vinculum (used as a symbol of aggregation).

e The base of the system of natural logarithms;
$\lim_{n\to\infty} (1+1/n)^n = 2.7182818285\cdots$

$\log a$, $\log_{10} a$ Common (Briggsian) logarithm of a; $\log a$ is used for $\log_{10} a$ when the context shows that the base is 10.

$\ln a$, $\log a$, $\log_e a$ Natural (Napierian) logarithm of a.

antilog Antilogarithm.

colog Cologarithm.

$\exp x$ e^x, where e is the base of the natural system of logarithms ($2.718\cdots$).

$a \propto b$ a varies directly as b; a is directly proportional to b (seldom used).

i (or j) Square root of -1; $\sqrt{-1}$; j is used in physics, where i denotes current, but i is almost universally used in mathematics.

$\omega_1, \omega_2, \omega_3$ or $1, \omega, \omega^2$ The three cube roots of unity.

$n!$ (or $\underline{|n}$) Factorial n; n factorial; $1 \cdot 2 \cdot 3 \cdots n$.

a' a prime.

a'' a double prime; a second.

$a^{[n]}$ a with n primes.

a_n a sub n, a subscript n.

$f(x)$, $F(x)$, $\phi(x)$, etc. Function of x.

$|z|$ Absolute value of z; numerical value of z; modulus of z.

$\bar{z}$ or $\operatorname{conj} z$ Conjugate of z.

$\arg z$ Argument, amplitude, or phase of z.

$\mathrm{R}(z)$, $\Re(z)$, $\mathrm{Re}(z)$ Real part of z; $\mathrm{R}(z) = x$, if $z = x + iy$ and x and y are real.

$\mathrm{I}(z)$, $\Im(z)$, $\mathrm{Im}(z)$ Imaginary part of z; $\mathrm{I}(z) = y$, if $z = x + iy$ and x and y are real.

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ Unit vectors along the coordinate axes.

$\mathbf{a} \cdot \mathbf{b}$, $(\mathbf{a}, \mathbf{b})$, $S\mathbf{ab}$, $(\mathbf{ab})$ Scalar product, or dot product, of the vectors $\mathbf{a}$ and $\mathbf{b}$.

$\mathbf{a} \times \mathbf{b}$, $V\mathbf{ab}$, $[\mathbf{ab}]$ Vector product, or cross product, of the vectors $\mathbf{a}$ and $\mathbf{b}$.

$[\mathbf{abc}]$ The *scalar triple product* of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, or $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$.

$P(n, r)$, ${}_nP_r$ The number of permutations of n things taken r at a time; $n!/(n-r)! = n(n-1)(n-2) \cdots (n-r+1)$.

${}_nC_r$, $\{{n \atop r}\}$, C_r^n, or $C(n, r)$ The number of combinations of n things taken r at a time; $n!/[r!(n-r)!]$; the $(r+1)$st binomial coefficient.

$|a_{ij}|$ The determinant whose element in the ith row and jth column is a_{ij}.

$\|a_{ij}\|$ or (a_{ij}) The matrix whose element in the ith row and the jth column is a_{ij}.

$|abc\cdots|$ The determinant $\begin{vmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \\ \cdot & \cdot & \cdots \end{vmatrix}$

$\|abc\cdots\|$ or $(abc\cdots)$ The matrix
$$\begin{Vmatrix} a_1\, a_2 \cdots \\ b_1\, b_2 \cdots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{Vmatrix} \quad \text{or} \quad \begin{pmatrix} a_1\, a_2 \cdots \\ b_1\, b_2 \cdots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{pmatrix}$$

$\begin{pmatrix} a\, b\, c \cdots \\ b\, c\, d \cdots \end{pmatrix}$ or $(abcd\cdots)$ The permutation which replaces a by b, b by c, c by d, etc.

adj A, $[A_{ji}]$, (A) Adjoint of the matrix $A=[a_{ij}]$.

$\bar{A}$ Complex conjugate of the matrix A.

I An identity matrix.

A^{-1} Inverse of the matrix A.

A^* Hermitian conjugate of the matrix A.

A', A^T Transpose of the matrix A.

A_{ij} Cofactor of the element a_{ij} in the matrix $[a_{ij}]$.

$\|A\|$ The norm of the matrix A.

$\oplus$ An operation in a postulational algebraic system, with $x \oplus y$ called the sum of x and y.

$\otimes$ An operation in a postulational algebraic system, with $x \otimes y$ called the product of x and y.

$\circ$, $*$ An operation in a postulational algebraic system, with $x \circ y$, or $x * y$ a third element of the system.

G.C.D. or g.c.d. Greatest common divisor.

L.C.D. or l.c.d. Least common denominator.

L.C.M. or l.c.m. Least common multiple.

(a, b) The G.C.D. of a and b; the open interval from a to b.

$[a, b]$ The L.C.M. of a and b; the closed interval from a to b.

$a|b$ a divides b.

$x \equiv a \pmod p$ $x-a$ is divisible by p, read: x is congruent to a modulus p, or modulo p.

$[x]$ The greatest integer not greater than x.

$\phi(n)$ Euler's ϕ-function (the number of positive integers prime to n and not greater than n).

$p(n)$ The number of partitions of n.

$d(n)$ The number of divisors of n.

$\nu(n)$ The number of different primes which divide n.

$\pi(n)$ The number of primes which are not greater than n.

$\lambda(n)$ Liouville's function.

$\mu(n)$ Möbius' function.

Trigonometry and Hyperbolic Functions

a° a degrees (angle).

a' a minutes (angle).

a'' a seconds (angle).

$a^{(r)}$ a radians, unusual.

s One-half the sum of the lengths of the sides of a triangle (plane or spherical).

S, σ One-half the sum of the angles of a spherical triangle.

E Spherical excess.

sin Sine.

cos Cosine.

tan Tangent.

ctn (or **cot**) Cotangent.

sec Secant.

csc Cosecant.

covers Coversed sine or coversine.

exsec Exsecant.

gd (or **amh**) Gudermannian (or hyperbolic amplitude).

hav Haversine.

vers Versed sine or versine.

$\sin^{-1} x$ (or arc sin x) The principal value of the angle whose sine is x (when x is real); antisine x; inverse sine x.

$\sin^2 x$, $\cos^2 x$, etc. $(\sin x)^2$, $(\cos x)^2$, etc.

sinh Hyperbolic sine.

cosh Hyperbolic cosine.

tanh Hyperbolic tangent.

ctnh (or **coth**) Hyperbolic cotangent.

sech Hyperbolic secant.

csch Hyperbolic cosecant.

$\sinh^{-1} x$ (or arc sinh x) The number whose hyperbolic sine is x; antihyperbolic sine of x; inverse hyperbolic sine of x.

Elementary and Analytic Geometry

$\angle$ Angle.

$\angle\!\!s$ Angles.

$\perp$ Perpendicular; is perpendicular to.

$\perp\!\!s$ Perpendiculars.

$\parallel$ Parallel; is parallel to.

$\parallel$s Parallels.

$\cong$, $\equiv$ Congruent; is congruent to.

$\therefore$ Therefore; hence.

$\triangle$ Triangle.
$\triangle$s Triangles.
$\square$ Parallelogram.
$\square$ Square.
$\bigcirc$ Circle.
Ⓢ Circles.
π The ratio of the circumference of a circle to the diameter, the Greek letter pi, equal to 3.1415926536-.
(x, y) Rectangular coordinates of a point in a plane.
(x, y, z) Rectangular coordinates of a point in space.
(r, θ) Polar coordinates.
χ The angle from the radius vector to the tangent to a curve.
(ρ, θ, ϕ) or (r, θ, ϕ) Spherical coordinates: ϕ = colatitude (or longitude), θ = longitude (or colatitude).
(r, θ, z) Cylindrical coordinates.
$\cos \alpha, \cos \beta, \cos \gamma$ Direction cosines.
l, m, n Direction numbers.
e Eccentricity of a conic.
p Half of the latus rectum of a parabola (usage general in U. S.).
m Slope.
$P(x, y)$ or $P{:}(x, y)$ Point P with coordinates x and y in the plane.
$P(x, y, z)$ or $P{:}(x, y, z)$ Point P with coordinates x, y, z in space.
(AB, CD) or $(AB \mid CD)$ The cross ratio of the elements (points, lines, etc.) A, B, C, and D, the quotient of the ratio in which C divides AB by the ratio in which D divides AB.
$[A] \overline{\wedge} [B]$ Indicates that there is a perspective correspondence between the ranges $[A]$ and $[B]$.
$[A] \wedge [B]$ Indicates a projective correspondence between ranges $[A]$ and $[B]$.

Calculus and Analysis

(a, b) The open interval $a < x < b$.
$[a, b]$ The closed interval $a \leqq x \leqq b$.
$(a, b]$ The interval $a < x \leqq b$.
$[a, b)$ The interval $a \leqq x < b$.
$\{a_n\}, [a_n], (a_n)$ The sequence $a_1, a_2, \cdots, a_n, \cdots$.
$\sum_1^n$ or $\sum_{i=1}^n$ Sum to n terms, one for each positive integer from 1 to n.
$\sum$ Sum of certain terms, the terms being indicated by the context or by added notation, as in $\sum_{i=1}^n X_i$ or $\sum_{\alpha \in A} X_\alpha$.
$\prod_1^n$ or $\prod_{i=1}^n$ Product of n terms, one for each positive integer from 1 to n.
$\prod$ Product of certain terms, the terms being indicated by the context or by added notation, as in $\sum_{i=1}^n X_i$ or $\sum_{\alpha \in A} X_\alpha$.
I Moment of inertia.
k Radius of gyration.
$\bar{x}, \bar{y}, \bar{z}$ Coordinates of the center of mass.
s (or σ) Length of arc.
ρ Radius of curvature.
κ Curvature of a curve.
τ Torsion of a curve.
l.u.b. or sup Least upper bound.
g.l.b. or inf Greatest lower bound.
$\lim_{x \to a} y = b$, or $\lim_{x=a} y = b$ The limit of y as x approaches a is b.
$\overline{\lim}_{n \to \infty} t_n$ The greatest of the limits of the sequence (t_n); the largest number which is a limit point of the set of numbers (t_n); limit superior of (t_n).
$\underline{\lim}_{n \to \infty} t_n$ The least of the limits of the sequence (t_n); the smallest number which is a limit point of the set of points (t_n); limit inferior of (t_n).
$\rightarrow$ Approaches, or implies.
lim sup or $\overline{\lim}$ Limit superior.
lim inf or $\underline{\lim}$ Limit inferior.
$f(a+0), f(a+), \lim_{x \downarrow a} f(x)$, or $\lim_{x \to a+} f(x)$ The limit on the right of $f(x)$ at $x = a$.
$f(a-0), f(a-)$, $\lim_{x \uparrow a} f(x)$, or $\lim_{x \to a-} f(x)$ The limit on the left of $f(x)$ at $x = a$.
$f'(a+)$ The derivative on the right of f at $x = a$.
$f'(a-)$ The derivative on the left of f at $x = a$
Δy An increment of y.
∂y A variation in y; an increment of y.
dy Differential of y.
$\dot{s}, ds/dt, v$ The derivative of s with respect to t; speed.
$\ddot{s}, dv/dt, d^2s/dt^2, a$ The second derivative of s with respect to the time t; acceleration.

ω, α Angular speed and angular acceleration, respectively.

$\frac{dy}{dx}, \frac{df(x)}{dx}, y', f'(x), D_x y$ The derivative of y with respect to x, where $y=f(x)$.

$\frac{d^n y}{dx^n}, y^{(n)}, f^{(n)}(x), D_x{}^n y$ The nth derivative of $y=f(x)$ with respect to x.

$\frac{\partial u}{\partial x}, u_x, f_x(x, y), D_x u$ The partial derivative of $u=f(x, y)$ with respect to x.

$\frac{\partial^2 u}{\partial y \partial x}, u_{xy}, f_{xy}(x, y), D_y(D_x u)$ The second partial derivative of $u=f(x, y)$, taken first with respect to x, and then with respect to y.

D The operator $\frac{d}{dx}$.

D_i, D_{ij}, etc. Partial differentiation operators $\left(e.g., D_{ij}=\frac{\partial^2}{\partial x_i \partial x_j}\right)$.

$D_s f$ Directional derivative of f in the direction s.

E The operator defined by $Ef(x)=f(x+h)$, for a specified constant h.

Δ The operator defined by $\Delta f(x)=f(x+h)-f(x)$, for a specified constant h (also see below, ∇^2 or Δ).

∇ Del: the operator

$$\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).$$

∇u or grad u Gradient of u:

$$\left(i\frac{\partial u}{\partial x}+j\frac{\partial u}{\partial y}+k\frac{\partial u}{\partial z}\right).$$

$\nabla \cdot \mathbf{v}$ or **div v** Divergence of **v**.

$\nabla \times \mathbf{F}$ Curl of **F**.

∇^2 or Δ The Laplacian operator:

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

δ^i_j Kronecker delta.

$\delta^{i_1, i_2, \cdots, i_k}_{j_1, j_2, \cdots, j_k}$ The generalized Kronecker delta.

$\epsilon^{i_1, i_2, \cdots i_k}, \epsilon_{i_1, i_2 \cdots i_k}$ The epsilon symbols.

g_{ij}, g^{ji} The components of the fundamental metric tensor of a *Riemannian space*, $ds^2=g_{ij}dx^i dx^j=g^{ij}dx_i dx_j$.

E, F, G The coefficients in the first fundamental quadratic form of a surface.

$F(x)]_a^b$ $F(b)-F(a)$.

$\int f(x)\,dx$ Integral of $f(x)$ with respect to x, the primitive of $f(x)$.

$\int_a^b f(x)\,dx$ The definite integral of $f(x)$ between the limits a and b.

$\overline{\int_a^b}$ The upper Darboux integral.

$\underline{\int_a^b}$ The lower Darboux integral.

$m_e(S), m^*(S), \mu^*(S)$ Exterior measure of S.

$m_i(S), m_*(S), \mu_*(S)$ Interior measure of S.

$m(S), \mu(S)$ Measure of S.

a.e. Almost everywhere; except for a set of measure zero.

G_δ set; F_σ set See BOREL—Borel set.

BV Of bounded variation.

$T_f(I), V_f(I)$, or $V(f, I)$ Total variation of f on the interval I.

$\Omega_f(I), \omega_f(I)$, or $o_f(I)$ Oscillation of f on I.

$\omega_f(x), o_f(x)$ Oscillation of f at the point x.

(f, g) Inner product of the functions f and g.

$\|f\|$ Norm of the function f; *i.e.*, $(f, f)^{1/2}$.

$f*g$ Convolution of f and g.

$W(u_1, u_2, \cdots, u_n)$ Wronskian of $u_1, u_2, \cdots, u_n$.

$\frac{\partial(f_1, f_2, \cdots, f_n)}{\partial(x_1, x_2, \cdots, x_n)}, \quad \frac{D(f_1, f_2, \cdots, f_n)}{D(x_1, x_2, \cdots, x_n)}$, or $J\left(\frac{f_1, f_2, \cdots, f_n}{x_1, x_2, \cdots, x_n}\right)$ Jacobian of the functions $f_i(x_1, x_2, \cdots, x_n)$.

$C_n, C^{(n)}$ See FUNCTION—function of class C^n.

$L_p, L^{(p)}$ See FUNCTION—function of class L_p.

$f(x)\sim\sum_0^\infty A_n$ The series is an asymptotic expansion of the function $f(x)$.

$x_n \sim y_n$ Limit $x_n/y_n=1$; x_n and y_n are *asymptotically equal*.

$u_n=O(v_n)$ u_n is *of the order of* v_n (u_n/v_n is bounded).

$u_n=o(v_n)$ $\lim_{n\to\infty} u_n/v_n=0$.

summable C_k, or $(C\ k)$ Summable by Cesàro's method of summation of order k.

γ Euler's constant.

$B_1, B_2, B_3, \cdots$ The Bernoullian numbers. The Bernoullian numbers are also sometimes taken as $B_1, B_3, B_5, \cdots$.

$|z|$ Absolute value of z.

$\bar{z}$ or **conj** z Conjugate of z.

arg z Argument of z.

$\mathrm{R}(z), \Re(z), \mathrm{Re}(z)$ Real part of z.

$\mathrm{I}(z)$ $\Im(z), \mathrm{Im}(z)$ Imaginary part of z.

$\operatorname{Res}_{z=a} f(z)$ Residue of f at a.

$\Gamma(z)$ The Gamma function.

$(\gamma a, x)$; $\Gamma(a, x)$ Incomplete gamma functions.

$B_n(x)$ The Bernoulli polynomial of degree n.

$F(a, b; c; z)$ A hypergeometric function.

$H_n(x)$ The Hermite polynomial of degree n.

$J_n(p, q; x)$ A Jacobi polynomial.

$J_n(x)$ The nth Bessel function.

$I_n(z)$; $K_n(z)$ Modified Bessel functions.

$H_n^{(1)}(z)$; $H_n^{(2)}(z)$ Hankel functions.

$N_p(z)$; $Y_n(z)$ Neumann's functions.

$\beta(m, n)$, $B(m, n)$ The beta function.

$B_x(m, n)$ Incomplete beta function.

$ber(z)$, $bei(z)$, $ker(z)$, $kei(z)$ See BER.

$J(\tau)$; $\lambda(\tau)$, $f(\tau)$, $g(\tau)$, $h(\tau)$ Modular functions.

$\vartheta_1(z)$, etc. Theta functions.

$\vartheta_1, \vartheta_2, \cdots; \vartheta_1', \cdots$ Theta functions and their derivatives with zero argument.

$\zeta(z)$ Riemann's zeta function.

$Erf(x)$ $\int_0^x e^{-t^2}\,dt = \frac{1}{2}\gamma(\frac{1}{2}, x^2)$; see ERROR—error function.

$Erfc(x)$ $\int_x^\infty e^{-t^2}\,dt = \frac{1}{2}\pi^{1/2} - Erf(x) = \frac{1}{2}\Gamma(\frac{1}{2}, x^2)$.

$Erfi(x)$ $\int_0^x e^{t^2}\,dt = -i\,Erf(ix)$.

$L_n(x)$ The Laguerre polynomial of degree n.

$L_n^k(x)$ An associated Laguerre polynomial.

$P_n(x)$ The Legendre polynomial of degree n.

$P_n^m(x)$ An associated Legendre function.

$T_n(x)$ The Tchebycheff polynomial of degree n.

$ce_n(x)$, $se_n(x)$ Mathieu functions.

$sn\, z$, $cn\, z$, $dn\, z$ The Jacobian elliptic functions.

$p(z)$, $p'(z)$ The Weierstrass elliptic functions.

Mathematical Logic and Set Theory

$\therefore$ Therefore.

$\ni$ Such that.

$\sim p$, $-p$, $\bar{p}$, p' Not p.

$p \wedge q$, $p \cdot q$, $p \,\&\, q$ Both p and q; p and q.

$p \vee q$, $p \vee q$ At least one of p and q; p or q.

$p|q$, p/q Not both p and q; not p or not q.

$p \downarrow q$, $p \Delta q$ Neither p nor q.

$p \rightarrow q$, $p \supset q$ If p, then q; p only if q.

$\leftrightarrow$, $\equiv$, $\sim$, iff If and only if.

V, V, I, 1 The universal class (containing all the members of some specific class, such as the set of all real numbers).

ϕ, $\wedge$, Λ, 0 The null class; the class containing no members.

., :, :., etc. Dots used in place of parentheses, n dots being stronger than $n-1$ dots, and n dots with a sign such as $\vee$, $\rightarrow$, or $\leftrightarrow$ being equivalent to $n+1$ dots.

(x), Π_x, A_x, V_x For all x.

$A_{x,y},\cdots$; $V_{x,y},\cdots$ For all $x, y, \cdots$.

$\exists$ There exists.

$(\exists x)$, (Ex), Σ_x There is an x such that.

$E_{x,y,\ldots}$ There exist $x, y, \cdots$ such that.

E_x, $\hat{x}$, C_x, $[x|\ ]$ The class of all objects x which satisfy the condition stated after the symbol (or after the vertical bar of the last symbol).

$x \epsilon M$, $x \in M$ The point x belongs to the set M.

$M = N$ The sets M and N coincide.

$M \subset N$ Each point of M belongs to N; M is a subset of N. Sometimes (but rarely) understood to mean that M is a proper subset of N.

$M \subseteqq N$ Each point of M belongs to N; M is a subset of N.

$M \supset N$ Each point of N belongs to M; M contains N as a subset. Sometimes (but rarely) understood to mean that N is a proper subset of M.

$M \supseteqq N$ Each point of N belongs to M; M contains N as a subset.

$M \cap N$, $M \cdot N$ The intersection of M and N.

$M \cup N$, $M + N$ The join (or sum) of M and N.

$\cap_{\alpha \epsilon A} M_\alpha$, $\Pi_{\alpha \epsilon A} M_\alpha$ The set of all points which belong to M_α for all α of A.

$\cup_{\alpha \epsilon A} M_\alpha$, $\Sigma_{\alpha \epsilon A} M_\alpha$ The set of all points which belong to M_α for some α of A.

$\sim M$, $C(M)$, $\bar{M}$, $\tilde{M}$ The complement of M.

$M - N$, $M \sim N$ The complement of N in M; all points of M not in N.

$M \sim N$ The sets M and N can be put into one-to-one correspondence.

$\aleph$ Aleph, the first letter of the Hebrew alphabet.

$\aleph_0$ Aleph-null, or aleph-zero. The cardinal number of the set of positive integers.

c The cardinal number of the set of all real numbers.

$\aleph_\alpha$ An infinite cardinal number, the least being $\aleph_0$, the next $\aleph_1$, the next $\aleph_2$, etc. The first cardinal number greater than $\aleph_\alpha$ is denoted by $\aleph_{\alpha+1}$.

$M \simeq N$ M and N are of the same ordinal type.

ω The ordinal number of the positive integers in their natural order.

ω^*, $^*\omega$ The ordinal number of the negative integers in their natural order.

π The ordinal number of all integers in their natural order.

η The ordinal number of the rational numbers in the open interval (0, 1).

θ The ordinal number of the real numbers of the closed interval [0, 1].

α^*, $^*\alpha$ The ordinal number of a simply ordered set whose ordering is exactly reversed from that of a set of ordinal type α.

Topology and Abstract Spaces

$\bar{M}$ The closure of M.

M' The derived set of M.

$d(x, y), \delta(x, y), \rho(x, y), (x, y)$ Distance from x to y.

$M \times N$ The Cartesian product of spaces M and N.

M/N The quotient space of M by N.

$\prec, <; \succ, >$ Symbols denoting an order relation.

T_0**-space** A topological space such that for distinct x and y there is either a neighborhood of x not containing y or a neighborhood of y not containing x.

T_1**-space** A topological space such that for distinct x and y there is a neighborhood of x not containing y.

T_2**-space** A Hausdorff topological space.

T_3**-space** A T_2-space which is regular.

T_4**-space** A T_2-space which is normal.

E_n, E^n, R_n, R^n Real n-dimensional Euclidean space.

Z_n, C_n Complex n-dimensional space.

$H, \mathfrak{H}$ Hilbert space.

(x, y) Inner product of the elements x and y of a vector space.

$\|x\|$ Norm of x (see VECTOR—vector space).

(B)**-space** A Banach space.

$(C), C$ The space of all continuous real-valued functions on some specified compact set, as on the closed interval [0, 1] (then sometimes denoted by $C[0, 1]$), with $\|f\|$ defined as $\sup|f(x)|$.

$(M), M$ The space of bounded functions on some set (particularly the interval [0, 1]), with $\|f\|$ defined as $\sup|f(x)|$.

$(m), m$ The space of all bounded sequences $x=(x_1, x_2, \cdots)$, with $\|x\|$ defined as $\sup|x_i|$.

$(c), c$ The space of all convergent sequences $x=(x_1, x_2, \cdots)$, with x defined as $\sup|x_i|$.

$(c_0), c_0$ The space of all sequences $x=(x_1, x_2, \cdots)$ with $\lim_{n\to\infty} x_n=0$ and $\|x\|$ defined as $\sup|x_i|$.

$l_p, l^{(p)}$ The space of all sequences $x=(x_1, x_2, \cdots)$ with $\Sigma|x_i|^p$ convergent $(p \geqq 1)$ and $\|x\|$ defined as $[\Sigma|x_i|^p]^{1/p}$.

$L_p, L^{(p)}$ The space of all measurable functions f on a specified set S with $|f(x)|^p$ integrable $(p \geqq 1)$ and $\|f\| = \left[\int_S |f(x)|^p\,dx\right]^{1/p}$; S is frequently taken as the interval [0, 1].

p The genus of an orientable surface (sometimes the number of "handles," whether the surface is orientable or not—see SURFACE).

q The number of cross-caps on a non-orientable surface (see SURFACE).

r The number of boundary curves on a surface.

χ Euler characteristic.

$\partial S, \Delta S, d(S)$ Boundary of the set S.

B^s_m An s-dimensional Betti group modulo m (m a prime).

B^s_0 An s-dimensional Betti group relative to the group of integers.

R^s_m An s-dimensional Betti number modulo m (m a prime).

R^s_0 An s-dimensional Betti number relative to the group of integers.

Mathematics of Finance

% Per cent.
\$ Dollar, dollars.
¢ Cent; cents.
@ At.
P Principal; present value.
$j_{(p)}$ Nominal rate (p conversion periods per year).
i, j, r Rate of interest.
s Compound amount of \$1 for n periods; $s=(1+i)^n$.
n Number of periods or years, usually years.
v^n Present value of \$1 (n periods); $v^n=1/(1+i)^n$.
l_x Number of persons living at age x (mortality table).
d_x Number of deaths per year of persons of age x (mortality table).
p_x Probability of a person of age x living one year.
q_x Probability of a person of age x dying within one year.
D_x $v^x l_x$.
C_x $v^{x+1} d_x$.
N_x $D_x+D_{x+1}+D_{x+2}+ \cdots$ to table limit.
M_x $C_x+C_{x+1}+C_{x+2}+ \cdots$ to table limit.
a_x Present value of a life annuity of \$1 at age x; N_{x+1}/D_x.
a_x, $\ddot{a}_x$ Present value of a (life) annuity due of \$1 at age x; $1+a_x$.
u_x D_x/D_{x+1}.
k_x C_x/D_{x+1}.
A_x Net single premium for \$1 of whole-life insurance taken out at age x.
P_x Net annual premium for an insurance of \$1 at age x on the ordinary life plan.
${}_nA_x$ Net single premium for \$1 of term insurance for n years for a person aged x.
${}_nP_x$ Premium for a limited payment life policy of \$1 with a term of n years at age x.
${}_nV_x$ Terminal reserve, at the end of n years after the policy was issued, on an ordinary life policy of \$1 for a life of age x.
${}_nE_x$ The present value of a pure endowment to be paid in n years to a person of age x.
R Annual rent.
$s_{\overline{n|}}$ or $s_{\overline{n|}i}$ Compound amount of \$1 per annum for n years at interest rate i; $[(1+i)^n-1]/i$.
$s_{\overline{n|}}^{(p)}$ or $s_{\overline{n|}i}^{(p)}$ Amount of \$1 per annum for n years at interest rate i when payable in p equal installments at intervals of $(1/p)$th part of a year.
$a_{\overline{n|}}$ or $a_{\overline{n|}i}$ Present value of \$1 per annum for n years at interest rate i; $[1-(1+i)^{-n}]/i$.
$a_{\overline{n|}}^{(p)}$ or $a_{\overline{n|}i}^{(p)}$ Present value of \$1 per annum for n years at interest rate i if payable in p installments at intervals of $(1/p)$th part of a year.
$A_{\overline{xn|}}$ Net single premium for an endowment insurance for \$1 for n years at age x.
$P_{\overline{xn|}}$ Net annual premium for an n-year endowment policy for \$1 taken out at age x.
$A^1_{\overline{xn|}}$ or $|_nA_x$ Net single premium for an endowment policy of \$1 for n years taken out at age x.
$P^1_{\overline{xn|}}$ or $|_nP_x$ Net annual premium for a term insurance of \$1 for n years at age x.

Statistics

χ^2 Chi-square.
d.f. Degrees of freedom.
F F ratio.
i Width of a class interval.
k Coefficient of alienation.
P.E. Probable error (same as probable deviation).
r Correlation coefficient (Pearson product moment correlation coefficient between two variables).
$r_{12\cdot34\cdots n}$ Partial correlation coefficient between variables 1 and 2 in a set of n variables.
$r_{1\cdot234\cdots n}$ Multiple correlation coefficient between variable 1 and remainder of a set of n variables.
s Standard deviation (from a sample).
σ_x Standard deviation of the population of x.

$\sigma_{x \cdot y}$ Standard error of estimate; also standard deviation of an x array for given value of y.

t Students' "t" statistic.

V Coefficient of variation.

$\bar{x}$ Arithmetic average of the variable x (from a sample).

μ Arithmetic mean of a population.

$\mu_2=\sigma^2$ Second moment about the mean.

μ_r The rth moment about the mean.

$\beta_1=\frac{\mu_3{}^2}{\mu_2{}^2}$ Coefficient of skewness.

$\beta_2=\frac{\mu_4}{\mu_2{}^2}$ Coefficient of kurtosis.

$\beta_{12 \cdot 34}$ Multiple-regression coefficient in terms of standard-deviation units.

η Correlation ratio.

z Fisher's z statistic.

Q_1 First quartile.

Q_3 Third quartile.

$E(x)$ Expectation of x.

$P(x_i)$ Probability that x assumes the value x_i.